Digital Communications and Simulation
ECE 5654

Section 3 – Modulation

Module 1 – Signal Space Representation of Signals

Set 2 – Signal Space Concepts

Modulation Principles

- Almost all communication systems transmit digital data using a sinusoidal carrier waveform.
  - Electromagnetic signals propagate well
  - Choice of carrier frequency allows placement of signal in arbitrary part of spectrum
- Physical system implements modulation by:
  - Processing digital information at baseband
  - Pulse shaping and filtering of digital waveform
  - Baseband signal is mixed with signal from oscillator
  - RF signal is filtered, amplified and coupled with antenna

Representation of Modulation Signals

- We can modify amplitude, phase or frequency.
- Amplitude Shift Keying (ASK) or On/Off Keying (OOK): $1 \Rightarrow A\cos(2\pi f_c t), 0 \Rightarrow 0$
- Frequency Shift Keying (FSK):

$$1 \Rightarrow A\cos(2\pi f_1 t), 0 \Rightarrow A\cos(2\pi f_0 t)$$

- Phase Shift Keying (PSK):

$$1 \Rightarrow A\cos(2\pi f_c t)$$

$$0 \Rightarrow A\cos(2\pi f_c t + \pi) = -A\cos(2\pi f_c t)$$

Representation of Bandpass Signals

Bandpass signals (signals with small bandwidth compared to carrier frequency) can be represented in any of three standard formats:

- **Quadrature Notation**

$$s(t) = x(t)\cos\left(2\pi f_c t\right) - y(t)\sin\left(2\pi f_c t\right)$$

where *x(t)* and *y(t)* are real-valued baseband signals called the in-phase and quadrature components of *s(t)*

Representation of Bandpass Signals (continued)

- **Complex Envelope Notation**

$$s(t) = \mathrm{Re}\left[\left(x(t) + jy(t)\right)e^{-j2\pi f_c t}\right] = \mathrm{Re}\left[s_l(t)e^{-j2\pi f_c t}\right]$$

where $s_l(t)$ is the <u>complex envelope</u> of *s(t)*.

- **Magnitude and Phase**

$$s(t) = a(t)\cos\left(2\pi f_c t + \theta(t)\right)$$

where $a(t) = \sqrt{x^2(t) + y^2(t)}$ is the <u>magnitude</u> of *s(t)*,

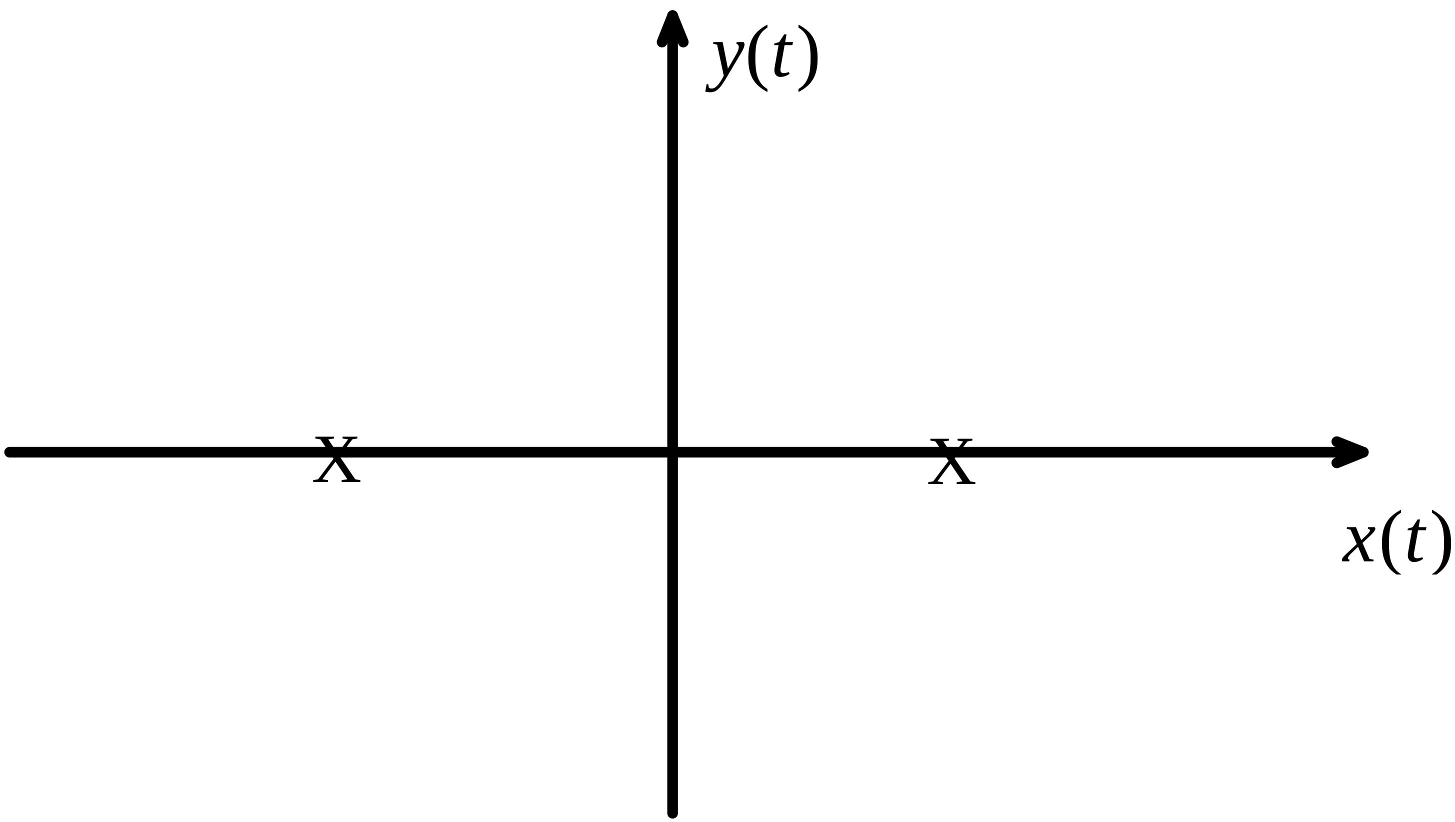
and $\theta(t) = \tan^{-1}\left[\frac{y(t)}{x(t)}\right]$ is the <u>phase</u> of *s(t)*.

Key Ideas from I/Q Representation of Signals

- We can represent bandpass signals independent of carrier frequency.
- The idea of quadrature sets up a coordinate system for looking at common modulation types.
- The coordinate system is sometimes called a <u>signal constellation diagram</u>.

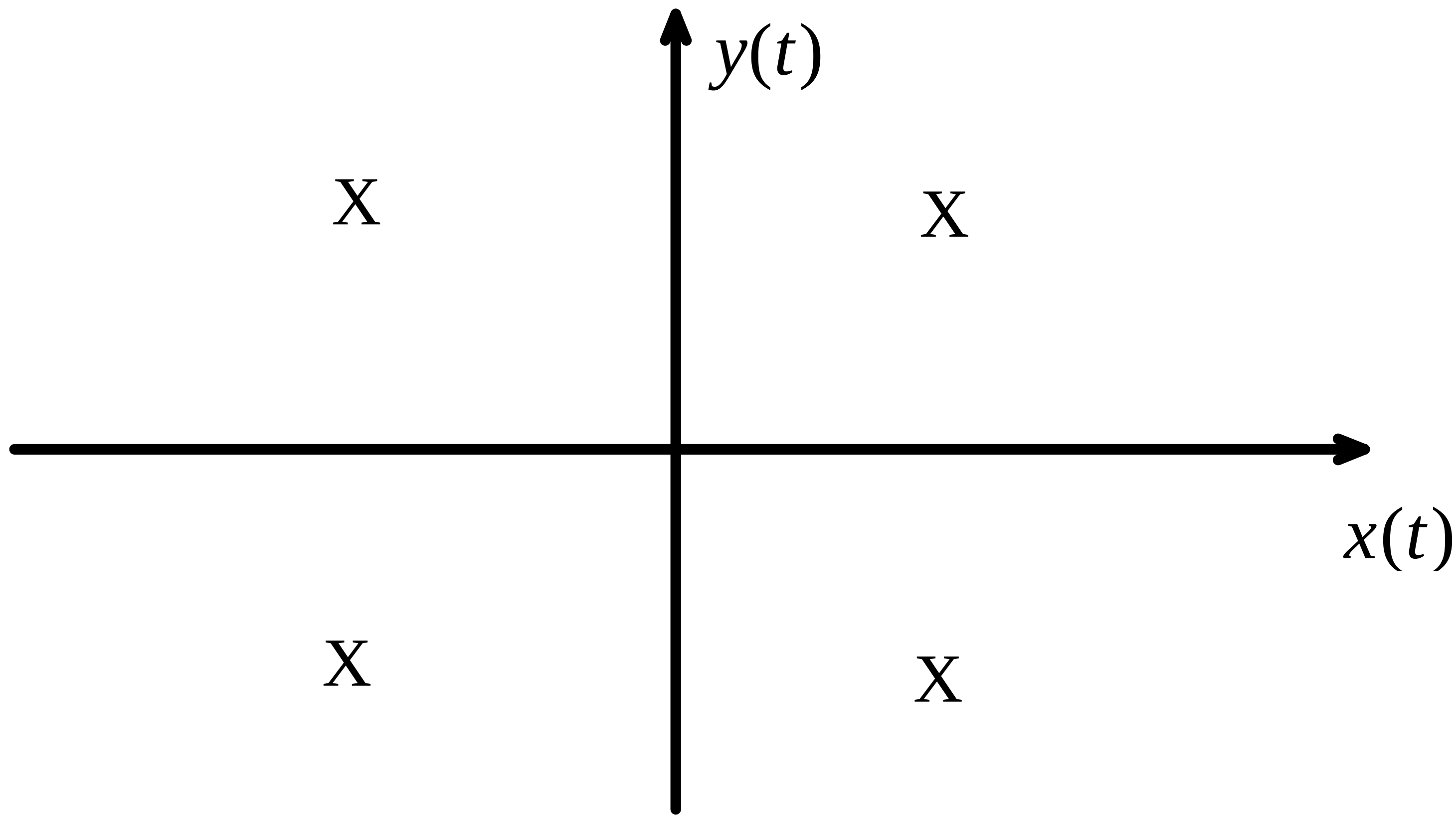
Example of Signal Constellation Diagram: BPSK

- $x(t) \in \{\pm 1\}, y(t) = 0$



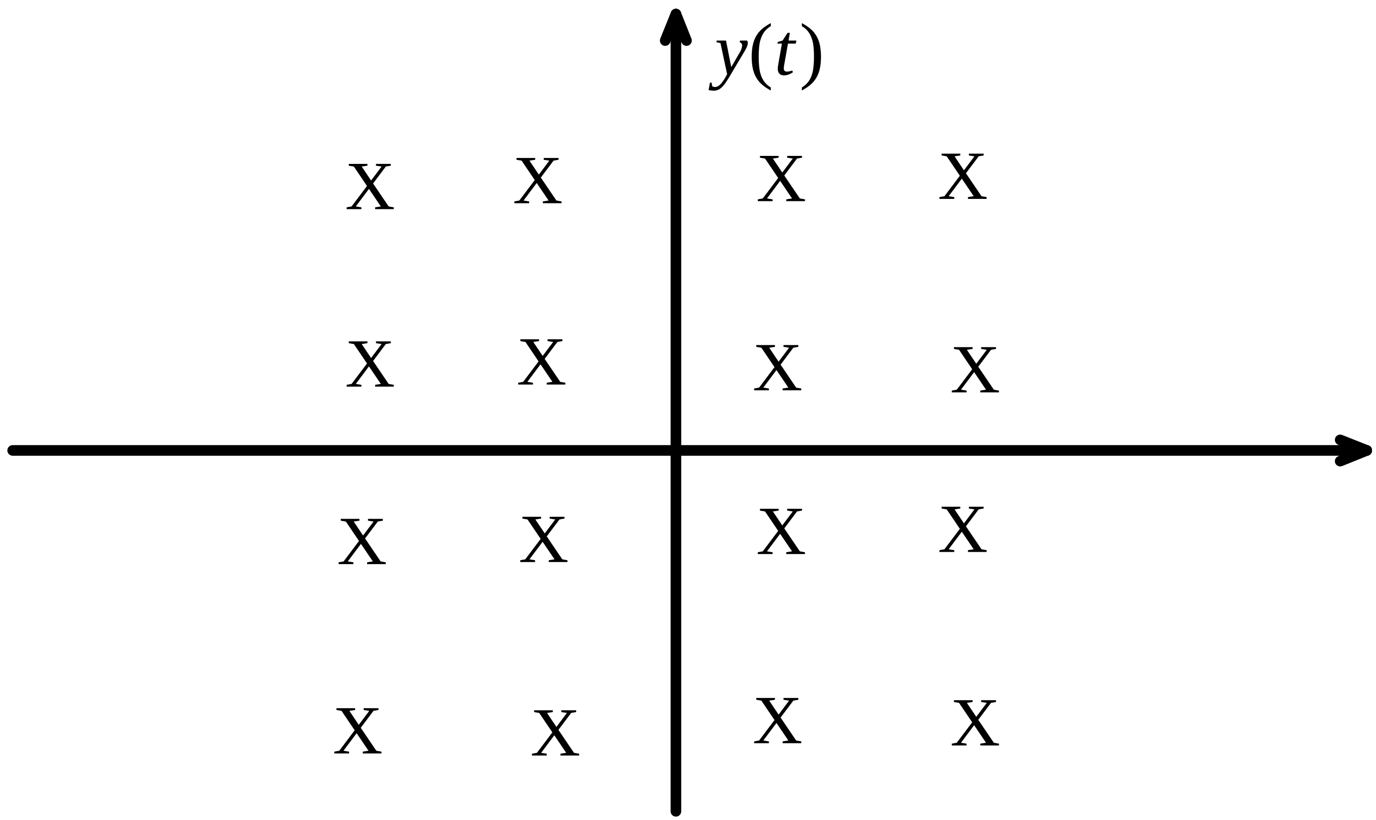
Example of Signal Constellation Diagram: QPSK

- $x(t) \in \{\pm 1\}, y(t) \in \{\pm 1\}$



Example of Signal Constellation Diagram: QAM

- $$x(t) \in \{-3,-1,+1,+3\},\ y(t) \in \{-3,-1,+1,+3\}$$



Interpretation of Signal Constellation Diagram

- Axis are labeled with $x(t)$ and $y(t)$
- Possible signals are plotted as points
- Signal power is proportional to distance from origin
- Probability of mistaking one signal for another is related to the distance between signal points
- Decisions are made on the received signal based on the distance to signal points in constellation

A New Way of Viewing Modulation

- The I/Q representation of modulation is very convenient for some modulation types.
- We will examine an even more general way of looking at modulation using signal spaces.
- By choosing an appropriate set of axis for our signal constellation, we will be able to:
  - Design modulation types which have desirable properties
  - Construct optimal receivers for a given type of modulation
  - Analyze the performance of modulation types using very general techniques.

Vector Spaces

- An n-dimensional vector $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ consists of n scalar components $\{v_1, v_2, \ldots, v_n\}$
- The norm (length) of a vector $\mathbf{v}$ is given by:

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} v_i^{\,2}}$$

- The inner product of two vectors $\mathbf{v}_1 = [v_{11}, v_{12}, \ldots, v_{1n}]$ and $\mathbf{v}_2 = [v_{21}, v_{22}, \ldots, v_{2n}]$ is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}$$

Basis Vectors

- A vector $\mathbf{v}$ may be expressed as a linear combination of it's basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$:

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$$

  where $v_i = \mathbf{e}_i \cdot \mathbf{v}$

- Think of the basis vectors as a coordinate system (x-y-z... axis) for describing the vector $\mathbf{v}$
- What makes a good choice of coordinate system?

A Complete Orthornomal Basis

- The set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ should be complete or span the vector space $\Re^n$. Any vector can be expressed as $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$ for some $\{v_i\}$
- Each basis vector should be orthogonal to all others:

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, \forall i \neq j$$

- Each basis vector should be normalized: $\|\mathbf{e}_i\| = 1, \forall i$
- A set of basis vectors which satisfies these three properties is said to be a complete orthonormal basis.

Signal Spaces

Signals can be treated in much the same way as vectors.

- The norm of a signal $x(t), t \in [a,b]$ is given by:

$$\|x(t)\| = \left(\int_a^b |x(t)|^2 dt \right)^{1/2} = \sqrt{E_x}$$

- The inner product of signals $x_1(t)$ and $x_2(t)$ is:

$$\langle x_1(t), x_2(t) \rangle = \int_a^b x_1(t) x_2^*(t) dt$$

- Signals can be represented as the sum of basis functions: $x(t) = \sum_{i=1}^{n} x_k f_k(t), \qquad x_k = \langle x(t), f_k(t) \rangle$

Basis Functions for a Signal Set

- One of *M* signals is transmitted: $\{s_1(t),\ldots,s_M(t)\}$
- The functions $\{f_1(t),\ldots,f_K(t)\}$ ($K \le M$) form a complete orthonormal basis for the signal set if
  - Any signal can be described by a linear combination:

$$s_i(t) = \sum_{k=1}^{K} s_{i,k} f_k(t), i = 1,\ldots, M$$

  - The basis functions are orthogonal to each other:

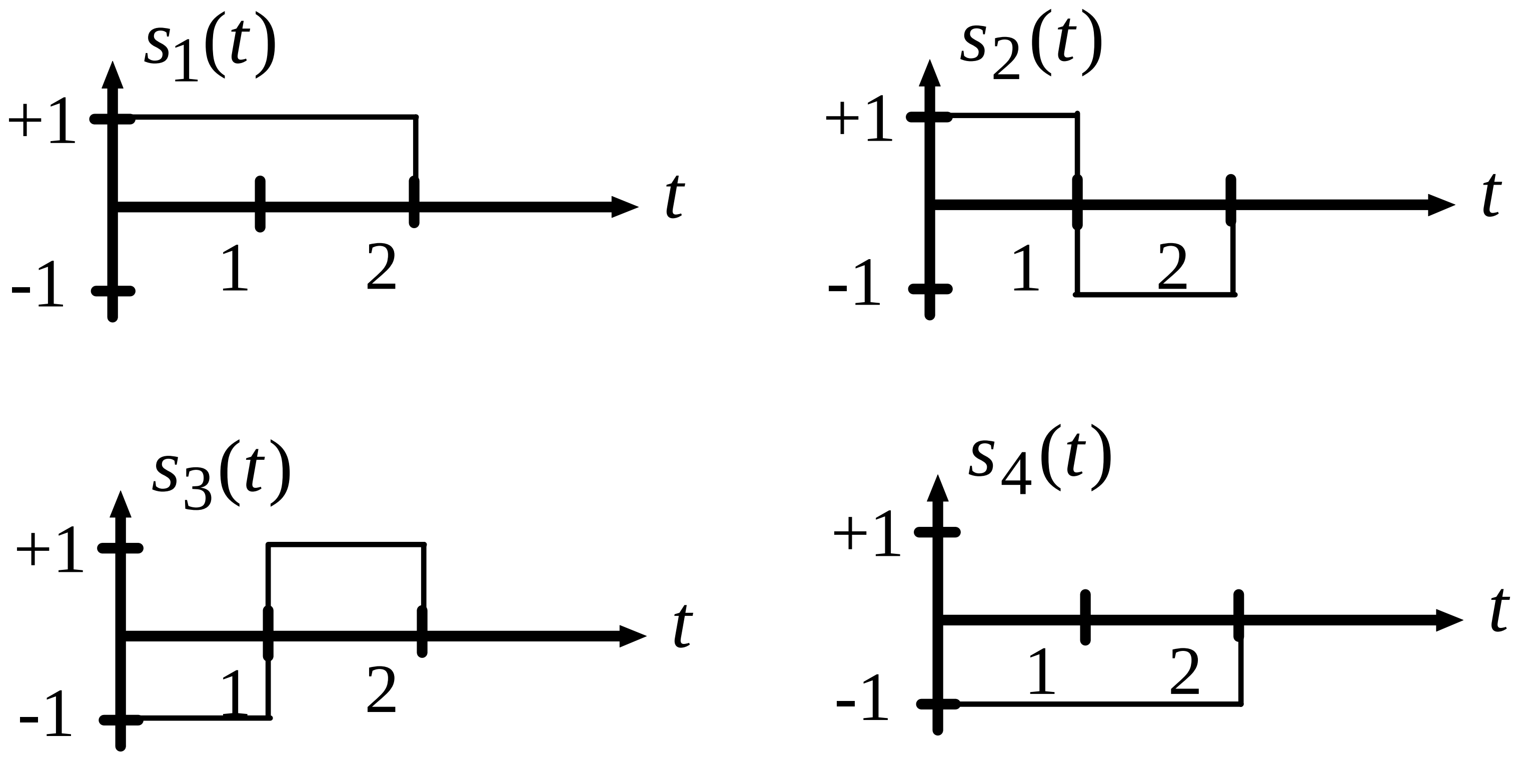
$$\int_a^b f_i(t) f_j^*(t)dt = 0, \forall i \ne j$$

  - The basis functions are normalized:

$$\int_a^b \left| f_k(t)\right|^2 dt = 1, \forall k$$

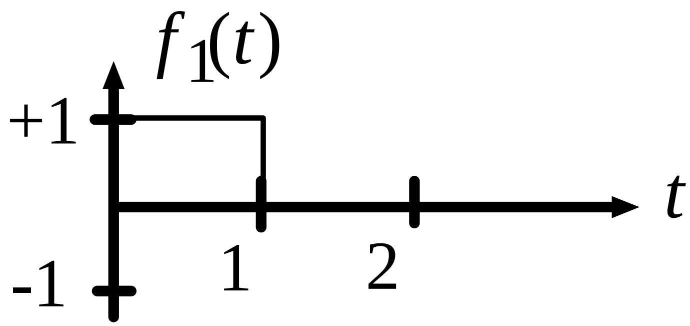
Example of Signal Space

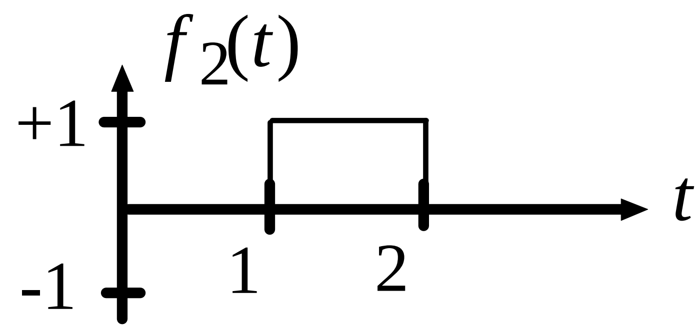
Consider the following signal signal set:



Example of Signal Space (continued)

- We can express each of the signals in terms of the following basis functions:





$$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$$

$$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$$

- Therefore the basis is complete

Example of Signal Space (continued)

- The basis is orthogonal:

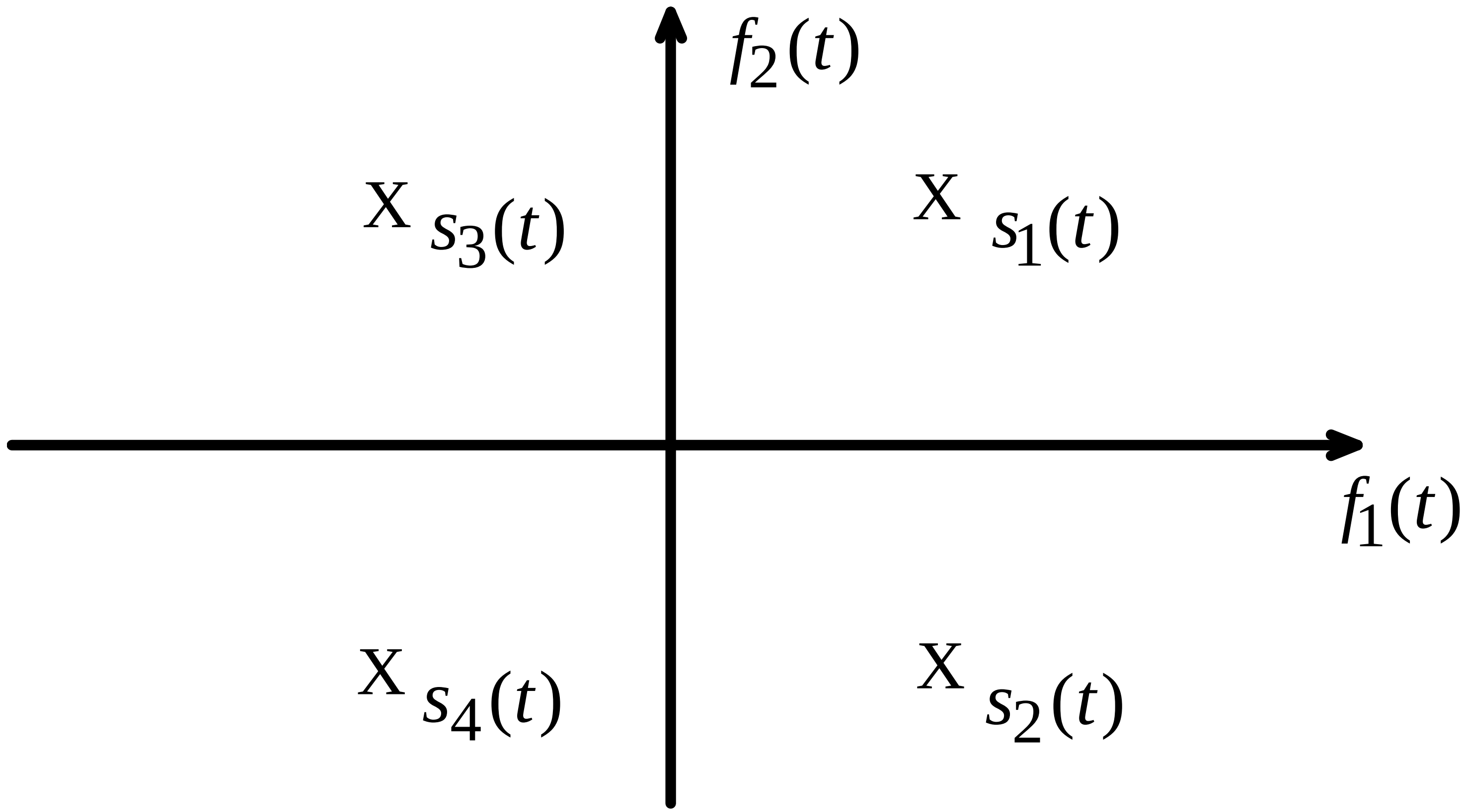
$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = 0$$

- The basis is normalized:

$$\int_{-\infty}^{\infty} \left| f_1(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| f_2(t) \right|^2 dt = 1$$

Signal Constellation for Example

- We've seen this signal constellation before



Notes on Signal Spaces

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In both of these cases we were fortunate enough to guess the correct basis functions.
- Is there a general method to find a complete orthonormal basis for an arbitrary signal set?
  - ***Gram-Schmidt Procedure***