Random signal and noise

Random Process
Autocorrelation
Power spectral density
Ex 6.31

\(X_n\) is iid sequence of \(N(0, \sigma^2)\) r.v.'s. \(Y_n\) is the average of two consecutive values

\[
Y_n = \frac{X_n + X_{n-1}}{2}
\]

\[
m_Y = 0.5E\left[ X_n + X_{n-1} \right] = 0
\]

\[
C_Y(i, j) = E\left[ Y_i Y_j \right] = 0.25E\left[ (X_i + X_{i-1})(X_j + X_{j-1}) \right]
\]

\[
= 0.25\left\{ E\left[ X_i X_j \right] + E\left[ X_i X_{j-1} \right] + E\left[ X_{i-1} X_j \right] + E\left[ X_{i-1} X_{j-1} \right] \right\}
\]

\[
= \frac{1}{2} \sigma^2 \delta_{i,j} + \frac{1}{4} \sigma^2 \delta_{i,j+1} + \frac{1}{4} \sigma^2 \delta_{i,j-1}
\]

\(Y_n\) is WSS

\(Y_n\) is a linear trafo of Gaussian r.v.'s, so \(Y_n\) is Gaussian

joint PDF is Gaussian, specified by \(m_Y\) and \(C_Y(i, j)\)
Time averages & ergodicity

$$\hat{m}_X(t) = \frac{1}{N} \sum_{i=1}^{N} X(t, \zeta_i)$$  
ensemble averaging  
repeating the experiment  
from many realizations

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t, \zeta) \, dt$$  
time averaging  
based on single realization

Ergodic $\longrightarrow$ Stationary but not Vice versa

an ergodic theorem states conditions under which a time average converges when the observation interval becomes large

we’re interested in ergodic theorems that state when time averages converge to the ensemble average or expected value
Mean, Autocorrelation of sine wave with random phase

\[ X(t) = A \cos(2\pi f_c t + \theta) \]

\( A \) and \( f_c \) are constants and \( \theta \) is a RV that is uniformly distributed over range of 0 and \( 2\pi \)

\[
m_x(t) = \int_0^{2\pi} A \cos(2\pi f_c t + \theta) \frac{1}{2\pi} d\theta
\]

\[
m_x(t) = A \left[ \sin(2\pi f_c t + \theta) \right]_0^{2\pi} = zero
\]

\[
< X(t)_T > = \int_0^T A \cos(2\pi f_c t + \theta) dt
\]

\[
< X(t)_T > = A \left[ \sin(2\pi f_c t + \theta) \right]_0^T = zero
\]

Ergodic
Mean, Autocorrelation of sine wave with random phase

\[ X(t) = A \cos(2\pi f_c t + \theta) \]

A and \( f_c \) are constants and \( \theta \) is a RV that is uniformly distributed over range of 0 and \( 2\pi \)

\[ R_x(t_1, t_2) = E[X(t_1)X(t_2)] = A^2 E[\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)] \]

\[ R_x(t_1, t_2) = \frac{A^2}{2} E[\cos(2\pi f_c (t_1 - t_2)) + \cos(2\pi f_c (t_1 + t_2) + 2\theta)] \]

\[ R_x(t_1, t_2) = \frac{A^2}{2} \cos(2\pi f_c (t_1 - t_2)) = R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c (\tau)) \quad \text{W.S Stationary} \]

\[ < X(t)X(t+\tau) > = A^2 \frac{1}{2T} \int_{-T}^{T} \cos(2\pi f_c t + \theta)\cos(2\pi f_c (t+\tau) + \theta)dt \]

\[ < X(t)X(t+\tau) > = A^2 \frac{1}{2T} \int_{-T}^{T} [\cos(2\pi f_c \tau) + \cos(2\pi f_c (2t + \tau) + 2\theta)]dt \]

\[ < X(t)X(t+\tau) > = \frac{A^2}{2} \cos(2\pi f_c (\tau)) = R_x(\tau) \quad \text{Ergodic} \]
Properties of Gaussian RP

1. If a Gaussian process $X(t)$ is applied to a LTIS then the output is also a Gaussian Process.
2. Gaussian Random process defined at a set of time instants is completely defined by the vector mean $M$ and the covariance matrix $C$
3. If $X(t)$ is a Gaussian WSS RP then it is a SS RP
4. If $x(t)$ is a Gaussian RP with uncorrelated RV then they are also independent.
Cross Correlation function

- Consider two random processes $X(t)$ and $Y(t)$ with autocorrelation functions $R_x(t_1,t_2)$, $R_y(t_1,t_2)$ respectively.
- The cross correlation function of $X(t)$ and $Y(t)$ is defined by $R_{xy}(t_1,t_2) = E[X(t_1)Y(t_2)]$ and $R_{yx}(t_1,t_2) = E[Y(t_1)X(t_2)]$.
- If $X(t)$ and $Y(t)$ are WSS then $R_{xy}(t_1,t_2) = R_{xy}(\tau)$
- $R_{xy}(\tau) = R_{yx}(-\tau)$
Power Spectral Density

From Communication Theory we know that Autocorrelation is the IFT of PSD

IF $X(t)$ is WSS RP

$$R_X(\tau) \overset{F.T.}{\leftrightarrow} S_X(f)$$

Properties of PSD

1. $S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau$

2. $E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$

3. $S_X(f) \geq 0$  Non Negative

4. $S_X(f) = S_X(-f)$  for real valued random process
Ex. PSD of sine wave with random phase

\[ R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c(\tau)) \] W.S Stationary

- Using the F.T

\[ S_x(f) = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)] \]
\[ R_x(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right) & |\tau| \leq T \\ 0 & \text{Otherwise} \end{cases} \]

Using the F.T

\[ S_x(f) = A^2 T \sin c^2(fT) \]

\[ Y(t) = X(t) \cos(2\pi f_c t + \theta) \]
\[ R_y(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(2\pi f_c t + \theta)X(t+\tau)\cos(2\pi f_c (t+\tau) + \theta)] \]
\[ R_y(\tau) = E[X(t)X(t+\tau)]E[\cos(2\pi f_c t + \theta)\cos(2\pi f_c (t+\tau) + \theta)] \]
\[ R_y(\tau) = R_x(\tau) \cos(2\pi f_c \tau) \]
\[ S_y(f) = \frac{1}{2} \left[ S_x(f - f_c) + S_x(f - f_c) \right] \]
Relation among PSD of the input and output RP of LTIS

Suppose RP X(t) is applied to a LTIS with impulse response h(t), producing RP Y(t).

What is the PSD of Y(t) w.r.t PSD of X(t) Assuming X(t) is WSS RP.

\[ S_Y (f) = S_X (f) |H(f)|^2 \]

Ex.

\[ H(f) = 1 - \exp(-j2\pi f T) \]
\[ |H(f)|^2 = H(f) H^*(f) \]
\[ |H(f)|^2 = (1 - \exp(-j2\pi f T))(1 - \exp(+j2\pi f T)) \]
\[ |H(f)|^2 = (1 + 1 - \exp(-j2\pi f T) - \exp(+j2\pi f T)) \]
\[ |H(f)|^2 = 2(1 - \cos(2\pi f T)) = 4 \sin^2(\pi f T) \]
\[ S_Y (f) = 4 \sin^2(\pi f T) S_X (f) \]
Cross Spectral Density

- Provides a measure of the frequency interrelationship between two random processes.
- $X(t)$ and $Y(t)$ are jointly WSS RP with $R_{XY}(\tau)$ and $R_{YX}(\tau)$
- Thus they have as a FT. $S_{XY}(f)$ and $S_{YX}(f)$
- $S_{XY}(f) = S_{YX}(-f)$
Example

- $X(t)$ and $Y(t)$ have zero mean and they are individually stationary in the wide sense.
- $Z(t) = X(t) + Y(t)$  find $S_Z(f)$

$$R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)]$$
$$= E[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))]$$
$$= E[X(t_1)X(t_2)] + E[X(t_2)Y(t_1)] + E[X(t_1)Y(t_2)] + E[Y(t_1)Y(t_2)]$$

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

$$S_Z(f) = S_X(f) + S_Y(f) + S_{XY}(f) + S_{YX}(f)$$

If $X$ and $Y$ are uncorrelated

$$S_Z(f) = S_X(f) + S_Y(f)$$
Noise

- Unwanted signal that tend to disturb the transmission and processing of signals in communication systems.
- Thermal Noise $\rightarrow$ random motion of electrons in a conductor.
- Shot noise $\rightarrow$ arises in electronic devices, sudden change in voltage or current.
White Noise

- White → occupies all frequencies → PSD is independent on the operating frequency
- Dimensions of $N_o$ is watt per Hertz, $N_o=KT$
  \[
  S_N(f) = \frac{N_o}{2}
  \]
  \[
  R_N(\tau) = \frac{N_o}{2} \delta(\tau)
  \]
- Any two different samples of white noise no matter how close they are will be uncorrelated.
- If white noise is Gaussian then they will also be independent
Ideal low pass filtered white noise

- A white Gaussian noise with zero mean and variance $N_0/2$ is applied to an ideal low pass filter of bandwidth $B$ and amplitude response of one.
- PSD of output $Y(t)$ is
  \[
  S_Y(f) = \begin{cases} 
  \frac{N_0}{2} & -B \leq f \leq B \\
  0 & \text{otherwise}
  \end{cases}
  \]
- The autocorrelation is
  \[
  R_Y(\tau) = N_0B \sin c(2B\tau)
  \]
- Autocorrelation maximum at $\tau$ equal zero equal $N_0B$ and passes through zero at $\tau=n/2B$ for $n=\text{integer values and variance } N_0B$
- If noise is sampled at rate $2B$ then they are uncorrelated and being Gaussian then statistically independent
RC low pass filtered white noise

- The $H(f)$ of RC filter is
  \[ H(f) = \frac{1}{1 + j2\pi fRC} \]

- The PSD of the o/p is
  \[ S_Y(f) = \frac{1}{1 + (2\pi fRC)^2} \]

- The autocorrelation of the output is
  \[ R_Y(\tau) = \frac{N_o}{4RC} \exp\left(-\frac{|\tau|}{RC}\right) \]

- If noise is samples at rate $0.217/RC$ then they are uncorrelated and being Gaussian then statistically independent
Ex sine wave plus white noise

\[ X(t) = A \cos(2\pi f_c t + \theta) + N(t) \]

\( \theta \) is Uniformly distributed, \( N(t) \) is WGN

\[ R_x(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2} \cos(2\pi f_c \tau) + \frac{N_0}{2} \delta(\tau) \]

\[ S_x(f) = \frac{A^2}{4} [\delta(f-f_c) + \delta(f+f_c)] + \frac{N_0}{2} \]
Noise Equivalent Bandwidth

- Output average power of ILPF $\rightarrow N_0B$
- Output average power of RCLPF $\rightarrow N_0/4RC$

- Output average power of any filter $\rightarrow N_0BH^2(0)$
- Equivalent Bandwidth $B = \frac{\int_0^\infty |H(f)|^2 df}{H^2(0)}$