Extra material

Gram Schmitd Procedure and signal space for digital modulation

3.1.3 A Vector View of Signals and Noise

We now present a geometric or vector view of signal waveforms that are useful for either baseband or bandpass signals. We define an N-dimensional *orthogonal space* as a space characterized by a set of N linearly independent functions $\{\phi_j(t)\}$, called *basis functions.* Any arbitrary function in the space can be generated by a linear combination of these basis functions. The basis functions must satisfy the conditions

$$\int_0^T \psi_j(t)\psi_k(t)\,dt = K_j\delta_{jk} \qquad 0 \le t \le T \qquad j,k = 1,\ldots,N \tag{3.8a}$$

where the operator

$$\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise} \end{cases} \tag{3.8b}$$

is called the *Kronecker delta function* and is defined by Equation (3.8b). When the K_j constants are nonzero, the signal space is called *orthogonal.* When the basis functions are normalized so that each $K_j = 1$, the space is called an *orthonormal* space. The principal requirement for orthogonality can be stated as follows. Each $\psi_j(t)$ function of the set of basis functions must be independent of the other members of the set. Each $\psi_j(t)$ must not interfere with any other members of the set in the detection process. From a geometric point of view, each $\psi_j(t)$ is mutually perpendicular to each of the other $\psi_k(t)$ for $j \neq k$. An example of such a space with $N = 3$ is shown in Figure 3.3, where the mutually perpendicular axes are designated $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$. If $\psi_j(t)$ corresponds to a real-valued voltage or current waveform component, associated with a 1-Ω resistive load, then using Equations (1.5) and (3.8), the normalized energy in joules dissipated in the load in T seconds, due to ψ_j, is

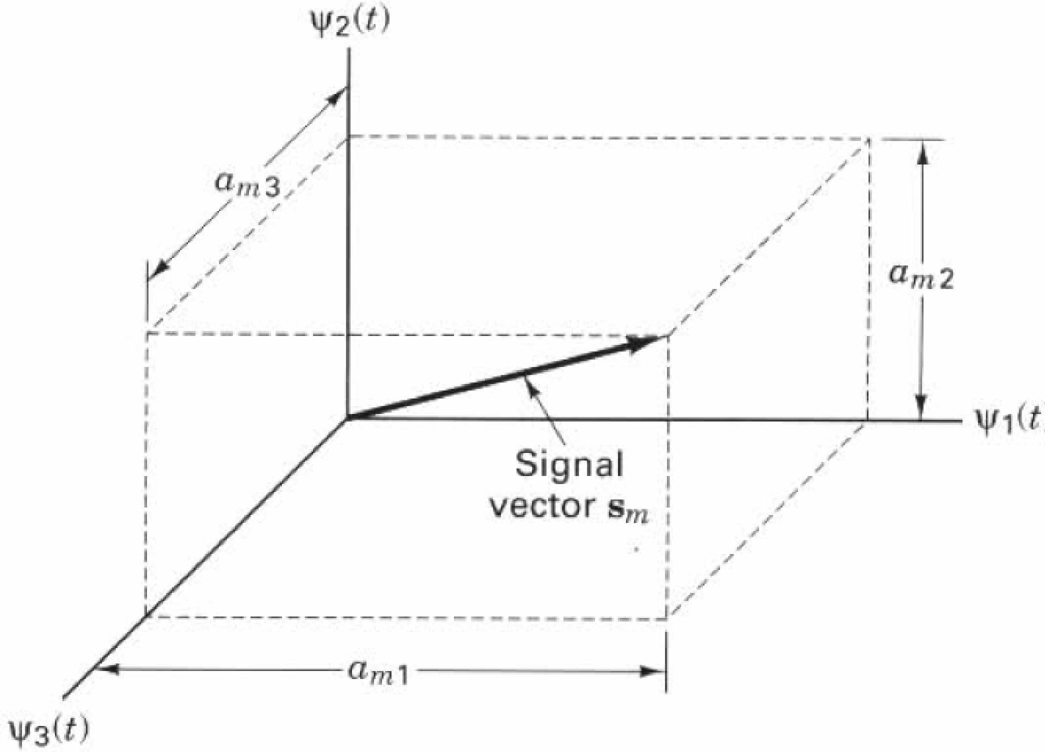

Figure 3.3 Vectorial representation of the signal waveform $s_m(t)$.

$$E_j = \int_0^T \psi_j^2(t)\, dt = K_j \tag{3.9}$$

One reason we focus on an *orthogonal signal space* is that Euclidean distance measurements, fundamental to the detection process, are easily formulated in such a space. However, even if the signaling waveforms do not make up such an orthogonal set, they can be transformed into linear combinations of orthogonal waveforms. It can be shown [3] that *any arbitrary* finite set of waveforms $\{s_i(t)\}$ $(i = 1, \ldots, M)$, where each member of the set is physically realizable and of duration T, can be expressed as a linear combination of N orthogonal waveforms $\psi_1(t)$, $\psi_2(t), \ldots, \psi_N(t)$, where $N \leq M$, such that

$$\begin{aligned}
s_1(t) &= a_{11}\psi_1(t) + a_{12}\psi_2(t) + \cdots + a_{1N}\psi_N(t) \\
s_2(t) &= a_{21}\psi_1(t) + a_{22}\psi_2(t) + \cdots + a_{2N}\psi_N(t) \\
&\vdots \\
s_M(t) &= a_{M1}\psi_1(t) + a_{M2}\psi_2(t) + \cdots + a_{MN}\psi_N(t)
\end{aligned}$$

These relationships are expressed in more compact notation as

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\,\psi_j(t) \quad i = 1, \ldots, M \quad N \leq M \tag{3.10}$$

where

$$a_{ij} = \frac{1}{K_j} \int_0^T s_i(t)\psi_j(t)\,dt \quad i = 1, \ldots, M \quad 0 \leq t \leq T \quad j = 1, \ldots, N \tag{3.11}$$

The coefficient a_{ij} is the value of the $\psi_j(t)$ component of signal $s_i(t)$. The form of the $\{\psi_j(t)\}$ is not specified; it is chosen for convenience and will depend on the form of the signal waveforms. The set of signal waveforms, $\{s_i(t)\}$, can be viewed as a set of vectors, $\{\mathbf{s}_i\} = \{a_{i1}, a_{i2}, \ldots, a_{iN}\}$. If, for example, $N = 3$, we may plot the vector $\mathbf{s}_m$ corresponding to the waveform

$$s_m(t) = a_{m1}\psi_1(t) + a_{m2}\psi_2(t) + a_{m3}\psi_3(t)$$

as a point in a three-dimensional Euclidean space with coordinates (a_{m1}, a_{m2}, a_{m3}), as shown in Figure 3.3. The orientation among the signal vectors describes the relation of the signals to one another (with respect to phase or frequency), and the amplitude of each vector in the set $\{\mathbf{s}_i\}$ is a measure of the signal energy transmitted during a symbol duration. In general, once a set of N orthogonal functions has been adopted, each of the transmitted signal waveforms, $s_i(t)$, is completely determined by the vector of its coefficients,

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN}) \qquad i = 1, \ldots, M \tag{3.12}$$

We shall employ the notation of signal vectors, {$\mathbf{s}$}, or signal waveforms, {$s(t)$}, as best suits the discussion. A typical detection problem, conveniently viewed in terms of signal vectors, is illustrated in Figure 3.4. Vectors $\mathbf{s}_j$ *and* $\mathbf{s}_k$ represent *prototype* or *reference signals* belonging to the set of M waveforms, {$s_i(t)$}. The receiver knows, a priori, the location in the signal space of each prototype vector belonging to the M-ary set. During the transmission of any signal, the signal is perturbed by noise so that the resultant vector that is actually received is a perturbed version (e.g., $\mathbf{s}_j + \mathbf{n}$ or $\mathbf{s}_k + \mathbf{n}$) of the original one, where $\mathbf{n}$ represents a noise vector. The noise is additive and has a Gaussian distribution; therefore, the resulting distribution of possible received signals is a cluster or cloud of points around $\mathbf{s}_j$ and $\mathbf{s}_k$. The cluster is dense in the center and becomes sparse with increasing distance from the prototype. The arrow marked "$\mathbf{r}$" represents a signal vector that might arrive at the receiver during some symbol interval. The task of the receiver is to decide whether $\mathbf{r}$ has a close "resem-

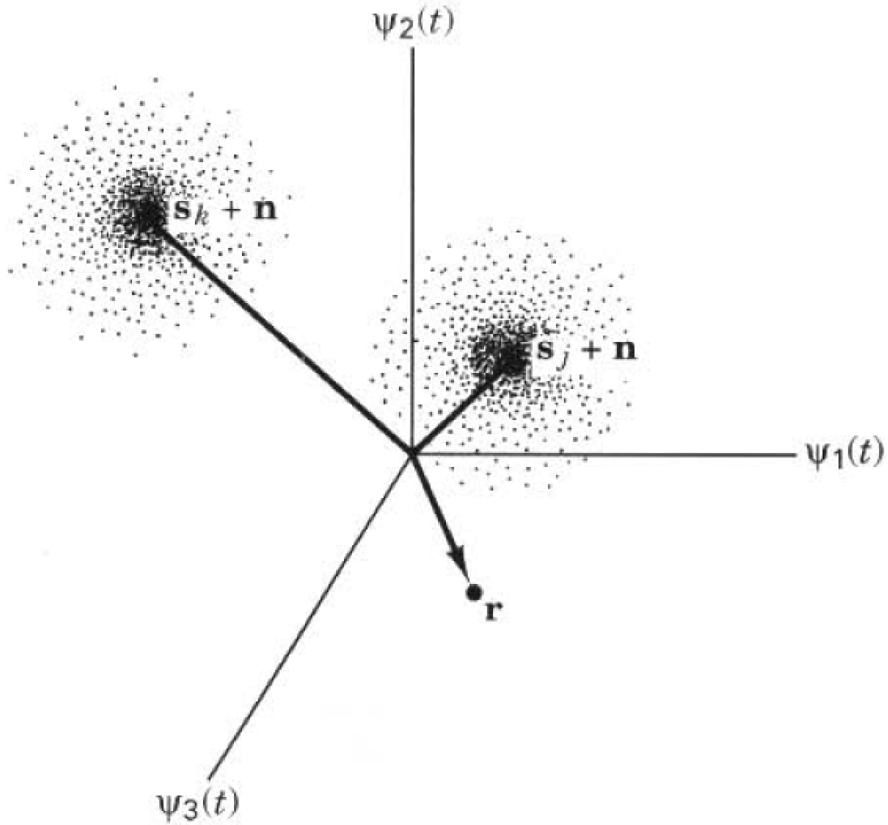

Figure 3.4 Signals and noise in a three-dimensional vector space.

blance" to the prototype $\mathbf{s}_j$, whether it more closely resembles $\mathbf{s}_k$, or whether it is closer to some other prototype signal in the M-ary set. The measurement can be thought of as a *distance* measurement. The receiver or detector must decide which of the prototypes within the signal space is *closest* in distance to the received vector $\mathbf{r}$. The analysis of all demodulation or detection schemes involves this concept of *distance* between a received waveform and a set of possible transmitted waveforms. A simple rule for the detector to follow is to decide that $\mathbf{r}$ belongs to the same class as its nearest neighbor (nearest prototype vector).

3.1.3.1 Waveform Energy

Using Equations (1.5), (3.10), and (3.8), the normalized energy E_i, associated with the waveform $s_i(t)$ over a symbol interval T can be expressed in terms of the orthogonal components of $s_i(t)$ as follows:

$$E_i = \int_0^T s_i^2(t)\, dt = \int_0^T \Big[\sum_j a_{ij}\, \psi_j(t) \Big]^2 dt \tag{3.13}$$

$$= \int_0^T \sum_j a_{ij}\, \psi_j(t) \sum_k a_{ik} \psi_k(t)\, dt \tag{3.14}$$

$$= \sum_j \sum_k a_{ij}\, a_{ik} \int_0^T \psi_j(t) \psi_k(t)\, dt \tag{3.15}$$

$$= \sum_j \sum_k a_{ij}\, a_{ik} K_j \delta_{jk} \tag{3.16}$$

$$= \sum_{j=1}^{N} a_{ij}^2 K_j \qquad i = 1, \ldots, M \tag{3.17}$$

Equation (3.17) is a special case of Parseval's theorem relating the integral of the square of the waveform $s_i(t)$ to the sum of the square of the orthogonal series coefficients. If orthonormal functions are used (i.e., $K_j = 1$), the normalized energy over a symbol duration T is given by

$$E_i = \sum_{j=1}^{N} a_{ij}^2 \tag{3.18}$$

If there is equal energy E in each of the $s_i(t)$ waveforms, we can write Equation (3.18) in the form

$$E = \sum_{j=1}^{N} a_{ij}^2 \qquad \text{for all } i \tag{3.19}$$

Example 3.1 Orthogonal Representation of Waveforms

Figure 3.5 illustrates the statement that any arbitrary integrable waveform set can be represented as a linear combination of orthogonal waveforms. Figure 3.5a shows a set of three waveforms, $s_1(t)$, $s_2(t)$, and $s_3(t)$.

(a) Demonstrate that these waveforms *do not* form an orthogonal set.

(b) Figure 3.5b shows a set of two waveforms, $\psi_1(t)$ and $\psi_2(t)$. Verify that these waveforms form an orthogonal set.

(c) Show how the nonorthogonal waveform set in part (a) can be expressed as a linear combination of the orthogonal set in part (b).

(d) Figure 3.5c illustrates another orthogonal set of two waveforms, $\psi_1'(t)$ and $\psi_2'(t)$. Show how the nonorthogonal set in Figure 3.5a can be expressed as a linear combination of the set in Figure 3.5c.

Solution

(a) $s_1(t)$, $s_2(t)$, and $s_3(t)$ are clearly not orthogonal, since they do not meet the requirements of Equation (3.8); that is, the time integrated value (over a symbol duration) of the cross-product of any two of the three waveforms is not zero. Let us verify this for $s_1(t)$ and $s_2(t)$:

$$\int_0^T s_1(t)s_2(t)\,dt = \int_0^{T/2} s_1(t)s_2(t)\,dt + \int_{T/2}^T s_1(t)s_2(t)\,dt$$

$$= \int_0^{T/2} (-1)(2)\,dt + \int_{T/2}^T (-3)(0)\,dt = -T$$

Similarly, the integral over the interval T of each of the cross-products $s_1(t)s_3(t)$ and $s_2(t)s_3(t)$ results in nonzero values. Hence, the waveform set $\{s_i(t)\}$ $(i = 1, 2, 3)$ in Figure 3.5a is not an orthogonal set.

(b) Using Equation (3.8), we verify that $\psi_1(t)$ and $\psi_2(t)$ form an orthogonal set as follows:

$$\int_0^T \psi_1(t)\psi_2(t)\,dt = \int_0^{T/2} (1)(1)\,dt + \int_{T/2}^T (-1)(1)\,dt = 0$$

(c) Using Equation (3.11) with $K_j = T$, we can express the nonorthogonal set $\{s_i(t)\}$ $(i = 1, 2, 3)$ as a linear combination of the orthogonal basis waveforms $\{\psi_j(t)\}$ $(j = 1, 2)$:

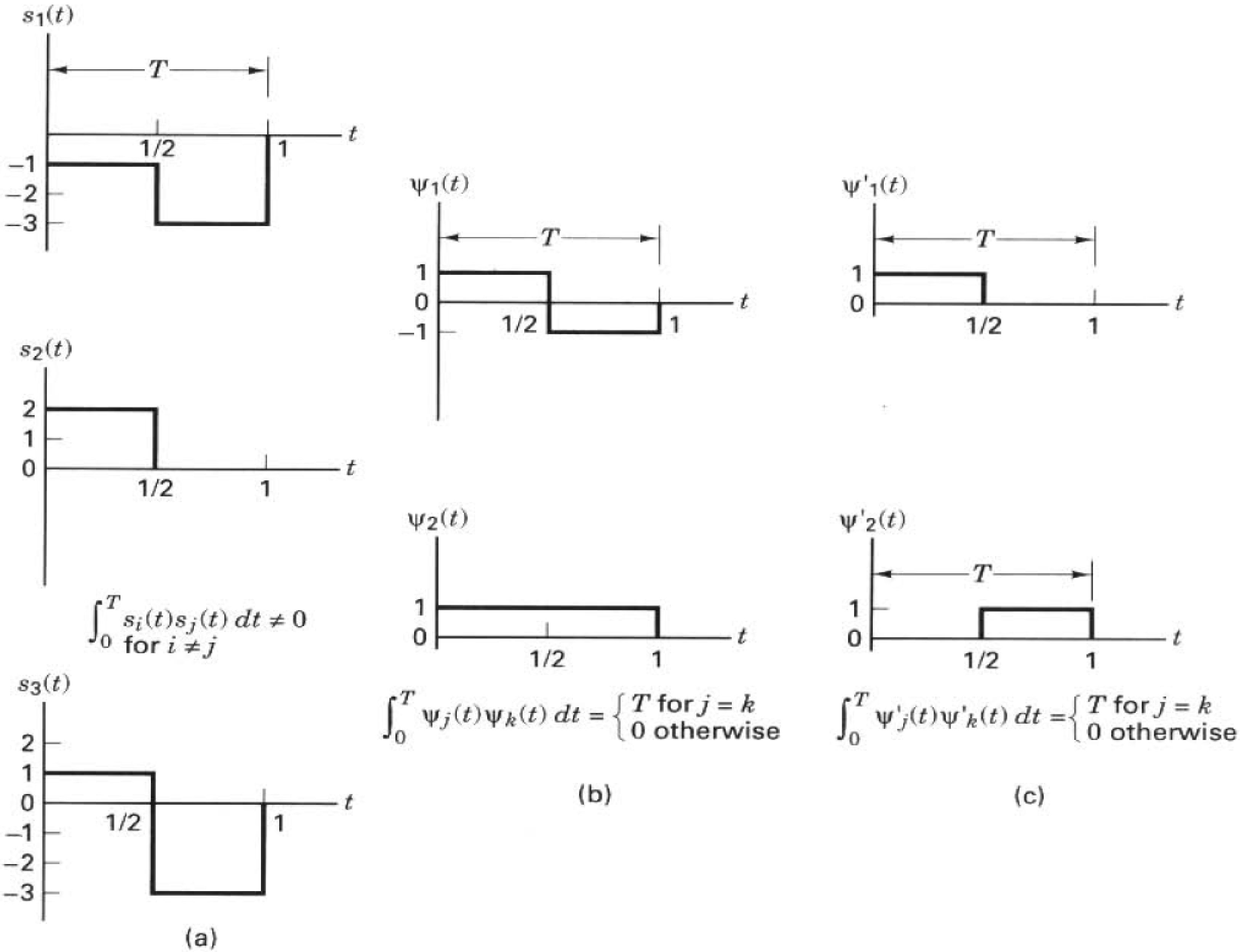

Figure 3.5 Example of an arbitrary signal set in terms of an orthogonal set. (a) Arbitrary signal set. (b) A set of orthogonal basis functions. (c) Another set of orthogonal basis functions.

$$s_1(t) = \psi_1(t) - 2\psi_2(t)$$
$$s_2(t) = \psi_1(t) + \psi_2(t)$$
$$s_3(t) = 2\psi_1(t) - \psi_2(t)$$

(d) Similar to part (c), the nonorthogonal set $\{s_i(t)\}$ $(i = 1, 2, 3)$ can be expressed in terms of the simple orthogonal basis set $\{\psi'_j(t)\}$ $(j = 1, 2)$ in Figure 3.5c, as follows:

$$s_1(t) = -\psi'_1(t) - 3\psi'_2(t)$$
$$s_2(t) = 2\psi'_1(t)$$
$$s_3(t) = \psi'_1(t) - 3\psi'_2(t)$$

These relationships illustrate how an arbitrary waveform set $\{s_i(t)\}$ can be expressed as a linear combination of an orthogonal set $\{\psi_j(t)\}$, as described in Equations (3.10) and (3.11). What are the practical applications of being able to describe $s_1(t)$, $s_2(t)$, and $s_3(t)$ in terms of $\psi_1(t)$, $\psi_2(t)$, and the appropriate coefficients? If we want a system for transmitting waveforms $s_1(t)$, $s_2(t)$, and $s_3(t)$, the transmitter and the receiver need only be implemented using the two basis functions $\psi_1(t)$ and $\psi_2(t)$ instead of the three original waveforms. The *Gram–Schmidt orthogonalization procedure* provides a convenient way in which an appropriate choice of a basis function set $\{\psi_j(t)\}$, can be obtained for any given signal set $\{s_i(t)\}$. (It is described in Appendix 4A of Reference [4].)