3.4 CIRCULAR WAVEGUIDE

A hollow metal tube of circular cross section also supports TE and TM waveguide modes. Figure 3.11 shows the cross-section geometry of such a circular waveguide of inner radius a. Since a cylindrical geometry is involved, it is appropriate to employ cylindrical coordinates. As in the rectangular coordinate case, the transverse fields in cylindrical coordinates can be derived from E_z or H_z field components, for TM and

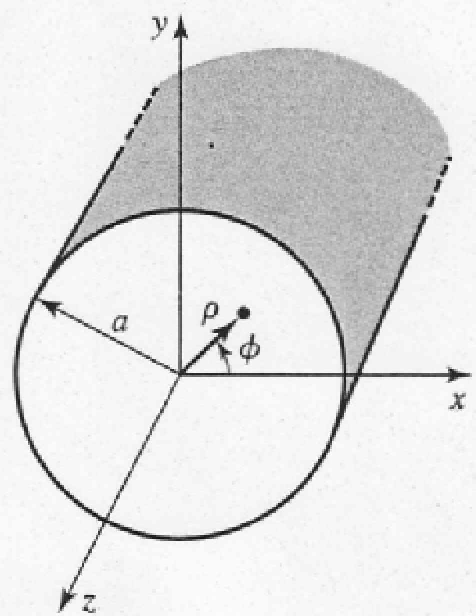


FIGURE 3.11 Geometry of a circular waveguide.

lecture #5

2.1 Circular WG

starting from Maxwell equations E_ρ, H_ρ, E_ϕ, H_ϕ

$$\nabla \times \bar{E} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} = -j\omega\mu \bar{H}$$

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + j\beta E_\phi = -j\omega\mu H_\rho \longrightarrow ①$$

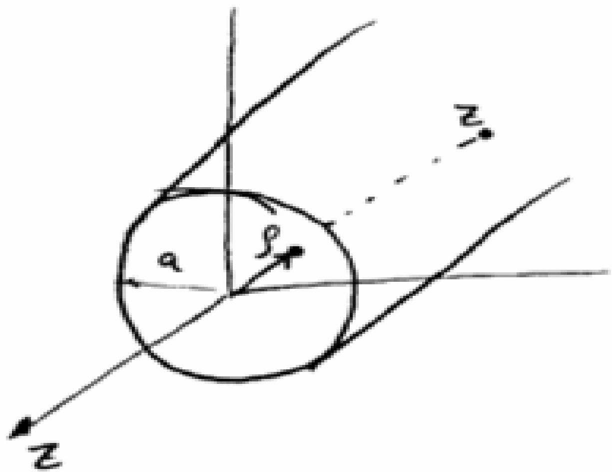
$$-j\beta E_\rho - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\phi \longrightarrow ②$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + j\beta H_\phi = j\omega\varepsilon E_\rho \longrightarrow ③$$

$$-j\beta H_\rho - \frac{\partial H_z}{\partial \rho} = j\omega\varepsilon E_\phi \longrightarrow ④$$

① & ④ Can be solved together to give E_ϕ & H_ρ

② & ③ ~ ~ ~ ~ ~ ~ H_ϕ & E_ρ



Cylinderical Coordinates

$$\bar{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$$
$$= \boxed{A\,\hat{r}}$$

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\therefore \quad E_\rho = \frac{-j}{k_c^2}\left(\beta \frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right) \longrightarrow (6)$$

$$E_\phi = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \omega\mu \frac{\partial H_z}{\partial \rho}\right) \longrightarrow (7)$$

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta \frac{\partial H_z}{\partial \rho}\right) \longrightarrow (8)$$

$$H_\phi = \frac{-j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right) \longrightarrow (9)$$

Find $E_\rho, E_\phi, H_\rho, H_\phi$ for TE & TM

* $k_c^2 = k^2 - \beta^2$ (lossless media inside WG)
* $e^{-j\beta z}$ propagation is assumed in $+z$ direction

$$\begin{bmatrix} E_\rho \\ H_\phi \\ E_\phi \\ H_\rho \end{bmatrix} = \frac{1}{k_c^2} \begin{bmatrix} -j\beta & -j\omega\mu & 0 & 0 \\ -j\omega\varepsilon & -j\beta & 0 & 0 \\ 0 & 0 & -j\beta & j\omega\mu \\ 0 & 0 & j\omega\varepsilon & -j\beta \end{bmatrix} \begin{bmatrix} \frac{\partial E_z}{\partial \rho} \\ \frac{\partial H_z}{\rho\,\partial \phi} \\ \frac{\partial E_z}{\rho\,\partial \phi} \\ \frac{\partial H_z}{\partial \rho} \end{bmatrix} \longrightarrow 5$$

(2)

Bessel functions: ?
are solutions to the differential eq. Appendix C

J_m : Bessel functions of 1st kind of order m
Y_m : ~ ~ ~ 2nd ~ ~ ~ m

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right)+\left(k^2-\frac{m^2}{\rho^2}\right)f=0$$

$\therefore f(\rho)=A J_m(k\rho)+B Y_m(k\rho)$

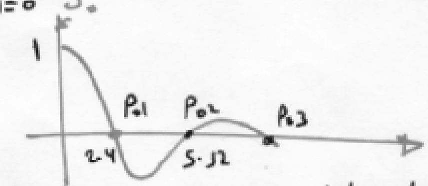
Zeros of Bessel fns of 1st kind $J_n(x)=0$

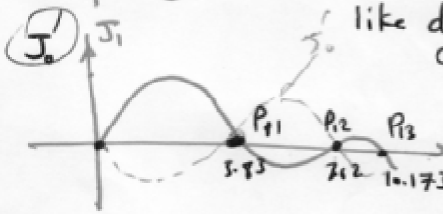
m \ n	1	2	3	4
0	2.4048	5.52	8.65	11.79
1	3.83	7.02	10.17	
2	5.13	8.41	11.61	

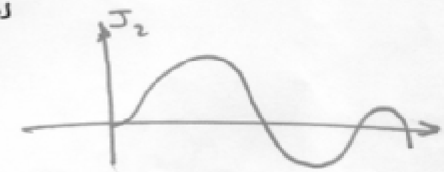
table (1)

$P_{mn} \Rightarrow P_{01} = 2.404$ is the lowest value of all zeros

$$\boxed{P_{mn} \text{ the } n^{th} \text{ root of } J_m}$$







Extrema of Bessel functions of 1st kind

$$\frac{dJ_m(x)}{dx}=0 \quad \text{for} \quad 0<x<\infty$$

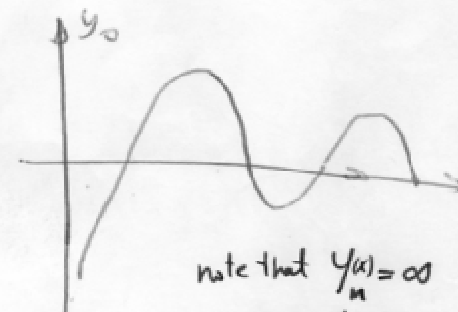
table (2)

m \ n	1	2	3	4
0	3.831	7.01	10.173	13.32
1	1.84	5.33	8.53	11.70
2	3.05	6.70	9.967	

$P'_{0n}=P_{1n}$

$$\boxed{P'_{mn} = \text{the } n^{th} \text{ root of } J'_m}$$

P_{11} is the lowest value for all zeros



note that $Y_m(x)=\infty$ at $x=0$

$J_0'(x) = -J_1(x)$
$P'_{0n} = P_{1n}$

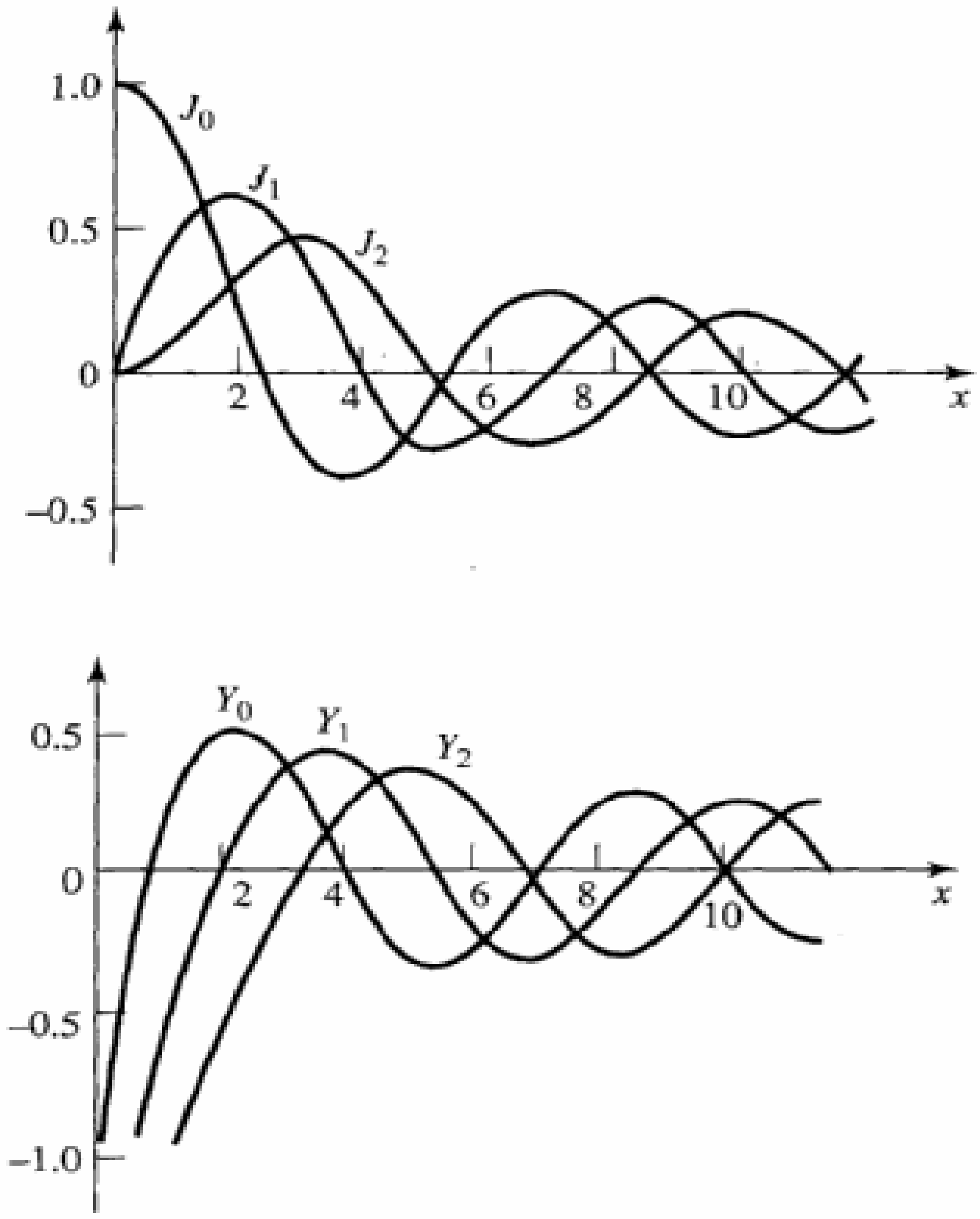


FIGURE C.1 Bessel functions of the first and second kind.

Zeros of Bessel functions of first kind: $J_n(x) = 0$ for $0 < x < 12$

n	1	2	3	4
0	2.4048	5.5200	8.6537	11.7951
1	3.8317	7.0155	10.1743	
2	5.1356	8.4172	11.6198	
3	6.3801	9.7610		
4	7.5883	11.0647		
5	8.7714			
6	9.9361			
7	11.0863			

Extrema of Bessel functions of first kind:
$dJ_n(x)/dx = 0$ for $0 < x < 12$

n	1	2	3	4
0	3.8317	7.0156	10.1735	13.3237
1	1.8412	5.3314	8.5363	11.7060
2	3.0542	6.7061	9.9695	
3	4.2012	8.0152	11.3459	
4	5.3175	9.2824		
5	6.4156	10.5199		
6	7.5013	11.7349		
7	8.5778			
8	9.6474			
9	10.7114			
10	11.7709			

22 TE modes

For TE modes $E_z = 0$ & $H_z \neq 0$

$$\nabla^2 H_z + k^2 H_z = 0$$

$$H_z(\rho, \phi, z) = h_z(\rho, \phi)\, e^{-j\beta z} \quad \& \quad k^2 = \omega^2 \mu \varepsilon$$

$$\overbrace{\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right.}^{\nabla^2}\left. + k_c^2\right) \underbrace{h_z(\rho, \phi)}_{R(\rho)\, \Phi(\phi)} = 0 \qquad \text{from wave eq.}$$

applying method of separation of variables, we get

$$\Phi(\phi) = A \sin m\phi + \cancel{B \cos m\phi}$$

$$R(\rho) = C\, J_m(k_c \rho) + \cancel{D\, Y_m(k_c \rho)} \qquad \text{(the solution of Bessel's DE)}$$

C is absorbed in A

$\therefore$ the general solution

$$\boxed{H_z(\rho, \phi, z) = \left\{ \begin{array}{l} A \sin(m\phi) \\ + B \cos(m\phi) \end{array} \right\} J_m(k_c \rho)\, e^{-j\beta z}} \longrightarrow (6)$$

notes where

① J_m : Bessel function of 1st kind for order m

Y_m : ~ ~ ~ 2nd ~ ~ ~

② $D = 0$ (because $Y_m(k_c\rho) = \infty$ at $\rho = 0$
unacceptable for the CWG)

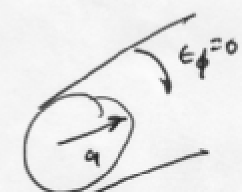
③ * the constants A & B control the amplitude of sin & cos terms

* the actual amplitude will be dependent on the excitation of WG

* coordinate system can be rotated about the z axis to obtain
h_z with either $A = 0$ or $B = 0$

[in our course ~~we with assume B=0~~ $A = 0$]

④ to get the constant k_c apply boundary condition CWG



∴ $E_{\phi_{tang}} = 0$ at $\rho = a$

$$\therefore \boxed{E_\phi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial \rho} = \frac{j\omega\mu}{k_c^2}\left(A \sin m\phi + B\cos(m\phi)\right) J_m'(k_c\rho)e^{-j\beta z}}$$

a

∴ $E_\phi = 0$ at $\rho = a$

$$\boxed{J_m'(k_c a) = 0}$$

if the roots of $J'_m(x)$ are defined as P'_{mn}

so that $J'_m(P'_{mn}) = 0$

<u>where</u> $\boxed{P'_{mn} \text{ is the } n^{th} \text{ root of } J'_m}$

<u>then</u>

cut off wave number:

$\therefore$ the TE_{mn} modes are thus defined by the cut off wave number

$$\boxed{k_{c_{mn}} = \frac{P'_{mn}}{a}} \longrightarrow (11)$$

See table 3.3 page $\boxed{135}$

<u>where</u>

m : refere to the number of circumferential (ϕ) variations

n : ~ ~ ~ ~ radial (ρ) ~

$\therefore$ the propagation constant $\beta_{mn} = \sqrt{k^2 - k_{c_{mn}}^2}$

the TE_{mn} cutoff freq.

$$\boxed{F_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{P'_{mn}}{2\pi a\sqrt{\mu\varepsilon}}} \longrightarrow (12)$$

the field components TE_{mn}

$$H_z = \left(A \sin(m\phi) + B\cos(m\phi)\right) J_m\left(\frac{p'_{mn}}{a}\rho\right) e^{-j\beta z} \quad \cancel{(13)}$$

$$E_\rho = \frac{-j\omega\mu m}{k_c^2 \rho}\left(A \cos(m\phi) \mp B\sin(m\phi)\right) J_m\left(\frac{p'_{mn}}{a}\rho\right) e^{-j\beta z} \longrightarrow (14)$$

matrix ←

$$E_\phi = \frac{j\omega\mu}{k_c}\left(A \sin(m\phi) + B\cos(m\phi)\right) J'_m(k_c \rho)\, e^{-j\beta z} \longrightarrow (15)$$

$$H_\rho = \frac{-j\beta}{k_c}\left(A \sin(m\phi) + B\cos m\phi\right) J'_m(k_c \rho)\, e^{-j\beta z} \longrightarrow (16)$$

$$H_\phi = \frac{-j\beta m}{k_c^2 \rho}\left(A \cos(m\phi) - B\sin m\phi\right) J_m(k_c \rho)\, e^{-j\beta z} \longrightarrow (17)$$

Note

$m=0$

$$\frac{dJ_0(x)}{dx} = -J_1(x)$$

$$\therefore \boxed{P'_{0n} = P_{1n}}$$

(*) the ~~too~~ 1st TE mode to propagate is the mode with the smallest P'_{mn} which is from the table $\boxed{P'_{mn} = P'_{11}}$

$\therefore$ the dominant mode : the mode with the lowest cutoff frequency for the CWG is the :

$$\boxed{TE_{11} \text{ mode}}$$

$P'_{11} = 1.84$

$$f_{c_{TE_{11}}} = \frac{v}{2\pi a} \times 1.84$$

$n \geq 1$

---

because $n \geq 1$ there is no TE_{n0} mode X TE_{10} TE_{20} TE_{30}

but there is TE_{01} ✓

the wave impedance

$$Z_{TE} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\eta k}{\beta} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

the dominant mode TE_{11} equations

$A=0$

or $B=0$

$$\boxed{H_z = A \, \overset{\cos}{\cancel{\sin}} \phi \, J_1(k_c \rho) \, e^{-j\beta z}}$$

final

$$E_\rho = \frac{-j\omega\mu}{k_c^2 \rho} A \cos\phi \, J_1$$

$$E_\phi = \frac{j\omega\mu}{k_c} A \sin\phi \, J_1'$$

$\boxed{2.2}$ TM_{mn} modes

$$H_z = 0 \quad \& \quad E_z \neq 0.$$

<u>assume</u> $E_z(\rho, \phi, z) = e_z(\rho, \phi) e^{-j\beta z}$ $\qquad k_c^2 = k^2 - \beta^2$

solving the wave eq.

$$\nabla^2 \bar{E}_z - \beta^2 \bar{E}_z = 0$$

<u>the general solution:</u>

$$\boxed{e_z(\rho, \phi) = (A \sin m\phi + B \cos m\phi)\ J_m(k_c \rho)}$$

from matrix & equations (6) → (9) $\quad \boxed{\& \text{ Considering } A = 0}$

$$\therefore \quad E_z = B \cos(m\phi)\ J_m(k_c \rho)\ e^{-j\beta z}$$

$$E_\rho = \frac{-j\beta}{k_c} B \cos(m\phi)\ J_m'(k_c \rho)\ e^{-j\beta z}$$

$$E_\phi = \frac{+j\beta m}{k_c^2 \rho} B \sin(m\phi)\ J_m(k_c \rho)\ e^{-j\beta z}$$

$$H_\rho = \frac{-j\omega\varepsilon m}{k_c^2 \rho} B \sin(m\phi)\ J_m(k_c \rho)\ e^{-j\beta z}$$

$$H_\phi = \frac{-j\omega\varepsilon}{k_c} B \cos(m\phi)\ J_m'(k_c \rho)\ e^{-j\beta z}$$

(1) Boundary Conditions $E_z(\rho, \phi) = 0$ at $\rho = 0$ $\longrightarrow J_m(k_c a) = 0 \longrightarrow \boxed{k_c = \frac{p_{mn}}{a}}$

(2) $\boxed{K_{c_{mn}} = \frac{p_{mn}}{a}}$ p_{mn} = n^{th} root of J_m

(3) $\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p_{mn}}{a}\right)^2}$

(4) $f_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{p_{mn}}{2\pi a\sqrt{\mu\varepsilon}}$

(5) 1st mode TM to propagates is the $TM_{01} \longrightarrow \boxed{p_{01} = 2.405}$ the lowest p_{mn}

However TE_{11} is the dominant $p'_{11} = 1.841$ took to p_{mn} table

(6) $n \geq 1$ $\boxed{\text{there is no } TM_{m0} \text{ mode}}$

Example 3.2 & table 3.5 pp. 139 TM_{01} :

TE modes, respectively. Paralleling the development of Section 3.1, the cylindrical components of the transverse fields can be derived from the longitudinal components as

$$E_\rho = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right), \qquad 3.110a$$

$$E_\phi = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial \rho}\right), \qquad 3.110b$$

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial \rho}\right), \qquad 3.110c$$

$$H_\phi = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right), \qquad 3.110d$$

where $k_c^2 = k^2 - \beta^2$, and $e^{-j\beta z}$ propagation has been assumed. For $e^{+j\beta z}$ propagation, replace β with $-\beta$ in all expressions.

TE Modes

For TE modes, $E_z = 0$, and H_z is a solution to the wave equation,

$$\nabla^2 H_z + k^2 H_z = 0. \tag{3.111}$$

If $H_z(\rho, \phi, z) = h_z(\rho, \phi)e^{-j\beta z}$, (3.111) can be expressed in cylindrical coordinates as

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + k_c^2 \right) h_z(\rho, \phi) = 0. \tag{3.112}$$

Again, a solution can be derived using the method of separation of variables. Thus, we let

$$h_z(\rho, \phi) = R(\rho)P(\phi), \tag{3.113}$$

and substitute into (3.112) to obtain

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 P}\frac{d^2P}{d\phi^2} + k_c^2 = 0, \qquad 3.114$$

or

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \rho^2 k_c^2 = \frac{-1}{P}\frac{d^2P}{d\phi^2}.$$

The left side of this equation depends on ρ (not ϕ), while the right side depends only on ϕ. Thus, each side must be equal to a constant, which we will call k_ϕ^2. Then,

$$\frac{-1}{P}\frac{d^2P}{d\phi^2} = k_\phi^2,$$

or

$$\frac{d^2P}{d\phi^2} + k_\phi^2 P = 0. \qquad 3.115$$

Also,

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left(\rho^2 k_c^2 - k_\phi^2\right) R = 0. \qquad 3.116$$

The general solution to (3.115) is

$$P(\phi) = A \sin k_\phi \phi + B \cos k_\phi \phi. \tag{3.117}$$

Since the solution to h_z must be periodic in ϕ (that is, $h_z(\rho, \phi) = h_z(\rho, \phi \pm 2m\pi)$), k_ϕ must be an integer, n. Thus (3.117) becomes

$$P(\phi) = A \sin n\phi + B \cos n\phi, \tag{3.118}$$

while (3.116) becomes

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left(\rho^2 k_c^2 - n^2\right) R = 0, \tag{3.119}$$

which is recognized as Bessel's differential equation. The solution is

$$R(\rho) = C J_n(k_c \rho) + D Y_n(k_c \rho), \tag{3.120}$$

where $J_n(x)$ and $Y_n(x)$ are the Bessel functions of first and second kinds, respectively. Since $Y_n(k_c\rho)$ becomes infinite at $\rho = 0$, this term is physically unacceptable for the circular waveguide problem, so that $D = 0$. The solution for h_z can then be written as

$$h_z(\rho, \phi) = (A \sin n\phi + B \cos n\phi) J_n(k_c \rho), \tag{3.121}$$

where the constant C of (3.120) has been absorbed into the constants A and B of (3.121).

We must still determine the cutoff wavenumber k_c, which we can do by enforcing the boundary condition that $E_{\tan} = 0$ on the waveguide wall. Since $E_z = 0$, we must have that

$$E_\phi(\rho, \phi) = 0, \qquad \text{at } \rho = a. \tag{3.122}$$

From (3.110b), we find E_ϕ from H_z as

$$E_\phi(\rho, \phi, z) = \frac{j\omega\mu}{k_c}(A \sin n\phi + B \cos n\phi) J'_n(k_c\rho) e^{-j\beta z}, \tag{3.123}$$

where the notation $J'_n(k_c\rho)$ refers to the derivative of J_n with respect to its argument. For E_ϕ to vanish at $\rho = a$, we must have

$$J'_n(k_c a) = 0. \tag{3.124}$$

If the roots of $J'_n(x)$ are defined as p'_{nm}, so that $J'_n(p'_{nm}) = 0$, where p'_{nm} is the mth root of J'_n, then k_c must have the value

$$k_{c_{nm}} = \frac{p'_{nm}}{a}. \tag{3.125}$$

Values of p'_{nm} are given in mathematical tables; the first few values are listed in Table 3.3.

The TE_{nm} modes are thus defined by the cutoff wavenumber, $k_{c_{nm}} = p'_{nm}/a$, where n refers to the number of circumferential (ϕ) variations, and m refers to the number of radial (ρ) variations. The propagation constant of the TE_{nm} mode is

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p'_{nm}}{a}\right)^2}, \qquad 3.126$$

TABLE 3.3 Values of p'_{nm} for TE Modes of a Circular Waveguide

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

with a cutoff frequency of

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}. \tag{3.127}$$

The first TE mode to propagate is the mode with the smallest p'_{nm}, which from Table 3.3 is seen to be the TE_{11} mode. This mode is then the dominant circular waveguide mode, and the one most frequently used. Because $m \geq 1$, there is no TE_{10} mode, but there is a TE_{01} mode.

The transverse field components are, from (3.110) and (3.121),

$$E_\rho = \frac{-j\omega\mu n}{k_c^2\rho}(A\cos n\phi - B\sin n\phi)\,J_n(k_c\rho)\,e^{-j\beta z}, \qquad 3.128a$$

$$E_\phi = \frac{j\omega\mu}{k_c}(A\sin n\phi + B\cos n\phi)\,J_n'(k_c\rho)e^{-j\beta z}, \qquad 3.128b$$

$$H_\rho = \frac{-j\beta}{k_c}(A\sin n\phi + B\cos n\phi)J_n'(k_c\rho)e^{-j\beta z}, \qquad 3.128c$$

$$H_\phi = \frac{-j\beta n}{k_c^2\rho}(A\cos n\phi - B\sin n\phi)J_n(k_c\rho)e^{-j\beta z}. \qquad 3.128d$$

The wave impedance is

$$Z_{\mathrm{TE}} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\eta k}{\beta}. \qquad 3.129$$

In the above solutions there are two remaining arbitrary amplitude constants, A and B. These constants control the amplitude of the $\sin n\phi$ and $\cos n\phi$ terms, which are independent. That is, because of the azimuthal symmetry of the circular waveguide, both the $\sin n\phi$ and $\cos n\phi$ terms are valid solutions, and can be present in a specific problem to any degree. The actual amplitudes of these terms will be dependent on the excitation of the waveguide. From a different viewpoint, the coordinate system can be rotated about the z-axis to obtain an h_z with either $A = 0$ or $B = 0$.

Now consider the dominant TE_{11} mode with an excitation such that $B = 0$. The fields can be written as

$$H_z = A \sin \phi J_1(k_c \rho) e^{-j\beta z}, \tag{3.130a}$$

$$E_\rho = \frac{-j\omega\mu}{k_c^2 \rho} A \cos \phi J_1(k_c \rho) e^{-j\beta z}, \tag{3.130b}$$

$$E_\phi = \frac{j\omega\mu}{k_c} A \sin \phi J_1'(k_c \rho) e^{-j\beta z}, \tag{3.130c}$$

$$H_\rho = \frac{-j\beta}{k_c} A \sin \phi J_1'(k_c \rho) e^{-j\beta z}, \tag{3.130d}$$

$$H_\phi = \frac{-j\beta}{k_c^2 \rho} A \cos \phi J_1(k_c \rho) e^{-j\beta z}, \tag{3.130e}$$

$$E_z = 0. \tag{3.130f}$$

The power flow down the guide can be computed as

$$
\begin{aligned}
P_o &= \frac{1}{2}\mathrm{Re}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} \bar{E}\times\bar{H}^*\cdot\hat{z}\rho\,d\phi\,d\rho \\
&= \frac{1}{2}\mathrm{Re}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} \left[E_\rho H_\phi^* - E_\phi H_\rho^*\right]\rho\,d\phi\,d\rho \\
&= \frac{\omega\mu|A|^2\mathrm{Re}(\beta)}{2k_c^4}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\left[\frac{1}{\rho^2}\cos^2\phi J_1^2(k_c\rho) + k_c^2\sin^2\phi J_1'^2(k_c\rho)\right]\rho\,d\phi\,d\rho \\
&= \frac{\pi\omega\mu|A|^2\mathrm{Re}(\beta)}{2k_c^4}\int_{\rho=0}^{a}\left[\frac{1}{\rho}J_1^2(k_c\rho) + \rho k_c^2 J_1'^2(k_c\rho)\right]d\rho \\
&= \frac{\pi\omega\mu|A|^2\mathrm{Re}(\beta)}{4k_c^4}\left(p_{11}'^2 - 1\right)J_1^2(k_c a),
\end{aligned}
\tag{3.131}
$$

which is seen to be nonzero only when β is real, corresponding to a propagating mode. (The required integral for this result is given in Appendix C.)

Attenuation due to dielectric loss is given by (3.29). The attenuation due to a lossy waveguide conductor can be found by computing the power loss per unit length of guide:

$$\begin{aligned}
P_\ell &= \frac{R_s}{2}\int_{\phi=0}^{2\pi} |\bar{J}_s|^2 a\, d\phi \\
&= \frac{R_s}{2}\int_{\phi=0}^{2\pi} \left[|H_\phi|^2 + |H_z|^2\right] a\, d\phi \\
&= \frac{|A|^2 R_s}{2}\int_{\phi=0}^{2\pi} \left[\frac{\beta^2}{k_c^4 a^2}\cos^2\phi + \sin^2\phi\right] J_1^2(k_c a) a\, d\phi \\
&= \frac{\pi |A|^2 R_s a}{2}\left(1 + \frac{\beta^2}{k_c^4 a^2}\right) J_1^2(k_c a). \qquad 3.132
\end{aligned}$$

The attenuation constant is then

$$\begin{aligned}
\alpha_c &= \frac{P_\ell}{2P_o} = \frac{R_s\left(k_c^4 a^2 + \beta^2\right)}{\eta k \beta a (p_{11}'^2 - 1)} \\
&= \frac{R_s}{ak\eta\beta}\left(k_c^2 + \frac{k^2}{p_{11}'^2 - 1}\right) \text{ Np/m.} \qquad 3.133
\end{aligned}$$

TM Modes

For the TM modes of the circular waveguide, we must solve for E_z from the wave equation in cylindrical coordinates:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_c^2\right) e_z = 0, \qquad 3.134$$

where $E_z(\rho, \phi, z) = e_z(\rho, \phi)e^{-j\beta z}$, and $k_c^2 = k^2 - \beta^2$. Since this equation is identical to (3.107), the general solutions are the same. Thus, from (3.121),

$$e_z(\rho, \phi) = (A \sin n\phi + B \cos n\phi)J_n(k_c\rho). \qquad 3.135$$

The difference between the TE solution and the present solution is that the boundary conditions can now be applied directly to e_z of (3.135), since

$$E_z(\rho, \phi) = 0, \qquad \text{at } \rho = a. \qquad 3.136$$

Thus, we must have

$$J_n(k_c a) = 0, \qquad 3.137$$

or

$$k_c = p_{nm}/a, \qquad 3.138$$

where p_{nm} is the mth root of $J_n(x)$; that is, $J_n(p_{nm}) = 0$. Values of p_{nm} are given in mathematical tables; the first few values are listed in Table 3.4.

The propagation constant of the TM_{nm} mode is

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - (p_{nm}/a)^2}. \qquad 3.139$$

The cutoff frequency is

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\epsilon}}. \qquad 3.140$$

Thus, the first TM mode to propagate is the TM_{01} mode, with $p_{01} = 2.405$. Since this is greater than $p'_{11} = 1.841$ of the lowest order TE_{11} mode, the TE_{11} mode is the dominant mode of the circular waveguide. As with the TE modes, $m \geq 1$, so there is no TM_{10} mode.

From (3.110), the transverse fields can be derived as

$$E_\rho = \frac{-j\beta}{k_c}(A \sin n\phi + B \cos n\phi) J'_n(k_c\rho) e^{-j\beta z}, \qquad 3.141a$$

TABLE 3.4 Values of p_{nm} for TM Modes of a Circular Waveguide

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

$$E_\phi = \frac{-j\beta n}{k_c^2 \rho}(A \cos n\phi - B \sin n\phi) J_n(k_c \rho) e^{-j\beta z}, \tag{3.141b}$$

$$H_\rho = \frac{j\omega\epsilon n}{k_c^2 \rho}(A \cos n\phi - B \sin n\phi) J_n(k_c \rho) e^{-j\beta z}, \tag{3.141c}$$

$$H_\phi = \frac{-j\omega\epsilon}{k_c}(A \sin n\phi + B \cos n\phi) J_n'(k_c \rho) e^{-j\beta z}. \tag{3.141d}$$

The wave impedance is

$$Z_{\mathrm{TM}} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\eta\beta}{k}. \tag{3.142}$$

Calculation of the attenuation for TM modes is left as a problem. Figure 3.12 shows the attenuation due to conductor loss versus frequency for various modes of a circular waveguide. Observe that the attenuation of the TE_{01} mode decreases to a very small value with increasing frequency. This property makes the TE_{01} mode of interest for low-loss transmission over long distances. Unfortunately, this mode is not the dominant mode of the circular waveguide, so in practice power can be lost from the TE_{01} mode to lower-order propagating modes.

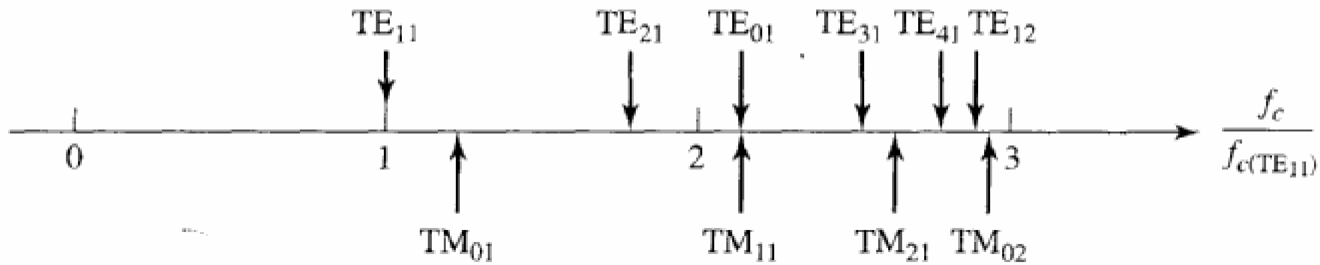


FIGURE 3.13 Cutoff frequencies of the first few TE and TM modes of a circular waveguide, relative to the cutoff frequency of the dominant TE_{11} mode.

Figure 3.13 shows the relative cutoff frequencies of the TE and TM modes, and Table 3.5 summarizes results for wave propagation in circular waveguide. Field lines for some of the lowest order TE and TM modes are shown in Figure 3.14.

EXAMPLE 3.2 **Characteristics of a Circular Waveguide**

Find the cutoff frequencies of the first two propagating modes of a circular waveguide with $a = 0.5$ cm and $\epsilon_r = 2.25$. If the guide is silver plated and the dielectric loss tangent is 0.001, calculate the attenuation in dB for a 50 cm length of guide operating at 13.0 GHz.

Solution

From Figure 3.13, the first two propagating modes of a circular waveguide are the TE_{11} and TM_{01} modes. The cutoff frequencies can be found using (3.127) and (3.140):

$$\text{TE}_{11}: \qquad f_c = \frac{p'_{11}c}{2\pi a\sqrt{\epsilon_r}} = \frac{1.841(3 \times 10^8)}{2\pi(0.005)\sqrt{2.25}} = 11.72 \text{ GHz},$$

$$\text{TM}_{01}: \qquad f_c = \frac{p_{01}c}{2\pi a\sqrt{\epsilon_r}} = \frac{2.405(3 \times 10^8)}{2\pi(0.005)\sqrt{2.25}} = 15.31 \text{ GHz}.$$

So only the TE_{11} mode is propagating at 13.0 GHz. The wavenumber is

$$k = \frac{2\pi f\sqrt{\epsilon_r}}{c} = \frac{2\pi(13 \times 10^9)\sqrt{2.25}}{3 \times 10^8} = 408.4 \text{ m}^{-1},$$

and the propagation constant of the TE_{11} mode is

$$\beta = \sqrt{k^2 - \left(\frac{p'_{11}}{a}\right)^2} = \sqrt{(408.4)^2 - \left(\frac{1.841}{0.005}\right)^2} = 176.7 \text{ m}^{-1}.$$

The attenuation due to dielectric loss is calculated from (3.29) as

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} = \frac{(408.4)^2(0.001)}{2(176.7)} = 0.47 \text{ Np/m}.$$

The conductivity of silver is $\sigma = 6.17 \times 10^7$ S/m, so the surface resistance is

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 0.029\,\Omega.$$

Then from (3.133) the attenuation due to metallic loss is

$$\alpha_c = \frac{R_s}{ak\eta\beta}\left(k_c^2 + \frac{k^2}{p_{11}'^2 - 1}\right) = 0.066 \text{ Np/m}.$$

So the total attenuation factor is

$$\alpha = \alpha_c + \alpha_d = 0.54 \text{ Np/m}.$$

Note that the dielectric loss dominates this result. The attenuation in the 50 cm long guide is

$$\text{attenuation (dB)} = -20\log e^{-\alpha\ell} = -20\log e^{-(0.547)(0.5)} = 2.38 \text{ dB}. \qquad \bigcirc$$

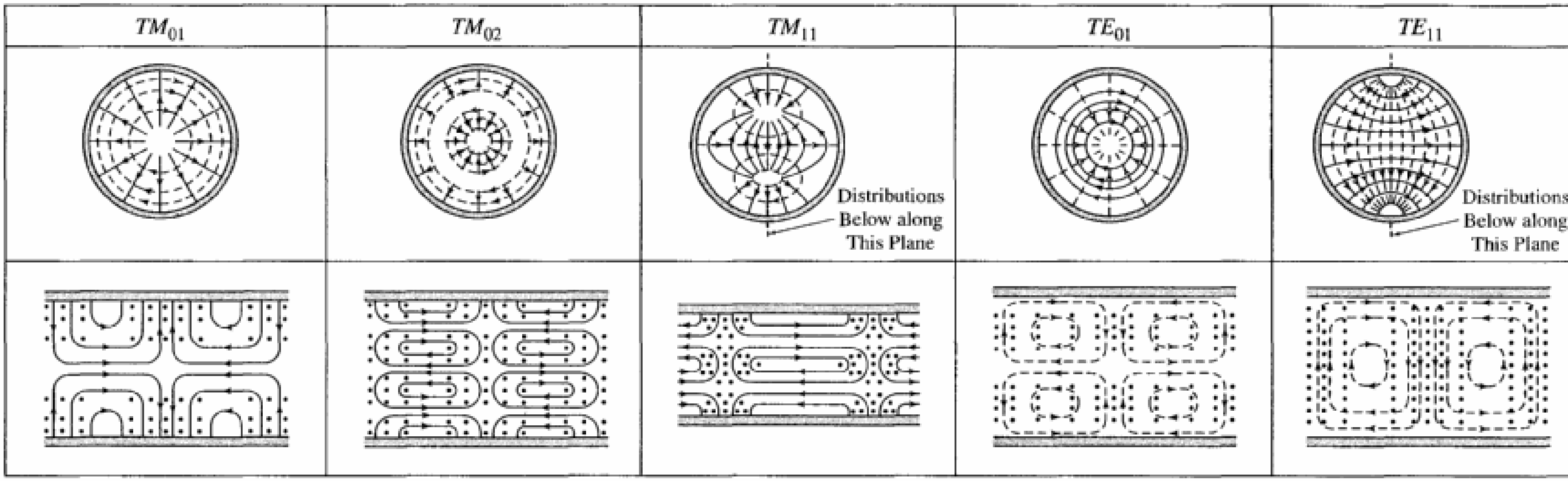


FIGURE 3.14 Field lines for some of the lower order modes of a circular waveguide.