

*Digital Communications and Simulation*  
**ECE 5654**

Section 3 – Modulation  
Module 3 – Performance Evaluation of Coherent Modulation  
Set 2 – Exact Performance Evaluation in AWGN

## *Modulation*

- Modulation is used to transmit digital data over a channel
- Gramm-Schmidt procedure allows vector representation of any signal constellation
- Optimal receiver consists of a correlator, weighted to adjust for signal energy and a priori probabilities
- We now turn to calculating performance

## *Analytical Form of MAP Receiver*

- Multiplying through by the constant  $N_0/2$  :

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \frac{N_0}{2} \ln[p_m] + \sum_{k=1}^K r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^K s_{m,k}^2$$

## ***Decision Regions***

- Optimal Decision Rule:

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \frac{N_0}{2} \ln[p_m] + \sum_{k=1}^K r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^K s_{m,k}^2$$

- Let  $R_i \subset \Re^K$  be the region in which

$$\frac{N_0}{2} \ln[p_i] + \sum_{k=1}^K r_k s_{i,k} - \frac{1}{2} \sum_{k=1}^K s_{i,k}^2$$

$$\geq \frac{N_0}{2} \ln[p_j] + \sum_{k=1}^K r_k s_{j,k} - \frac{1}{2} \sum_{k=1}^K s_{j,k}^2, \forall i \neq j$$

- Then  $R_i$  is the  $i$ th “Decision Region”

## *Symbol Error Probability*

- $P_s(e) = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}]$  is the average probability of symbol error

$$P_s(e) = \sum_{i=1}^M \Pr[\mathbf{s}_i] \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i]$$

- where  $\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i]$  is the conditional probability of the receiver not deciding on  $\mathbf{s}_i$  given  $\mathbf{s}_i$  was transmitted

$$\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i] = 1 - \int_{R_i} p(\mathbf{r} | \mathbf{s}_i) d\mathbf{r}$$

- This may be a multidimensional integral over the decision region  $R_i$

$$p(\mathbf{r} | \mathbf{s}_i) = (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^K (r_k - s_{i,k})^2 / N_0\right)$$

## *Symbol Error Probability Calculation for BPSK*

- Two antipodal signals ( $M = 2$ ):

$$s_1(t) = \sqrt{2P} \cos(2\pi f_c t) \Big|_0^T \quad s_2(t) = -\sqrt{2P} \cos(2\pi f_c t) \Big|_0^T$$

- One basis function:

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \Big|_0^T$$

- Signal Space Representation:

$$s_1(t) = \sqrt{P \cdot T} f_1(t) = \sqrt{E_b} f_1(t), s_2(t) = -\sqrt{E_b} f_1(t)$$

$$\mathbf{s}_1 = \sqrt{E_b}, \mathbf{s}_2 = -\sqrt{E_b}$$

$$\Pr[\mathbf{s}_1] = \Pr[\mathbf{s}_2] = \frac{1}{2}$$

## *Boundaries of Decision Regions for BPSK*

- Decision rule says choose  $s_1$  if:

$$p(\mathbf{r}|s_2) \Pr(s_2) \leq p(\mathbf{r}|s_1) \Pr(s_1)$$

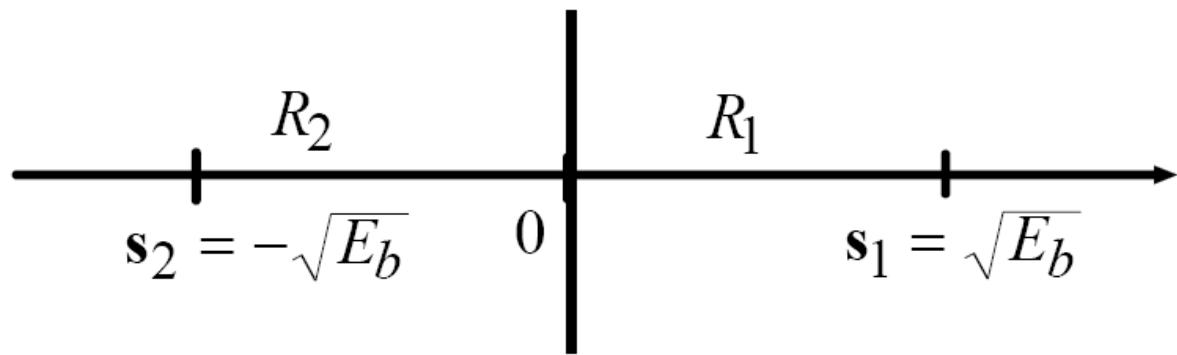
$$\Leftrightarrow \frac{1}{2} \cdot \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r - \sqrt{E_b})^2}{N_0}\right) \geq \frac{1}{2} \cdot \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r + \sqrt{E_b})^2}{N_0}\right)$$

$$\Leftrightarrow \exp\left(-\frac{(r - \sqrt{E_b})^2}{N_0}\right) \geq \exp\left(-\frac{(r + \sqrt{E_b})^2}{N_0}\right)$$

$$\Leftrightarrow (r - \sqrt{E_b})^2 \leq (r + \sqrt{E_b})^2$$

$$\Leftrightarrow r \geq 0$$

## *Decision Region for BPSK*



## *Calculation of Error Probability for BPSK*

$$\begin{aligned}\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_2 | \mathbf{s} = \mathbf{s}_2] &= 1 - \int_{-\infty}^0 \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r + \sqrt{E_b})^2}{N_0}\right) dr \\ &= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r + \sqrt{E_b})^2}{N_0}\right) dr \\ &= \int_{\sqrt{E_b}}^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(-y^2/N_0\right) dy, \quad y = r + \sqrt{E_b} \\ &= \int_{\sqrt{2E_b/N_0}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) dx, \quad x = y/\sqrt{\frac{N_0}{2}}\end{aligned}$$

## ***Error Probability for BPSK***

- $\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_2 | \mathbf{s} = \mathbf{s}_2] = Q\left(\sqrt{2E_b/N_o}\right)$ ,

where

$$Q(u) = \int_u^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx$$

- By symmetry:

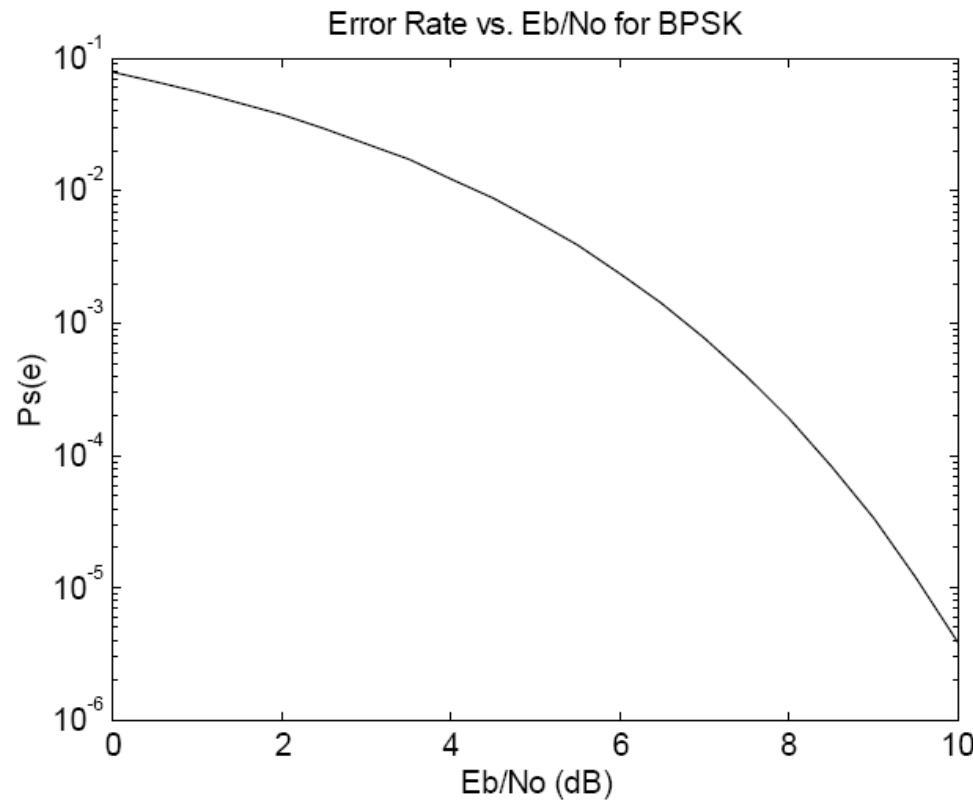
$$\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_1 | \mathbf{s} = \mathbf{s}_1] = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_2 | \mathbf{s} = \mathbf{s}_2] = P_s(e) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

- Although we have derived the result with BPSK in mind, the result holds for any signal set with this constellation.

## *Average Energy Per Bit: $E_b$*

- $E_i = \sum_{k=1}^K s_{i,k}^2$  is the energy of the  $i$ th signal
- $E_s = \frac{1}{M} \sum_{i=1}^M E_i$  is the average energy per symbol
- $\log_2 M$  is the number of bits transmitted per symbol
- $E_b = \frac{E_s}{\log_2 M}$  is the average energy per bit
  - allows fair comparisons of the energy requirements of different sized signal constellations

## *Error Probability Curve for BPSK*



## ***Symbol Error Probability Calculation for Binary Coherent FSK***

- Two orthogonal signals ( $M = 2$ ):

$$s_1(t) = \sqrt{2P} \cos(2\pi f_1 t) \Big|_0^T \quad s_2(t) = \sqrt{2P} \cos(2\pi f_2 t) \Big|_0^T$$

- Two basis functions:

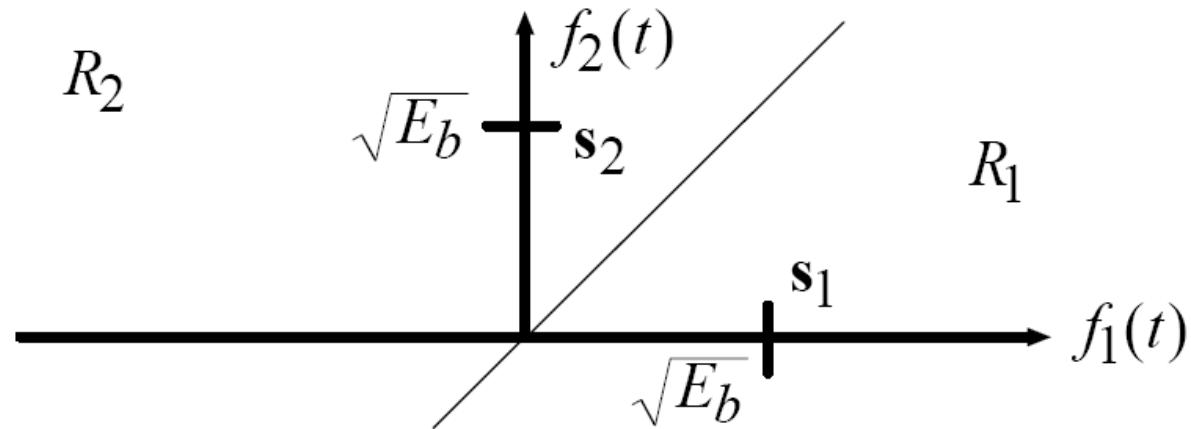
$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1 t) \Big|_0^T \quad f_2(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_2 t) \Big|_0^T$$

- Signal space representation:

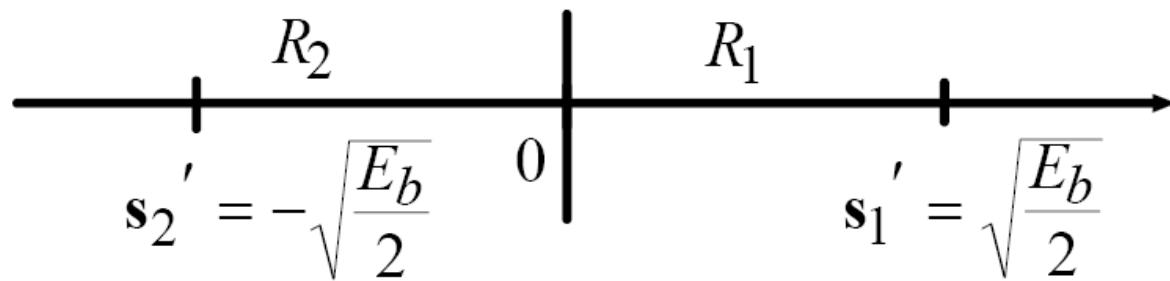
$$s_1(t) = \sqrt{E_b} f_1(t), \quad s_2(t) = \sqrt{E_b} f_2(t)$$

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} & 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{E_b} \end{bmatrix}$$

## *Decision Regions for Binary Coherent FSK*



- Rotating and translating the coordinates gives:



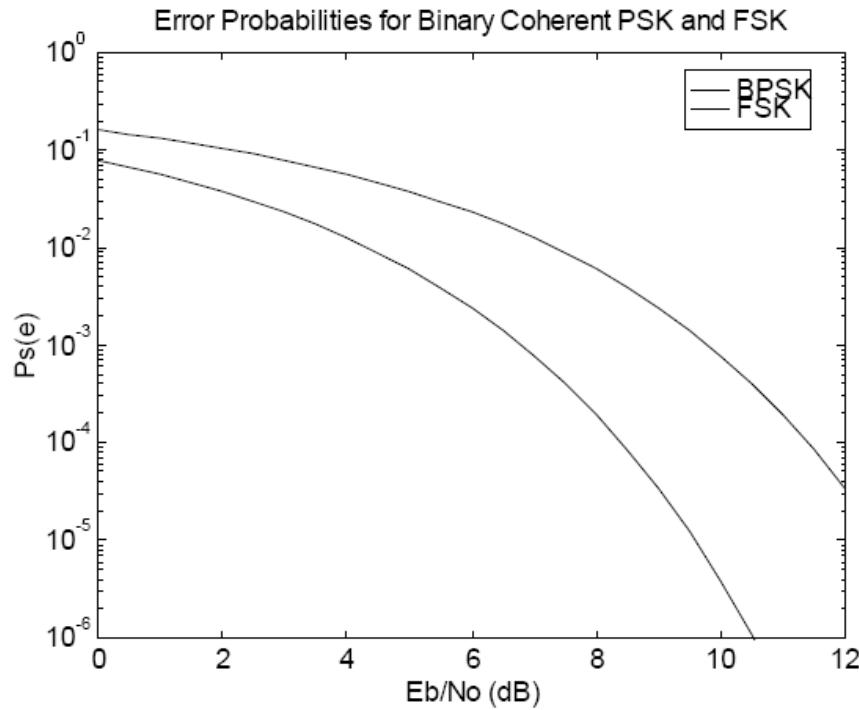
## *Error Calculation for Binary Coherent FSK*

- Any translation, rotation, or reflection operation on the coordinates which does not change the distance between signals will not effect the error probability.
- By repeating the calculation for BPSK (with  $\sqrt{E_b}/2$  substituted for  $\sqrt{E_b}$  ), we find that:

$$P_s(e) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

## *BER Curves for BPSK and FSK*

- FSK is approximately 3dB worse than BPSK



## **2-D Symbol Error Probability Example: QPSK**

- Consider a QPSK signal set:

$$s_1(t) = \sqrt{2P} \cos(2\pi f_c t) \Big|_0^T, s_2(t) = \sqrt{2P} \sin(2\pi f_c t) \Big|_0^T,$$

$$s_3(t) = -\sqrt{2P} \cos(2\pi f_c t) \Big|_0^T, s_4(t) = -\sqrt{2P} \sin(2\pi f_c t) \Big|_0^T$$

- These can be represented with the basis functions:

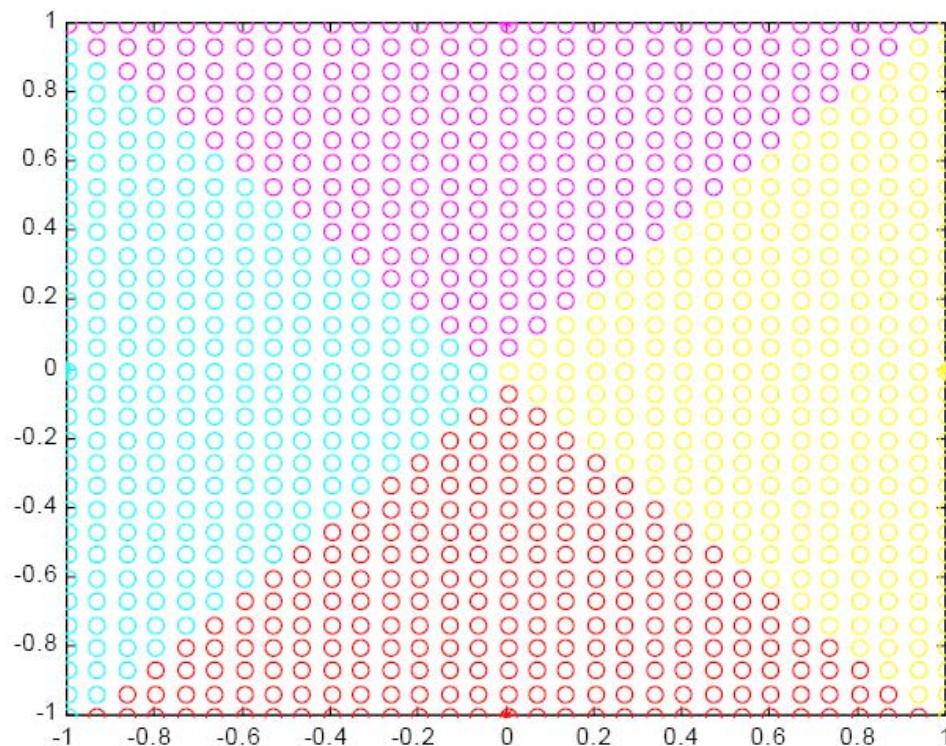
$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \Big|_0^T, f_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \Big|_0^T$$

- This representation results in the signal vectors:

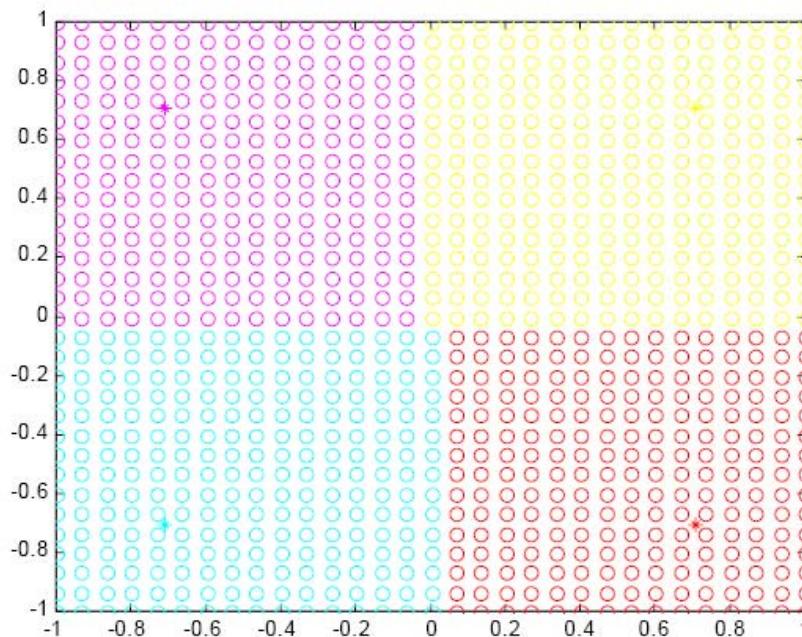
$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{P \cdot T} = \sqrt{E_s} & 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{E_s} \end{bmatrix},$$

$$\mathbf{s}_3 = \begin{bmatrix} -\sqrt{E_s} & 0 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 0 & -\sqrt{E_s} \end{bmatrix}$$

## *QPSK Signal Constellation and Decision Regions*



## *QPSK Signal Constellation after 45 Degree Rotation*



## ***Symbol Error Probability Calculation for QPSK***

$$\begin{aligned} \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_1 | \mathbf{s} = \mathbf{s}_1] &= 1 - \int_{R_1} p(\mathbf{r} | \mathbf{s}_1) d\mathbf{r} \\ &= 1 - \int_0^{\infty} \int_0^{\infty} \frac{1}{\pi N_0} e^{-\left(x - \sqrt{\frac{E_s}{2}}\right)^2 / N_0} \cdot e^{-\left(y - \sqrt{\frac{E_s}{2}}\right)^2 / N_0} dx dy \\ &= 1 - \int_{-\sqrt{\frac{E_s}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{-\sqrt{\frac{E_s}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

## *Symbol Error Probability Calculation for QPSK (continued)*

$$\begin{aligned} &= 1 - \left[ 1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right) \right] \cdot \left[ 1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right) \right] \\ &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \left[ Q\left(\sqrt{\frac{E_s}{N_0}}\right) \right]^2 \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[ Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \right]^2 \quad (\text{Because } E_s = 2E_b) \\ &\approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (\text{Because } Q^2(\cdot) \ll Q(\cdot)) \end{aligned}$$

## *Symbol Error Probability of QPSK*

- The conditional error probability given any of the four signals is identical:

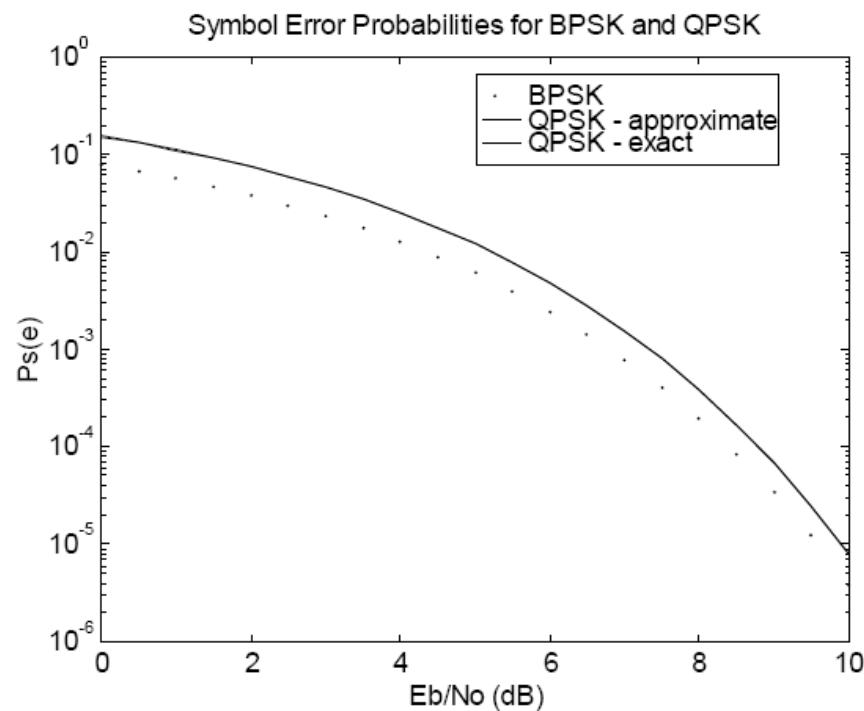
$$P_s(e) = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_1 | \mathbf{s} = \mathbf{s}_1] = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_2 | \mathbf{s} = \mathbf{s}_2] = \dots$$

- The symbol error probability of QPSK is approximately twice that of BPSK:

$$P_s(e) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- When we discuss Bit Error Rate, we will find that the BER of QPSK and BPSK are identical

## *Symbol Error Probabilities for BPSK and QPSK*



## *Notes on Error Probability Calculations*

- Error probability is found by integrating conditional probability of error over the decision region
  - This becomes difficult for a large number of dimensions
  - Difficult multidimensional integrations can be simplified by appropriate rotation, translation or reflection of coordinates
- Error performance depends only on the distance properties of the signal constellation.
- Calculations for nonbinary signal constellations may be reduced to a set of binary calculations via the “Union Bound”