Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:
  - Finite impulse response (FIR) transfer function
  - Infinite impulse response (IIR) transfer function

Types of Transfer Functions

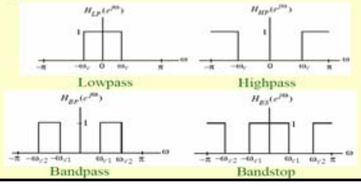
- In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications
- (1) Classification based on the shape of the magnitude function $|H(e^{j\omega})|$
- (2) Classification based on the the form of the phase function $\theta(\omega)$

Classification Based on Magnitude Characteristics

- One common classification is based on an ideal magnitude response
- A digital filter designed to pass signal components of certain frequencies without distortion should have a frequency response equal to one at these frequencies, and should have a frequency response equal to zero at all other frequencies

Ideal Filters

- The range of frequencies where the frequency response takes the value of one is called the **passband**
- The range of frequencies where the frequency response takes the value of zero is called the **stopband**
- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:



Ideal Filters

- Lowpass filter: Passband - $0 \le \omega \le \omega_c$
  Stopband - $\omega_c < \omega \le \pi$
- Highpass filter: Passband - $\omega_c \le \omega \le \pi$
  Stopband - $0 \le \omega < \omega_c$
- Bandpass filter: Passband - $\omega_{c1} \le \omega \le \omega_{c2}$
  Stopband - $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi$
- Bandstop filter: Stopband - $\omega_{c1} < \omega < \omega_{c2}$
  Passband - $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$

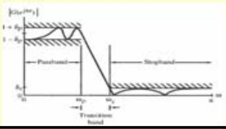
- The frequencies ω_c, ω_{c1}, and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere

- Earlier in the course we derived the inverse DTFT of the frequency response $\boxed{H_{LP}(e^{j\omega})}$ of the ideal lowpass filter:

$$\boxed{h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty}$$

- We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable
- Also, $h_{LP}[n]$ is not causal and is of doubly infinite length
- The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable
- Thus, the ideal filters with the ideal "brick wall" frequency responses cannot be realized with finite dimensional LTI filter

- To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a **transition band** between the passband and the stopband
- This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband
- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband
- Typical magnitude response specifications of a lowpass filter are shown below



Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a **bounded real** (BR) **transfer function** if

$$\boxed{|H(e^{j\omega})| \le 1 \quad \text{for all values of } \omega}$$

- Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs

- Then the condition $|H(e^{j\omega})| \le 1$ implies that

$$\boxed{\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2}$$

- Integrating the above from $-\pi$ to π, and applying Parseval's relation we get

$$\boxed{\sum_{n=-\infty}^{\infty} |y[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2}$$

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a **passive structure**
- If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a **lossless system**

- Example – Consider the causal stable IIR transfer function

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

where K is a real constant

- Its square-magnitude function is given by

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

- The maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha\cos\omega$ in the denominator is a maximum and the minimum value is obtained when $2\alpha\cos\omega$ is a minimum
- For $\alpha > 0$, maximum value of $2\alpha\cos\omega$ is equal to 2α at $\omega = 0$, and minimum value is -2α at $\omega = \pi$

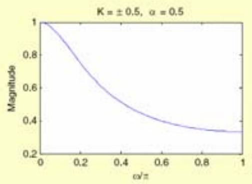
- Thus, for $\alpha > 0$, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$
- On the other hand, for $\alpha < 0$, the maximum value of $2\alpha\cos\omega$ is equal to -2α at $\omega = \pi$ and the minimum value is equal to 2α at $\omega = 0$
- Here, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = \pi$ and the minimum value is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$
- Hence, the maximum value can be made equal to 1 by choosing $K = \pm(1-\alpha)$, in which case the minimum value becomes $(1-\alpha)^2/(1+\alpha)^2$

- Hence,

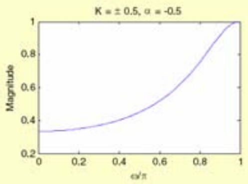
$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

is a BR function for $K = \pm(1-\alpha)$

- Plots of the magnitude function for $\alpha = \pm 0.5$ with values of K chosen to make $H(z)$ a BR function are shown on the next slide



Lowpass Filter



Highpass Filter

Allpass Transfer Function

Definition

- An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|A(e^{j\omega})|^2 = 1, \quad \text{for all } \omega$$

  is called an **allpass transfer function**
- An M-th order causal real-coefficient allpass transfer function is of the form

$$\boxed{A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}}}$$

- If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:
  $D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}$
  then it follows that $A_M(z)$ can be written as:

$$\boxed{A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}}$$

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r}e^{-j\phi}$
- The numerator of a real-coefficient allpass transfer function is said to be the **mirror-image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-M polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z^{-1})$$

- The expression

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

  implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** in the z-plane

$$\boxed{A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}}$$

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- To show that $|A_M(e^{j\omega})| = 1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$

- Hence

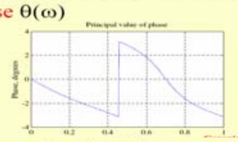
$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

- Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle

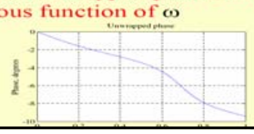
- Figure below shows the principal value of the phase of the 3rd-order allpass function

$$A_3(z) = \frac{-0.2+0.18z^{-1}+0.4z^{-2}+z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

- Note the discontinuity by the amount of 2π in the phase $\theta(\omega)$



- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below
- Note: The unwrapped phase function is a continuous function of ω



- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω

Properties

- (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure
- (2) The magnitude function of a stable allpass function $A(z)$ satisfies:

$$|A(z)| \begin{cases} <1, & \text{for } |z|>1 \\ =1, & \text{for } |z|=1 \\ >1, & \text{for } |z|<1 \end{cases}$$

- (3) Let $\tau(\omega)$ denote the group delay function of an allpass filter $A(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an M-th order stable real-coefficient allpass transfer function satisfies:

$$\int_0^{\pi} \tau(\omega) d\omega = M\pi$$

Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

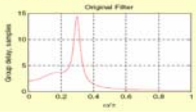
$$\longrightarrow \boxed{G(z)} \longrightarrow \boxed{A(z)} \longrightarrow$$

- Since $|A(e^{j\omega})| = 1$, we have

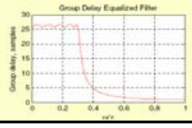
$$|G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $A(z)$

- Example – Figure below shows the group delay of a 4th order elliptic filter with the following specifications: $\omega_p = 0.3\pi$, $\delta_p = 1$ dB, $\delta_s = 35$ dB



- Figure below shows the group delay of the original elliptic filter cascaded with an 8th order allpass section designed to equalize the group delay in the passband



(2) Classification based on the the form of the phase function $\theta(\omega)$

- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

Zero-Phase Transfer Function

- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is not possible to design a causal digital filter with a zero phase

- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement
- One zero-phase filtering scheme is sketched below

$$x[n] \rightarrow \boxed{H(z)} \rightarrow v[n] \qquad u[n] \rightarrow \boxed{H(z)} \rightarrow w[n]$$

$$u[n] = v[-n], \qquad y[n] = w[-n]$$

- It is easy to verify the above scheme in the frequency domain
- Let $X(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively
- From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide

$$x[n] \rightarrow \boxed{H(z)} \rightarrow v[n] \qquad u[n] \rightarrow \boxed{H(z)} \rightarrow w[n]$$

$$u[n] = v[-n], \qquad y[n] = w[-n]$$

$$V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(e^{j\omega}), \qquad Y(e^{j\omega}) = W^*(e^{j\omega})$$

- Combining the above equations we get

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega})$$

$$= H^*(e^{j\omega})V(e^{j\omega}) = H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega})$$

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$$= \left|H(e^{j\omega})\right|^2 X(e^{j\omega})$$

- The function `filtfilt` implements the above zero-phase filtering scheme
- In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a **linear-phase** characteristic in the frequency band of interest

- The most general type of a filter with a linear phase has a frequency response given by

$$\boxed{H(e^{j\omega}) = e^{-j\omega D}}$$

which has a linear phase from ω = 0 to ω = 2π

- Note also $\boxed{\begin{array}{c} |H(e^{j\omega})| = 1 \\ \tau(\omega) = D \end{array}}$

Linear-Phase Transfer Function

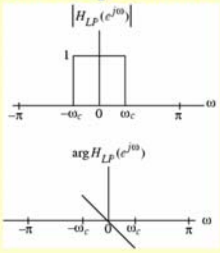
- The output $y[n]$ of this filter to an input $x[n] = Ae^{j\omega n}$ is then given by

$$\boxed{y[n] = Ae^{-j\omega D}e^{j\omega n} = Ae^{j\omega(n-D)}}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D

- If D is an integer, then $y[n]$ is identical to $x[n]$, but delayed by D samples
- If D is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed D units of time
- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest

- Figure below shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband



- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape
- Example - Determine the impulse response of an ideal lowpass filter with a linear phase response:

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_o}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_o)}{\pi (n - n_o)}, \quad -\infty < n < \infty$$

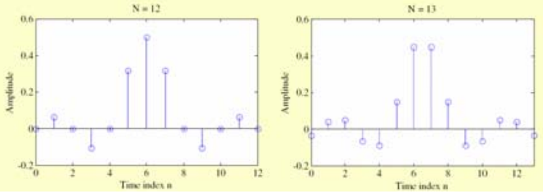
- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable
- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed
- The truncated approximation may or may not exhibit linear phase, depending on the value of n_o chosen

- If we choose $n_o = N/2$ with N a positive integer, the truncated and shifted approximation

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi (n - N/2)}, \quad 0 \le n \le N$$

will be a length N+1 causal linear-phase FIR filter

- Figure below shows the filter coefficients obtained using the function `sinc` for two different values of N



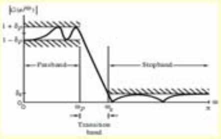
Chapter 9

Digital Filter Design

- Objective - Determination of a realizable transfer function $G(z)$ approximating a given frequency response specification is an important step in the development of a digital filter
- If an IIR filter is desired, $G(z)$ should be a stable real rational function
- Digital filter design is the process of deriving the transfer function $G(z)$

Digital Filter Specifications

- For example, the magnitude response $|G(e^{j\omega})|$ of a digital lowpass filter may be given as indicated below



- As indicated in the figure, in the **passband**, defined by $0 \le \omega \le \omega_p$, we require that $|G(e^{j\omega})| \cong 1$ with an error $\pm\delta_p$, i.e.,

$$1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- In the **stopband**, defined by $\omega_s \le \omega \le \pi$, we require that $|G(e^{j\omega})| \cong 0$ with an error δ_s, i.e.,

$$|G(e^{j\omega})| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

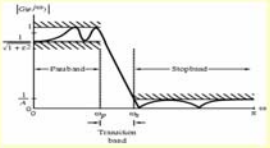
- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- Since $G(e^{j\omega})$ is a periodic function of ω, and $|G(e^{j\omega})|$ of a real-coefficient digital filter is an even function of ω
- As a result, filter specifications are given only for the frequency range $0 \le \omega \le \pi$
- Specifications are often given in terms of **loss function** $\mathcal{A}(\omega) = -20\log_{10}|G(e^{j\omega})|$ in dB
- **Peak passband ripple**

$$\alpha_p = -20\log_{10}(1-\delta_p) \ \text{dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s) \ \text{dB}$$

- Magnitude specifications may alternately be given in a normalized form as indicated below



- Here, the maximum value of the magnitude in the passband is assumed to be unity
- $1/\sqrt{1+\varepsilon^2}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband
- $\frac{1}{A}$ - Maximum stopband magnitude

- For the normalized specification, maximum value of the gain function or the minimum value of the loss function is 0 dB
- **Maximum passband attenuation** -
$$\alpha_{\max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right) \text{ dB}$$
- For $\delta_p << 1$, it can be shown that
$$\alpha_{\max} \cong -20\log_{10}(1-2\delta_p) \text{dB}$$
- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using
$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$
$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

Digital Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then
$$\omega_p = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$
$$\omega_s = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

Selection of Filter Type

- The transfer function $H(z)$ meeting the frequency response specifications should be a causal transfer function
- For IIR digital filter design, the IIR transfer function is a real rational function of z^{-1}:

$$\boxed{H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}, \quad M \le N}$$

- $H(z)$ must be a stable transfer function and must be of lowest order N for reduced computational complexity

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- For FIR digital filter design, the FIR transfer function is a polynomial in z^{-1} with real coefficients:

$$\boxed{H(z) = \sum_{n=0}^{N} h[n] z^{-n}}$$

- For reduced computational complexity, degree N of $H(z)$ must be as small as possible
- If a linear phase is desired, the filter coefficients must satisfy the constraint:

$$\boxed{h[n] = \pm h[N-n]}$$

- Advantages in using an FIR filter -
  (1) Can be designed with exact linear phase,
  (2) Filter structure always stable with quantized coefficients
- Disadvantages in using an FIR filter - Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

Digital Filter Design: Basic Approaches

- Most common approach to IIR filter design - (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications
- (2) Determine the analog lowpass filter transfer function $H_a(s)$
- (3) Transform $H_a(s)$ into the desired digital transfer function $G(z)$
- This approach has been widely used for the following reasons:
  (1) Analog approximation techniques are highly advanced
  (2) They usually yield closed-form solutions
  (3) Extensive tables are available for analog filter design
  (4) Many applications require digital simulation of analog systems

- An analog transfer function to be denoted as

$$\boxed{H_a(s) = \frac{P_a(s)}{D_a(s)}}$$

where the subscript "a" specifically indicates the analog domain

- A digital transfer function derived from $H_a(s)$ shall be denoted as

$$\boxed{G(z) = \frac{P(z)}{D(z)}}$$

- Basic idea behind the conversion of $H_a(s)$ into $G(z)$ is to apply a mapping from the s-domain to the z-domain so that essential properties of the analog frequency response are preserved
- Thus mapping function should be such that
  - Imaginary ($j\Omega$) axis in the s-plane be mapped onto the unit circle of the z-plane
  - A stable analog transfer function be mapped into a stable digital transfer function

Digital Filter Design: Basic Approaches

- Three commonly used approaches to FIR filter design -

(1) Windowed Fourier series approach

(2) Frequency sampling approach

(3) Computer-based optimization methods

Chapter 6 Home work:

5, 8, 20, 26, 38, 40 + book examples