Digital Communications and Simulation
ECE 5654

Section 3 – Modulation

Module 2 – Optimum Receiver Structures

Set 2 – MAP and ML Receivers

Set 4 – Matched Filter Form of Optimum Receiver

Modulation

- We want to modulate digital data using signal sets which are:
  - bandwidth efficient
  - energy efficient
- A signal space representation is a convenient form for viewing modulation which allows us to:
  - design energy and bandwidth efficient signal constellations
  - *determine the form of the optimal receiver for a given constellation*
  - evaluate the performance of a modulation type

Problem Statement

- We transmit a signal $\boxed{s(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\},}$ where $s(t)$ is nonzero only on $t \in [0,T]$.
- Let the various signals be transmitted with probability:

$$\boxed{p_1 = \Pr[s_1(t)], \ldots, p_M = \Pr[s_M(t)]}$$

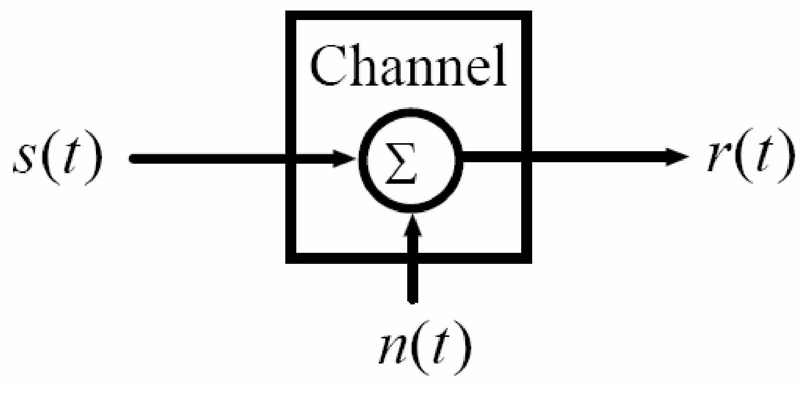
- The received signal is corrupted by noise:

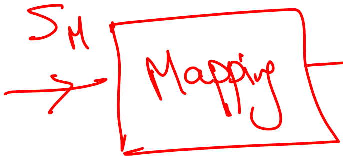
$$r(t) = s(t) + n(t)$$

- Given $r(t)$, the receiver forms an estimate $\hat{s}(t)$ of the signal $s(t)$ with the goal of minimizing symbol error probability $\underline{P_s = \Pr[\hat{s}(t) \neq s(t)]}$

Noise Model

- The signal is corrupted by Additive White Gaussian Noise (AWGN) $n(t)$
- The noise $n(t)$ has autocorrelation $\phi_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$ and power spectral density $\Phi_{nn}(f) = N_0/2$
- Any linear function of $n(t)$ will be a Gaussian random variable





C_{km} (the transmitted vector of K elements)

Signal Space Representation

- The transmitted signal can be represented as:

$$\boxed{s_m(t) = \sum_{k=1}^{K} s_{m,k} f_k(t)\ ,}$$

C_{Km}

previous lecture

where $s_{m,k} = \int_0^T s_m(t) f_k(t)\,dt$.

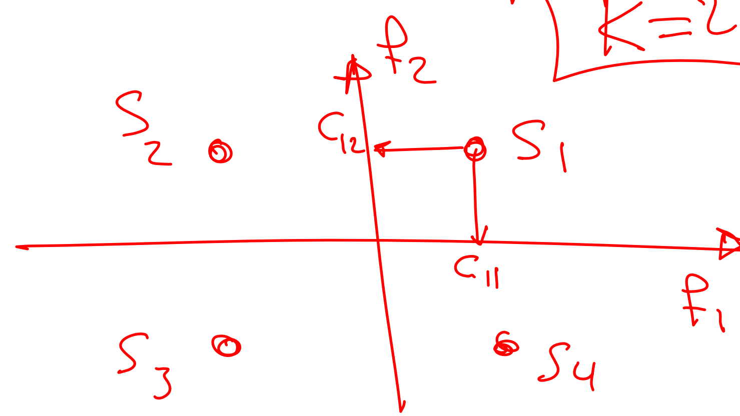
$f_k(t)$ are the basis functions

- The noise can be respresented as: $\boxed{n(t) = n'(t) + \sum_{k=1}^{K} n_k f_k(t)}$

where $n_k = \int_0^T n(t) f_k(t)\,dt$

and $n'(t) = n(t) - \sum_{k=1}^{K} n_k f_k(t)$

for example $\boxed{K=2}$



$S_1(t) = \sum_{k=1}^{2} C_{km} f_k(t)$

$S_1(t) = C_{11} f_1 + C_{21} f_2$

Signal Space Representation (continued)

- The received signal can be represented as:

$$\boxed{r(t)} = \sum_{k=1}^{K} s_{m,k} f_k(t) + \sum_{k=1}^{K} n_k f_k(t) + n'(t) = \boxed{\sum_{k=1}^{K} r_k f_k(t) + n'(t)}$$

where $\quad r_k = s_{m,k} + n_k$

zero

prove

The Orthogonal Noise: $n'(t)$

- The noise $n'(t)$ can be disregarded by the receiver

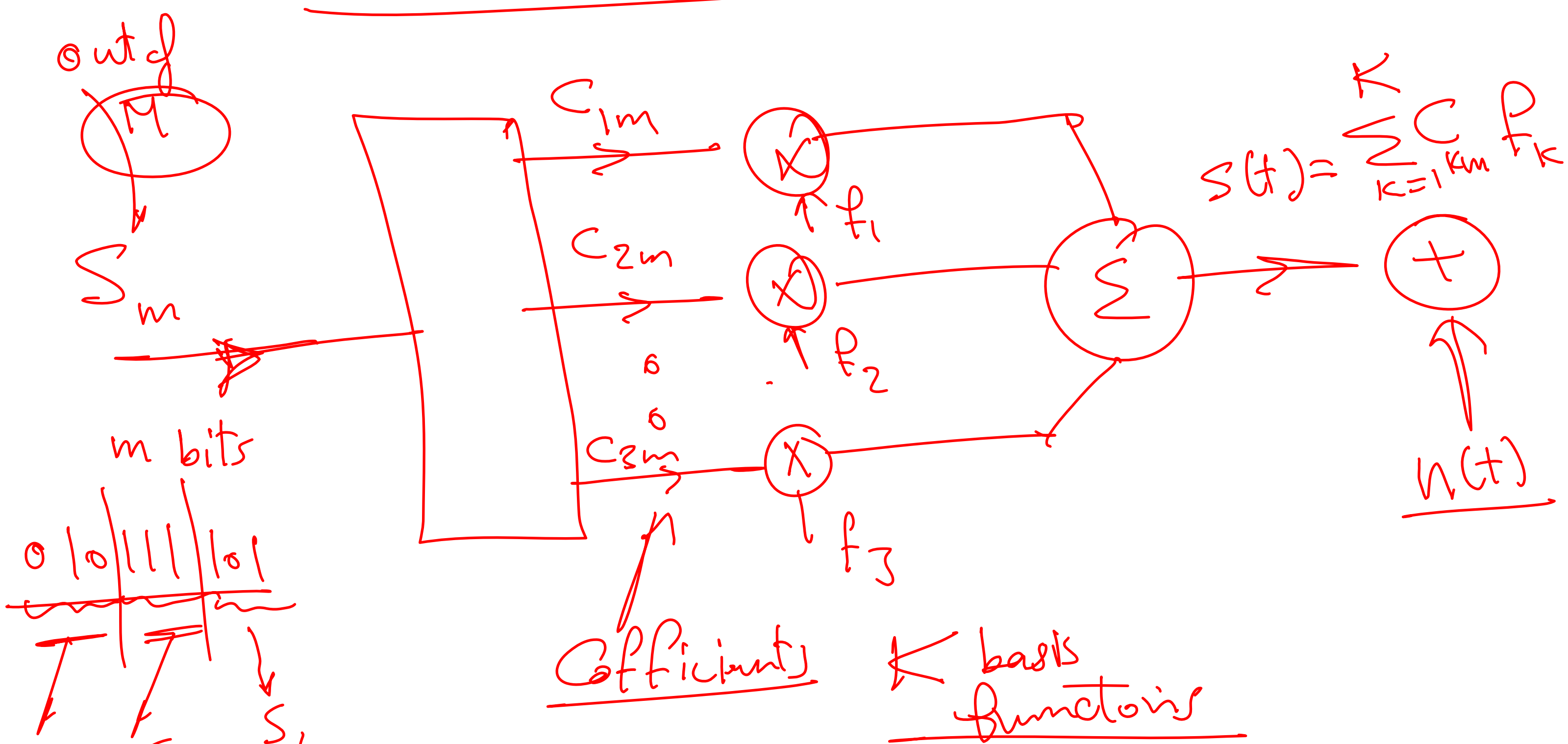
$$\int_0^T s_m(t)n'(t)dt = \int_0^T s_m(t)\left(n(t) - \sum_{k=1}^{K} n_k f_k(t)\right)dt$$

$$= \int_0^T \sum_{k=1}^{K} s_{m,k} f_k(t)\left(n(t) - \sum_{k=1}^{K} n_k f_k(t)\right)dt$$

$$= \sum_{k=1}^{K} s_{m,k} \int_0^T f_k(t)n(t)dt - \sum_{k=1}^{K} s_{m,k} n_k \int_0^T f_k^2(t)dt$$

$$= \sum_{k=1}^{K} s_{m,k} n_k - \sum_{k=1}^{K} s_{m,k} n_k = 0$$

The generalized digital modulation Transmitter



$$K \leq M$$

We can reduce the decision to a finite dimensional space!

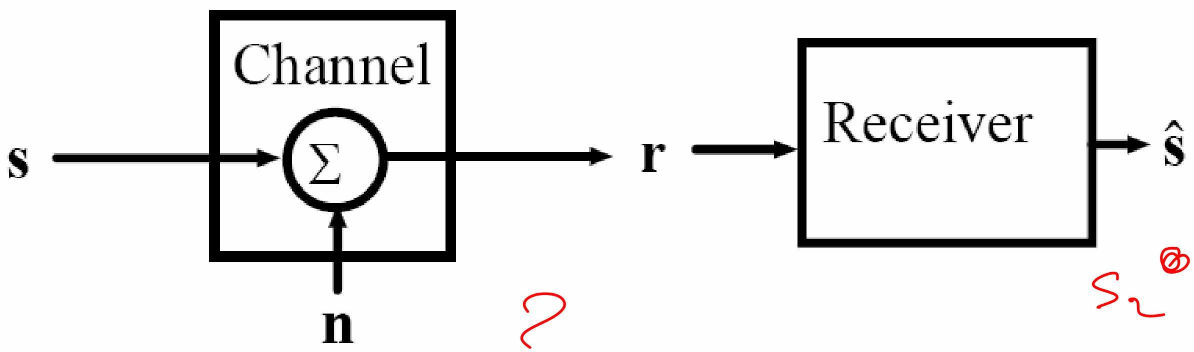
- We transmit a K dimensional signal vector:

  Cofficients $\quad \mathbf{s} = [s_1, s_2, \ldots, s_K] \in \{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$

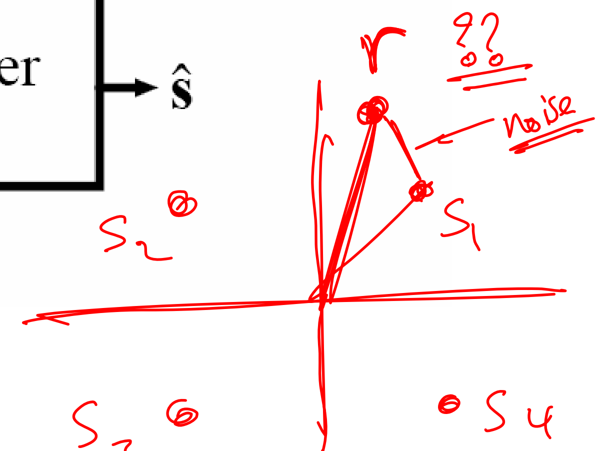
  $c_{m1} \quad c_{m2} \quad c_{m3}$

c_{km}

- We receive a vector $\boxed{\mathbf{r} = [r_1, \ldots, r_K] = \mathbf{s} + \mathbf{n}}$ which is the sum of the signal vector and noise vector $\mathbf{n} = [n_1, \ldots, n_K]$
- Given $\mathbf{r}$, we wish to form an estimate $\hat{\mathbf{s}}$ of the transmitted signal vector which minimizes $P_s = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}]$



$\hat{s} \in \{\underbrace{s_1, s_2, s_3, s_4}_{?}\}$



$\Pr(s_1|r) \geq \Pr(s_2|r)$ s_3 s_4 $\forall m \neq 1$

MAP (Maximum a posteriori probability) Decision Rule

- Suppose that signal vectors $\{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$ are transmitted with probabilities $\{p_1, \ldots, p_m\}$ respectively, and the signal vector $\mathbf{r}$ is received
- We minimize symbol error probability by choosing the signal $\mathbf{s}_m$ which satisfies: $\boxed{\Pr(\mathbf{s}_m|\mathbf{r}) \geq \Pr(\mathbf{s}_i|\mathbf{r}), \forall m \neq i}$
- Equivalently: $$\frac{p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m)}{p(\mathbf{r})} \geq \frac{p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i)}{p(\mathbf{r})}, \forall m \neq i$$

or $$\boxed{p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i), \forall m \neq i}$$

Maximum Likelihood (ML) Decision Rule

- If $p_1 = \cdots = p_m$ or the a priori probabilities are unknown, then the MAP rule simplifies to the ML Rule
- We minimize symbol error probability by choosing the signal $\mathbf{s}_m$ which satisfies $p(\mathbf{r}|\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i), \forall m \neq i$

Evaluation of Probabilities

- In order to apply either the MAP or ML rules, we need to evaluate: $\boxed{p(\mathbf{r}|\mathbf{s}_m)}$
- Since $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$ where $\mathbf{s}_m$ is constant, it is equivalent to evaluate: $\underline{p(\mathbf{n}) = p(n_1, \ldots, n_k)}$
- $n(t)$ is a Gaussian random process
  - Therefore $n_k = \int_0^T n(t) f_k(t) dt$ is a Gaussian random variable
  - Therefore $p(n_1, \ldots, n_K)$ will be a Gaussian p.d.f.

Final Form of MAP Receiver

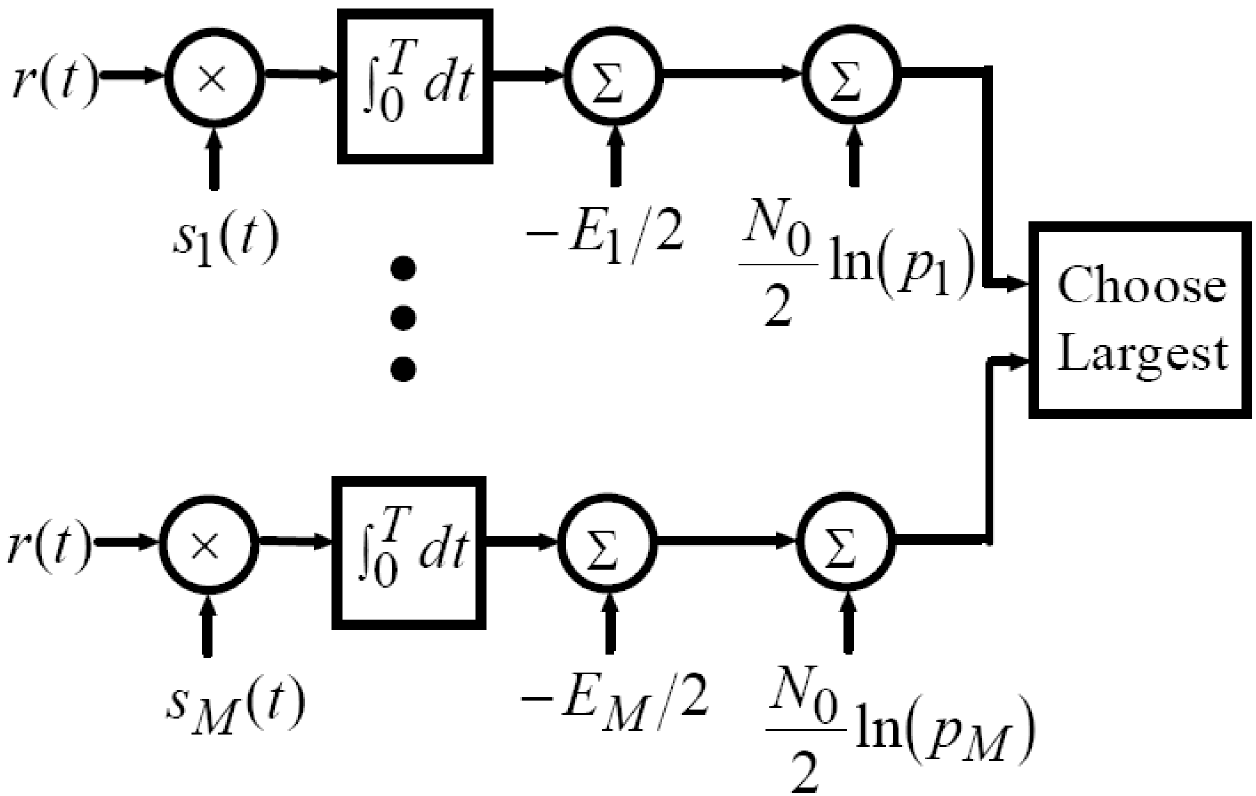
- Multiplying through by the constant $N_0/2$:

$$\boxed{\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2}$$

Interpreting This Result

- $\boxed{\frac{N_0}{2}\ln[p_m]}$ weights the a priori probabilities
  - If the noise is large, p_m counts a lot
  - If the noise is small, our received signal will be an accurate estimate and p_m counts less
- $\boxed{\sum_{k=1}^{K} r_k s_{m,k} = \int_0^T s_m(t) r(t) dt}$
  - represents the correlation with the received signal
- $\boxed{\frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2 = \frac{1}{2}\int_0^T s_m{}^2(t) dt = \frac{E_m}{2}}$
  - represents signal energy

An Implementation of the Optimal Receiver - Correlation Receiver

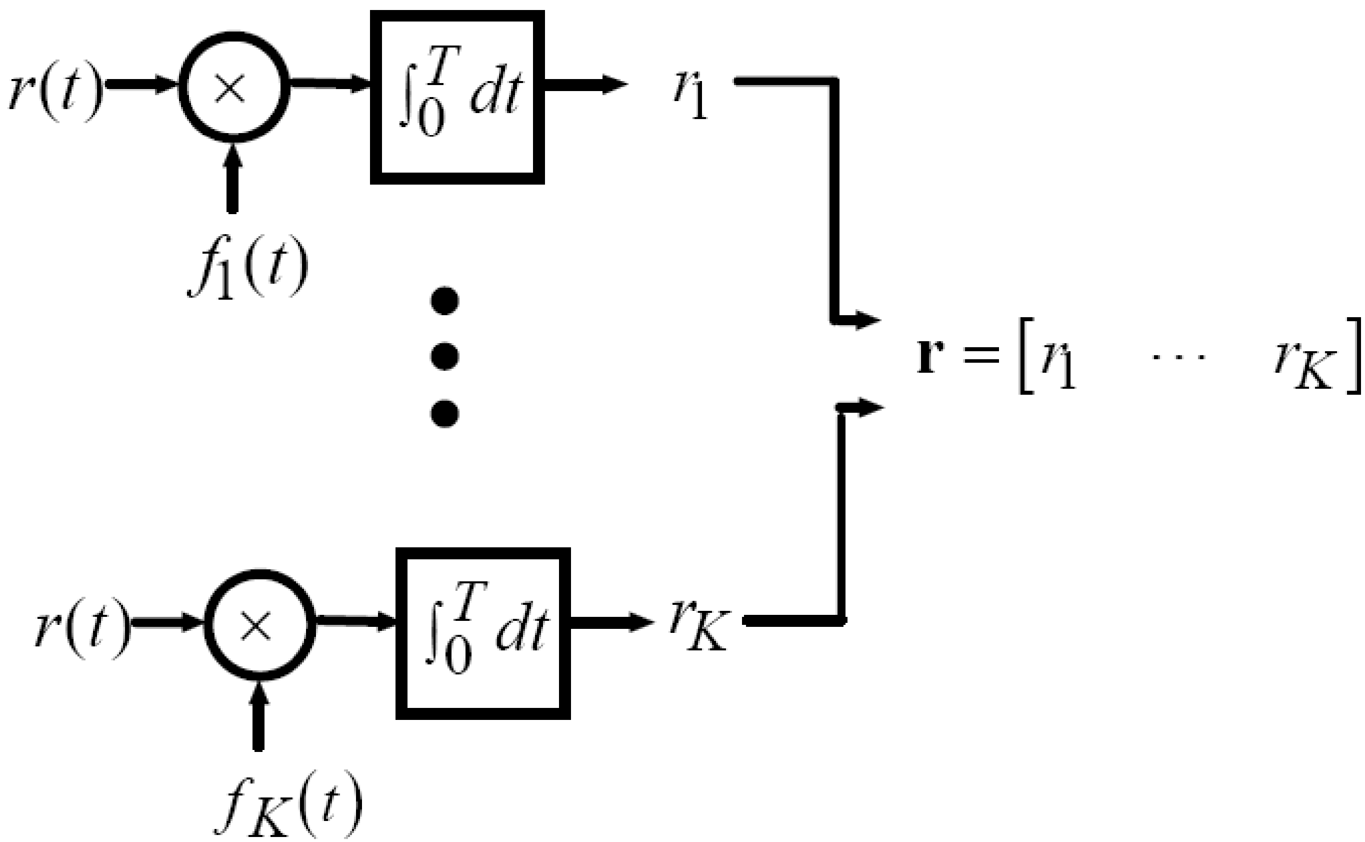


Simplifications for Special Cases

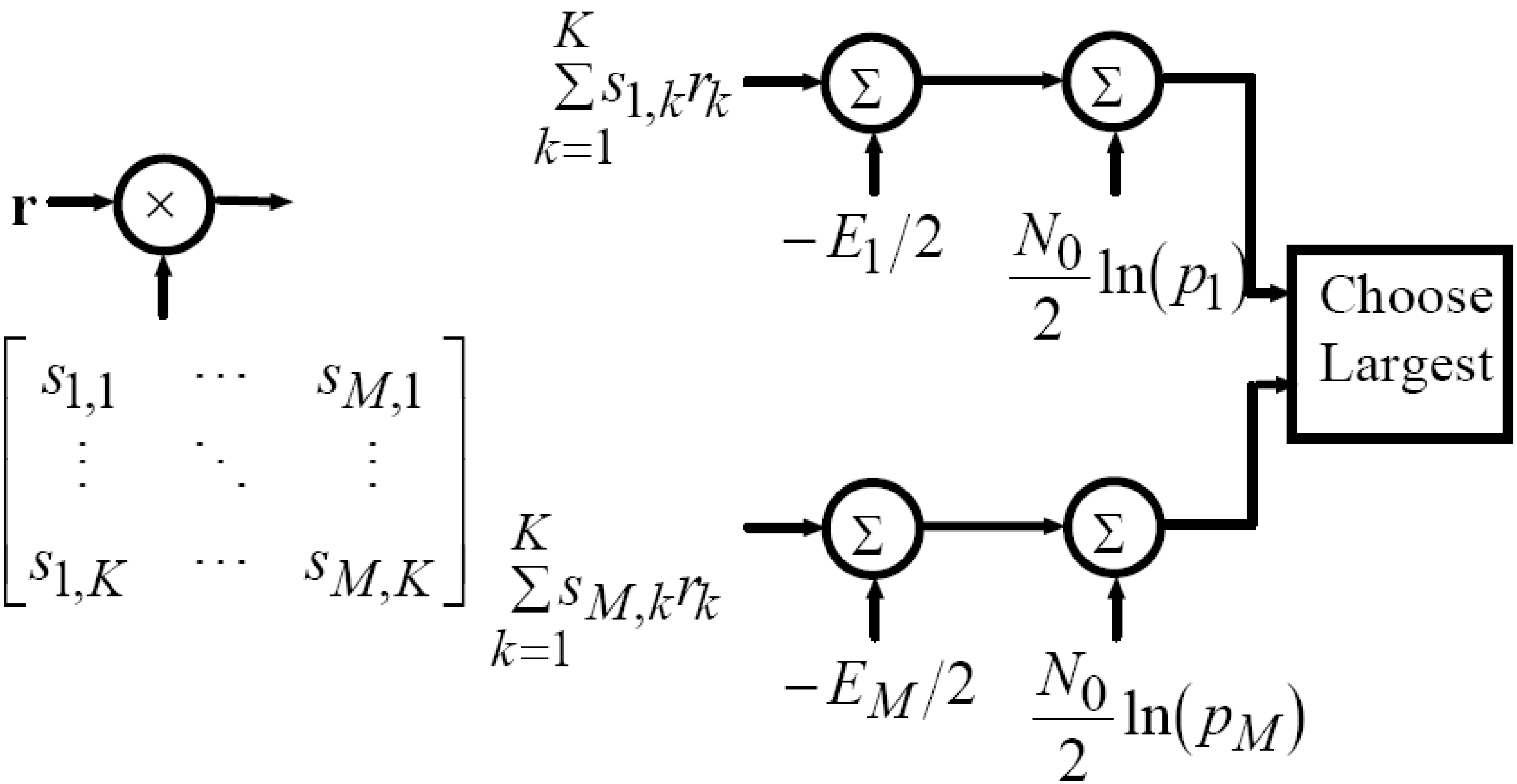
- ML case: All signals are equally likely ($p_1 = \cdots = p_M$).
  - A priori probabilities can be ignored.
- All signals have equal energy ($E_1 = \cdots = E_M$).
  - Energy terms can be ignored.
- We can reduce the number of correlations by directly implementing:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2$$

Reduced Complexity Implementation: Correlation Stage



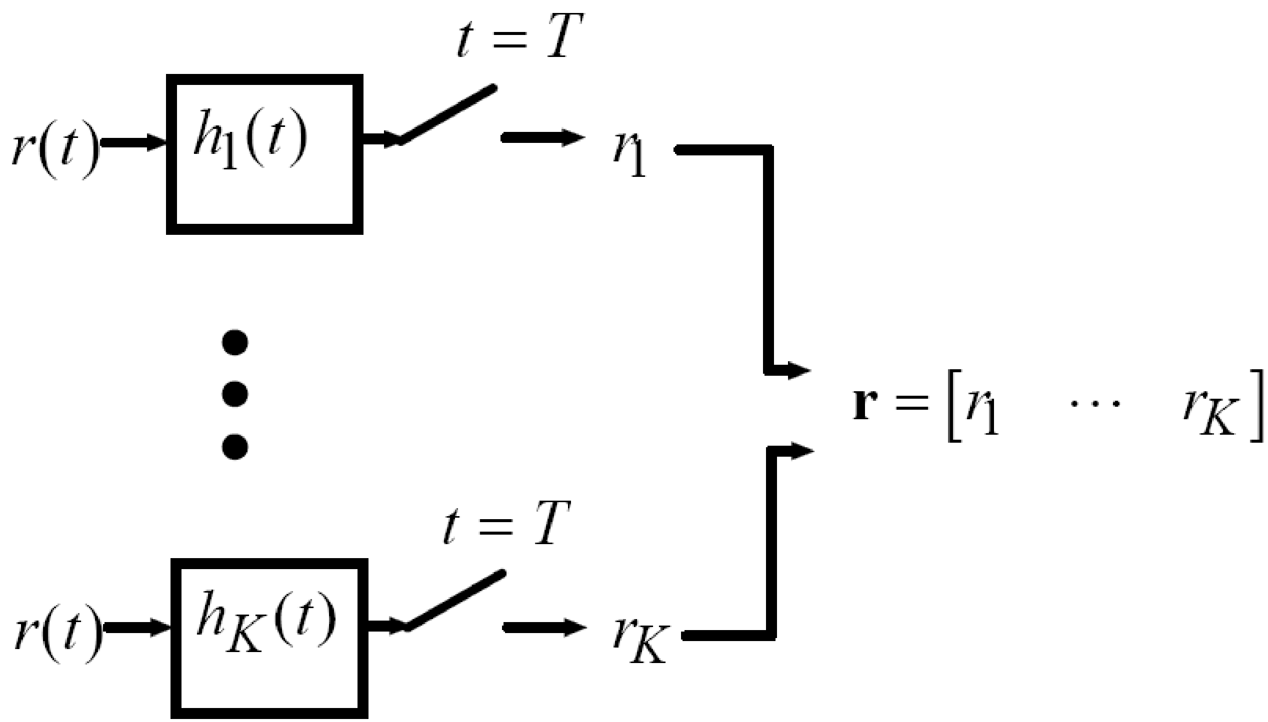
Reduced Complexity Implementation - Processing Stage



Matched Filter Implementation

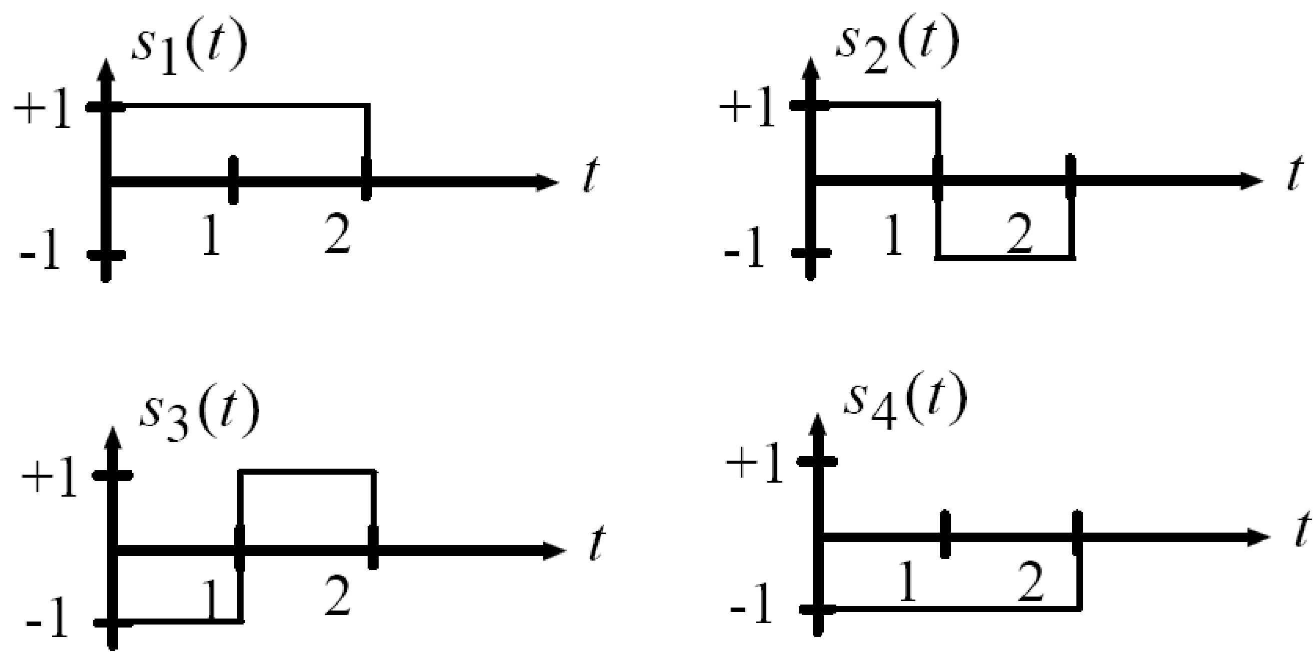
- Assume $f_k(t)$ is time-limited to $t \in [0,T]$, and let $h_k(t) = f_k(T-t)$
- Then $$r_k = \int_0^T r(t) f_k(t) dt = \int_0^T r(t) f_k(T-(T-t)) dt$$ $$= \int_0^T r(t) h_k(T-t) dt = r(t) \otimes h_k(t)\big|_{t=T}$$ where $r(t) \otimes h_k(t)\big|_{t=T}$ denotes the convolution of the signals $r(t)$ and $h_k(t)$ evaluated at time T
- We can implement each correlation by passing $r(t)$ through a filter with impulse response $h_k(t)$

Matched Filter Implementation of Correlation



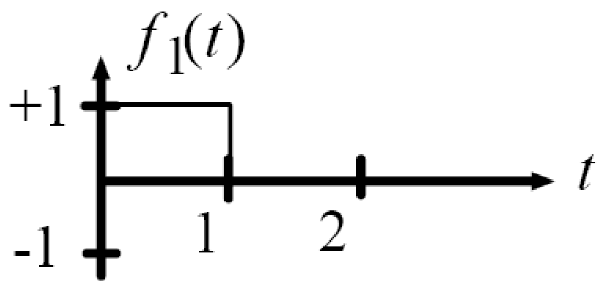
Example of Optimal Receiver Design

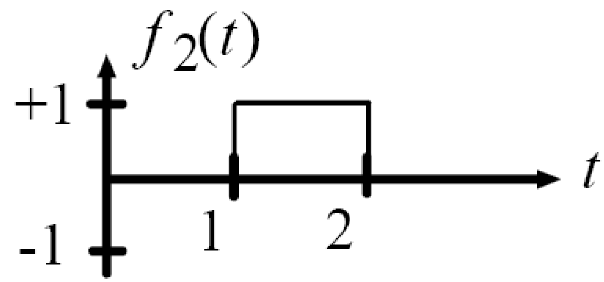
- Consider the signal set:



Example of Optimal Receiver Design (continued)

- Suppose we use the basis functions:





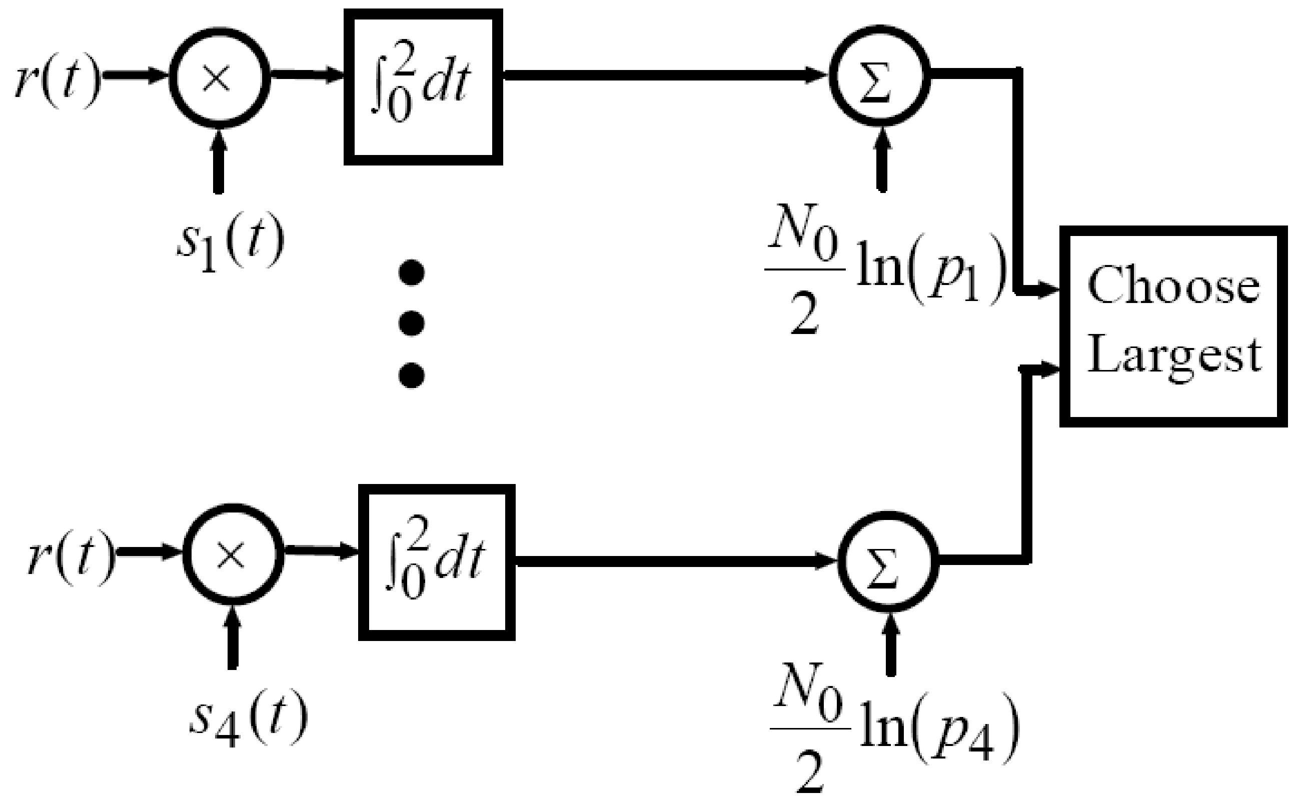
$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t)$ $\qquad$ $s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$

$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t)$ $\qquad$ $s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$

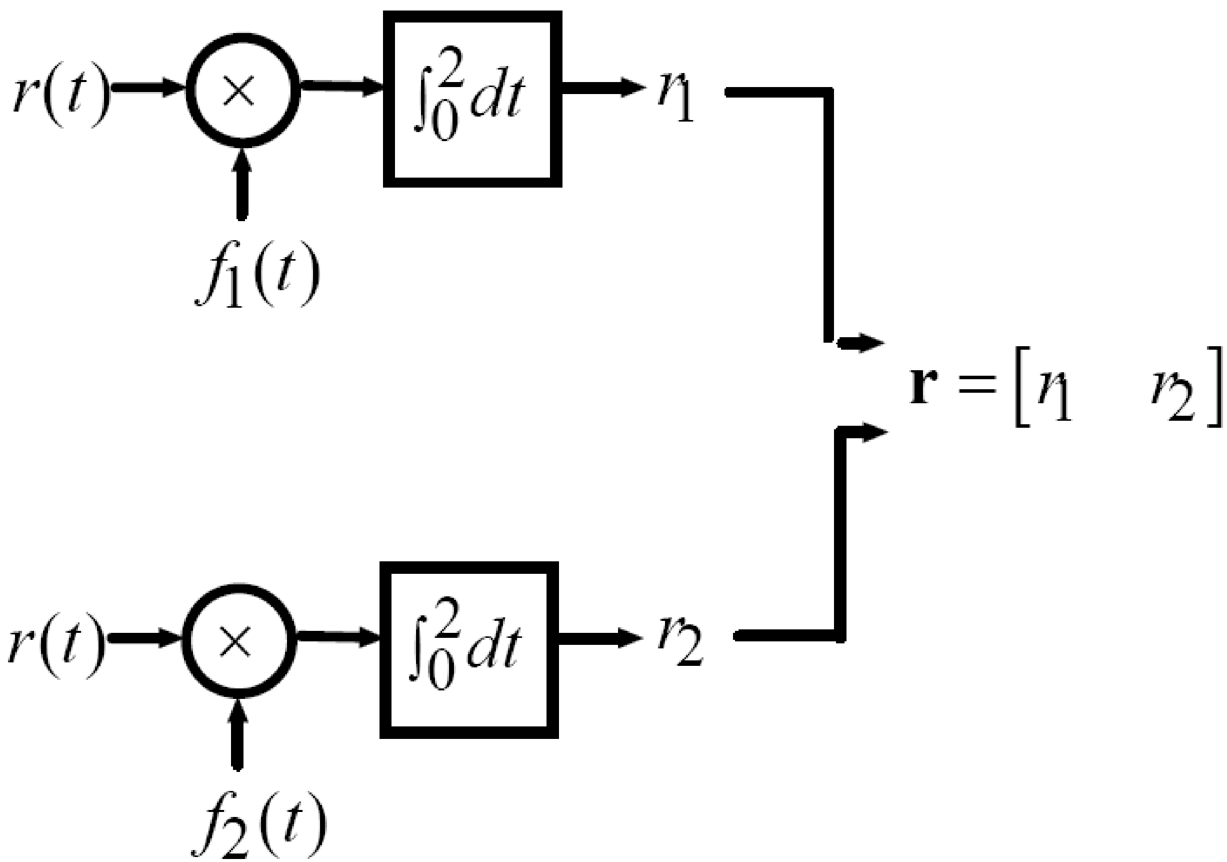
$T = 2$

$E_1 = E_2 = E_3 = E_4 = 2$

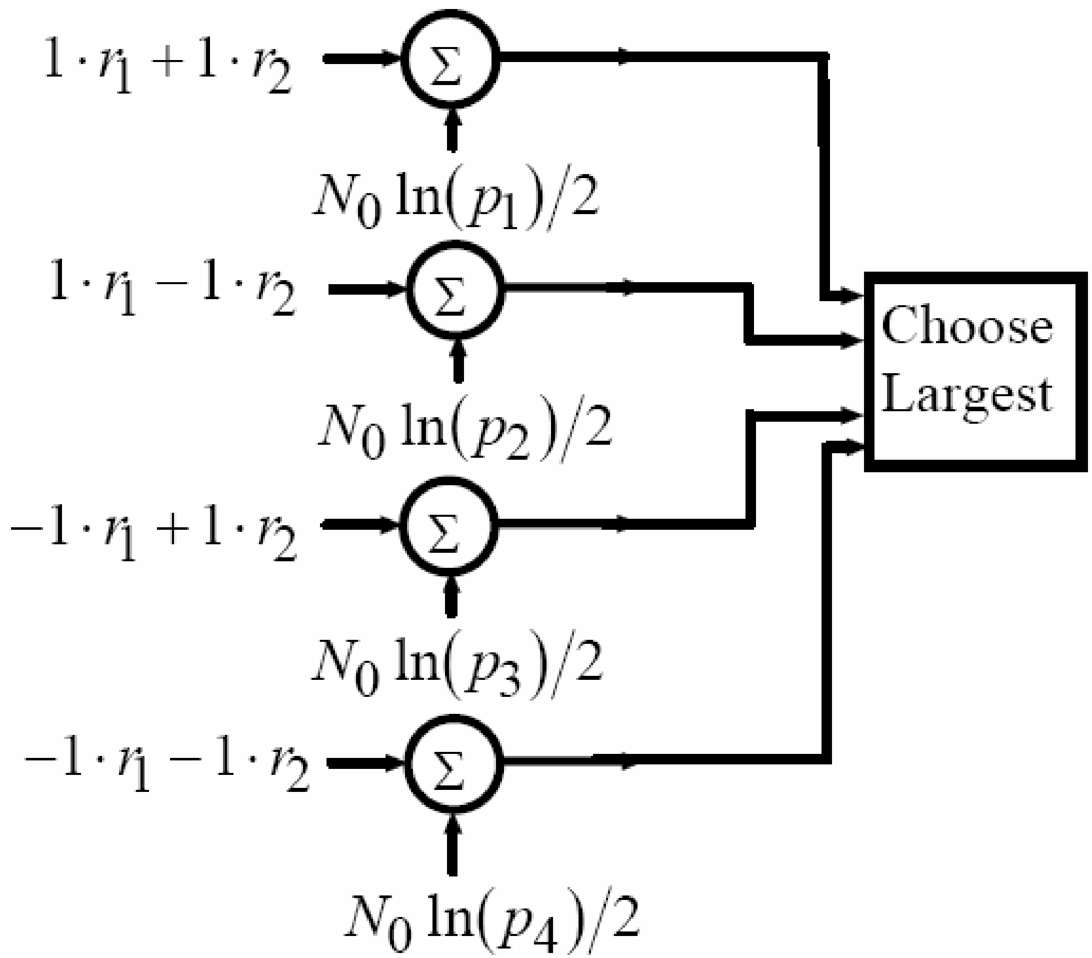
1st Implementation of Correlation Receiver



Reduced Complexity Correlation Receiver - Correlation Stage

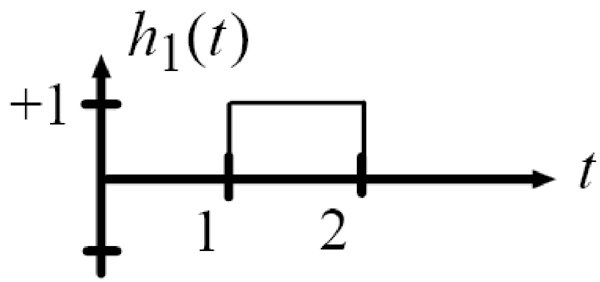


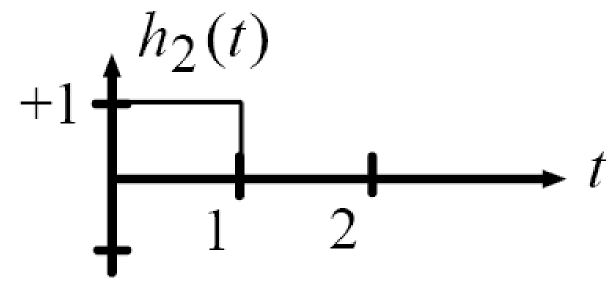
Reduced Complexity Correlation Receiver - Processing Stage

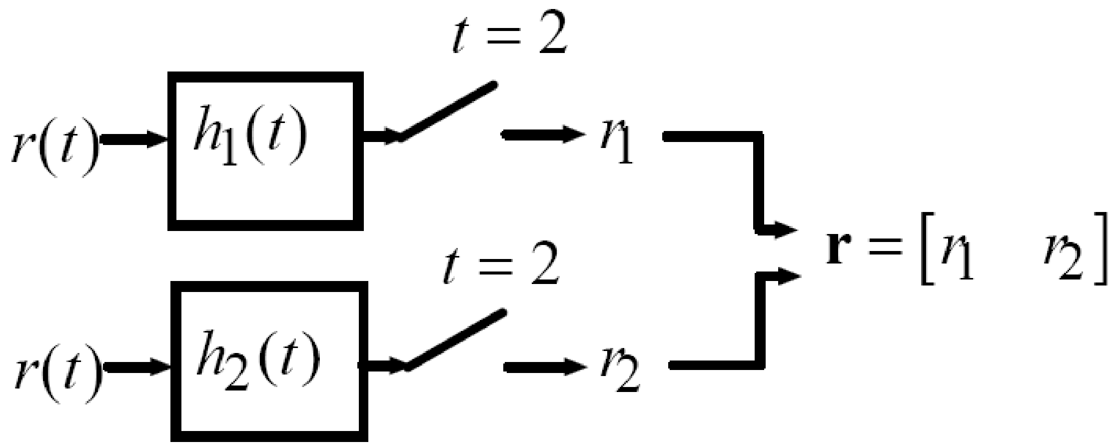


Matched Filter Implementation of Correlations

$$h_k(t) = f_k(2-t)$$







Summary of Optimal Receiver Design

- Optimal coherent receiver for AWGN has three parts:
  - Correlates the received signal with each possible transmitted signal signal
  - Normalizes the correlation to account for energy
  - Weights the a priori probabilities according to noise power
- This receiver is completely general for any signal set
- Simplifications are possible under many circumstances