Chapter (3)

Continuous time Fourier Transform

Discrete time Fourier Transform

In Section 2.2.3, we pointed out that any arbitrary sequence can be represented in the time domain as a weighted linear combination of delayed unit sample sequences $\{\delta[n-k]\}$. An important consequence of this representation, derived in Section 2.5.1, is the input–output characterization of a linear, time-invariant (LTI) digital filter in the time domain by means of the convolution sum describing the output sequence in terms of a weighted linear combination of its delayed impulse responses. In many applications, it is convenient to consider an alternate description of a sequence in terms of complex exponential sequences of the form $\{e^{-j\omega n}\}$, where ω is the normalized frequency variable in radians. This leads to a particularly useful representation of discrete-time sequences and LTI discrete-time systems in the frequency domain.[1]

discrete-time sequences and LTI discrete-time systems in the frequency domain.

The frequency-domain representation of a discrete-time sequence discussed in this chapter is the discrete-time Fourier transform by which a time-domain sequence is mapped into a continuous function of the frequency variable ω. Because of the periodicity of the discrete-time Fourier transform, the corresponding discrete-time sequence can be simply obtained by computing its Fourier series representation. Since the representation is in terms of an infinite series, existence of the discrete-time Fourier transform is examined along with its properties.

The frequency-domain representation of a linear, time-invariant (LTI) discrete-time system is its frequency response given by the discrete-time Fourier transform of its impulse response.

The Continuous-Time Fourier Transform

Probably the easiest way to introduce the discrete-time Fourier transform (DTFT) is through its counterpart, the continuous-time Fourier transform (CTFT). As you will recall, the CTFT and its inverse can be expressed as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}dt \tag{1}$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{2}$$

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega).$$

Note that we use the upper case variable Ω to indicate continuous-time angular frequency in this course. As we have discused, the first equation (which is actually the inverse CTFT) expresses the fact that a finite-energy time function $x(t)$ can be represented as a weighted linear combination of complex exponentials $e^{j\Omega t}$. The second equation (the actual CTFT) tells you how to compute the weights $X(j\Omega)$. Note that the frequencies Ω include all real numbers.

general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in polar form as

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)},$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}.$$

The quantity $|X_a(j\Omega)|$ is called the *magnitude spectrum*, and the quantity $\theta_a(\Omega)$ is called the *phase spectrum*, with both spectrums being real functions of Ω.

Condition for CTFT

In general, the CTFT $X_a(j\Omega)$ defined by Eq. (3.1) exists if the continuous-time signal $x_a(t)$ satisfies the *Dirichlet conditions:*

(a) The signal has a finite number of finite discontinuities and a finite number of maxima and minima in any finite interval.

(b) The signal is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |x_a(t)|dt < \infty. \tag{3.3}$$

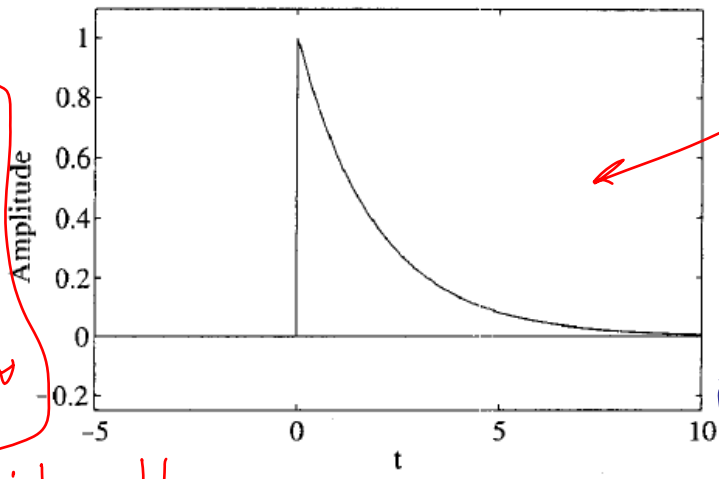
We illustrate the CTFT computation in Example 3.1.

$$x_a(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |x_a(t)|\,dt = \int_0^{\infty} e^{-\alpha t}\,dt$$

$$= -\frac{e^{-\alpha t}}{\alpha}\Big|_0^{\infty} = \frac{1}{\alpha} < \infty$$

∴ $x_a(t)$ is absolutly integrable

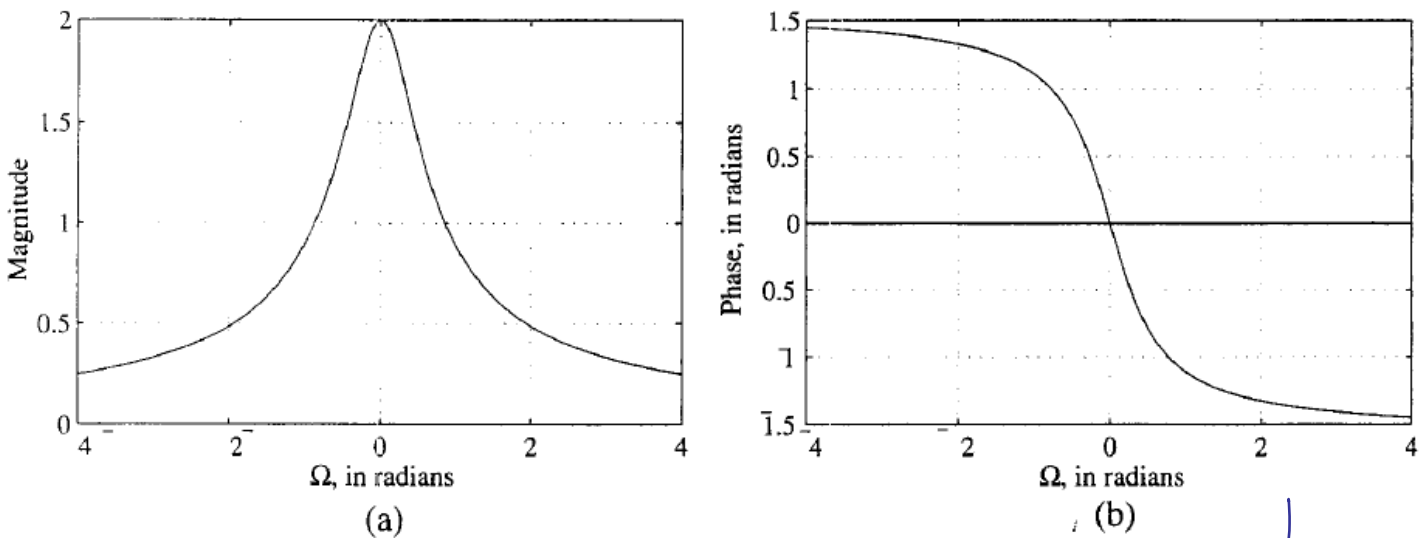


$$X_a(j\Omega) = \int_0^{\infty} e^{-\alpha t} e^{-j\Omega t}\,dt$$

$$= \frac{-1}{\alpha + j\Omega} e^{-(\alpha + j\Omega)t}\Big|_0^{\infty}$$

$$X_a(j\Omega) = \frac{1}{\alpha + j\Omega}$$

Figure 3.1: Plot of the continuous-time function of Eq. (3.4) for $\alpha = 0.5$.



$$\frac{1}{\sqrt{\alpha^2 + \Omega^2}} \qquad e^{-j\tan^{-1}\left(\frac{\Omega}{\alpha}\right)}$$

Figure 3.2: (a) Magnitude and (b) phase of $X_a(j\Omega) = 1/(0.5 + j\Omega)$.

3.1.2 Energy Density Spectrum

The total energy $\mathcal{E}_x$ of a finite-energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) \, dt. \tag{3.7}$$

The energy can also be expressed in terms of the CTFT $X_a(j\Omega)$. To this end, we first replace $x_a^*(t)$ in the above equation by its inverse CTFT expression as shown below:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \, d\Omega \right] dt.$$

Interchanging the order of the integrations in the above, we get

$$\begin{aligned} \mathcal{E}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} \, dt \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) \, d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega. \end{aligned} \tag{3.8}$$

Combining Eqs. (3.7) and (3.8), we arrive at

$$\int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega, \tag{3.9}$$

which is more commonly known as the *Parseval's relation* for finite-energy continuous-time signals.

The integrand $|X_a(j\Omega)|^2$ on the right-hand side of Eq. (3.8) is called the *energy density spectrum* of the continuous-time signal $x_a(t)$ and is usually denoted by the symbol $S_{xx}(\Omega)$; that is,

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2.$$

The energy $\mathcal{E}_{x,r}$ over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ of the signal $x_a(t)$ can be computed by integrating $S_{xx}(\Omega)$ over this range:

$$\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\, d\Omega.$$

3.1.3 Band-limited Continuous-Time Signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$, whereas a *band-limited* continuous-time signal has a spectrum that is limited to a portion of the above frequency range. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$; that is,

$$X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_a, \\ 0, & \Omega_b < |\Omega| < \infty. \end{cases}$$

However, an ideal band-limited signal cannot be generated in practice, and, for practical purposes, it is sufficient to ensure that for a band-limited signal outside the specified frequency range, the signal energy is arbitrarily small.

Band-limited signals are classified according to the frequency range where most of the signal's energy is concentrated. A *lowpass* continuous-time signal has a spectrum occupying the frequency range $0 \leq |\Omega| \leq \Omega_p < \infty$, where Ω_p is called the *bandwidth* of the signal. Likewise, a *highpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \leq |\Omega| < \infty$, where the *bandwidth* of the signal is from Ω_p to ∞. Finally, a *bandpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is its *bandwidth*. A precise definition of the bandwidth depends on applications. As can be seen from Figure 3.2(a), the continuous-time signal of Eq. (3.4) is a lowpass signal. It can be shown that 80% of the energy of this signal is contained in the frequency range $0 \leq |\Omega| \leq 0.4898\pi$, and hence, we can define the 80% bandwidth of the signal to be 0.4898π radians (Problem 3.10).

3.1.4 The Frequency Response of an LTI Continuous-Time System

As indicated by Eq. (1.2), the output response $y_a(t)$ of an initially relaxed linear, time-invariant continuous-time system characterized by an impulse response $h_a(t)$ for an input signal $x_a(t)$ is given by the convolution integral

$$y_a(t) = \int_{-\infty}^{\infty} h_a(t-\tau)x_a(\tau)d\tau. \tag{3.10}$$

$y_a(t) = h_a(t) \circledast x_a(t)$

LTI

$y_n = x_n \circledast h_n$

$y_n = \sum_{k=-\infty}^{\infty} h_n x_{n-k}$

discrete

Applying the CTFT to both sides of Eq. (3.10), we have

$$Y_a(j\Omega) = H_a(j\Omega)X_a(j\Omega), \tag{3.11}$$

where $Y_a(j\Omega)$, $X_a(j\Omega)$ and $H_a(j\Omega)$ are, respectively, the CTFTs of $y_a(t)$, $x_a(t)$, and $h_a(t)$. The function $H_a(j\Omega)$ is called the *frequency response* of the LTI continuous-time system.

3.2 The Discrete-Time Fourier Transform

The *discrete-time Fourier transform* (DTFT) of a discrete-time sequence $x[n]$ is a representation of the sequence in terms of the complex exponential sequence $\{e^{j\omega n}\}$, where ω is the real frequency variable. If there is no ambiguity, for brevity often the discrete-time Fourier transform is called simply the *Fourier transform* (FT). The Fourier transform representation of a sequence, if it exists, is unique, and the original sequence can be computed from its transform representation by an inverse transform operation. We first define the forward transform and derive its inverse transform. We then describe the condition for its existence and review its important properties.

The *discrete-time Fourier transform* $X(e^{j\omega})$ of a sequence $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

$\mathcal{F}\{x[n]\}$

the *analysis equation,*

$\mathcal{F}^{-1}\{X(e^{j\omega})\}$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{3.16}$$

synthesis equation,

called the *inverse discrete-time Fourier transform.* It should be noted that even though the integration in Eq. (3.16) can be carried out over any interval of duration 2π, it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.16) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π.

$\mathcal{F}\{x[n]\}$

to denote the $X(e^{j\omega})$ of the sequence $x[n]$. Likewise, we shall use the operator symbol

$\mathcal{F}^{-1}\{X(e^{j\omega})\}$

to denote the inverse Fourier transform $x[n]$ of the transform $X(e^{j\omega})$. A discrete-time Fourier transform pair will be denoted as

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}). \tag{3.17}$$

Example (1)

$x_n = \delta_n \qquad X(e^{j\omega}) = ??$

$\delta_0 = 1$

$\delta_n = 0 \quad n \neq 0$

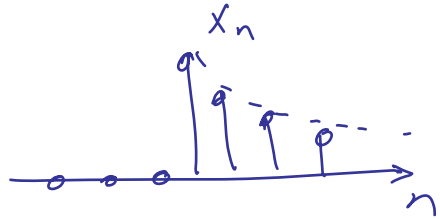
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta_n e^{-j\omega n} = 1$$

$$\therefore \quad \delta_n \overset{\text{DTFT}}{\rightleftharpoons} 1$$

Example (2)

$x_n = \alpha^n u(n) \qquad |\alpha| < 1$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$\sum_{k=0}^{\infty} d^k = \frac{1}{1-d},$$

with $0 < |d| < 1$.

As can be seen from the definition, the discrete-time Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω. However, unlike the continuous-time Fourier transform, it is a periodic function in ω with a period 2π. To verify this latter property, observe that for any integer k,

periodicity property of the transform.

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}), \qquad \text{for all values of } k,$$

where we have used the fact $e^{-j2\pi kn} = 1$. It therefore follows that Eq. (3.12) represents the Fourier series expansion of the periodic function $X(e^{j\omega})$. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral given by

IDFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{3.16}$$

called the *inverse discrete-time Fourier transform*. It should be noted that even though the integration in Eq. (3.16) can be carried out over any interval of duration 2π, it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.16) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π.

Proof of IDFT equation

To verify that the integral on the right-hand side of Eq. (3.16) indeed results in the inverse FT $x[n]$, we substitute the expression for $X(e^{j\omega})$ from Eq. (3.12) in Eq. (3.16), arriving at

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\omega\ell}\right) e^{j\omega n} d\omega.$$

The order of integration and the summation on the right-hand side of the above equation can be interchanged if the summation inside the brackets converges uniformly, that is, if $X(e^{j\omega})$ exists. Under this condition, we get from the above

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)} d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\,\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\,\mathrm{sinc}(n-\ell).$$

For $n \neq \ell$, $\sin\pi(n-\ell) = 0$, and as a result, $\mathrm{sinc}(n-\ell) = 0$. For $n = \ell$, $\mathrm{sinc}(n-\ell) = 0/0$. In this case, using L'Hospital rule, we get

$$\lim_{n\to\ell}\frac{\sin\pi(n-\ell)}{\pi(n-\ell)} = \frac{\pi\cos(\pi n)}{\pi} = 1.$$

Thus,

$$\mathrm{sinc}(n-\ell) = \begin{cases} 1, & n = \ell, \\ 0, & n \neq \ell, \end{cases}$$

$$= \delta[n-\ell]. \qquad (3.18)$$

Hence,

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\,\mathrm{sinc}(n-\ell) = \sum_{\ell=-\infty}^{\infty} x[\ell]\,\delta[n-\ell] = x[n],$$

using the sampling property of the unit sample sequence.

3.2.2 Basic Properties

$X(e^{j\omega})$ is a complex function of the real variable ω

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}),$$

$$\begin{cases} X_{re}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}, \\ X_{im}(e^{j\omega}) = \frac{1}{2j}\{X(e^{j\omega}) - X^*(e^{j\omega})\}, \end{cases}$$

the polar form

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j[\theta(\omega)+2\pi k]} = |X(e^{j\omega})|e^{j\theta(\omega)},$$

magnitude function *phase function*

the *Fourier spectrum* and likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are referred to as the *magnitude spectrum* and *phase spectrum,* respectively.

The relations between the rectangular and polar forms

$$X_{re}(e^{j\omega}) = |X(e^{j\omega})| \cos\ \theta(\omega),$$

$$X_{im}(e^{j\omega}) = |X(e^{j\omega})| \sin\ \theta(\omega),$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega}) = X_{re}^2(e^{j\omega}) + X_{im}^2(e^{j\omega}),$$

$$\tan\theta(\omega) = \frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})}.$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)},$$

$$\theta(\omega) = \arg\{X(e^{j\omega})\}.$$

3.2.3 Symmetry Relations

$$x[n] = x_{\text{re}}[n] + jx_{\text{im}}[n]$$

Table 3.1: Symmetry relations of the discrete-time Fourier transform of a complex sequence.

Sequence		Discrete-Time Fourier Transform
$x[n]$		$X(e^{j\omega})$
$x[-n]$	$\longleftrightarrow$	$X(e^{-j\omega})$
$x^*[-n]$	$\longleftrightarrow$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$		$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\text{Im}\{x[n]\}$		$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$		$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$		$jX_{\text{im}}(e^{j\omega})$

Note: $X_{\text{cs}}(e^{j\omega})$ and $X_{\text{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\text{cs}}[n]$ and $x_{\text{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

For real sequence, $x_{\text{im}}[n] = 0$. Hence, from Eq. (3.33), $X_{\text{ca}}(e^{j\omega}) = 0$, and thus, from Eq. (3.31b), we have $X(e^{j\omega}) = X^*(e^{j\omega})$, implying that $X(e^{j\omega})$ is a conjugate-symmetric function. As a result, the real part $X_{\text{re}}(e^{j\omega})$ and imaginary part $X_{\text{im}}(e^{j\omega})$ of the Fourier transform of a real sequence are, respectively, even and odd functions of ω.

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$
	$X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{\text{ev}}[n]$ and $x_{\text{od}}[n]$ denote the even and odd parts of $x[n]$, respectively.

3.2.4 Convergence Condition

Now, an infinite series of the form of Eq. (3.12) may or may not converge. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is said to exist if the series in Eq. (3.12) converges in some sense. Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}, \tag{3.42}$$

denote the partial sum of the weighted complex exponentials in Eq. (3.12). Then for *uniform convergence* of $X(e^{j\omega})$,

$$\lim_{K\to\infty} X_K(e^{j\omega}) = X(e^{j\omega}).$$

Now, if $x[n]$ is an *absolutely summable sequence*, that is, if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty, \tag{3.43}$$

then

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]|\,\left|e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

for all values of ω guaranteeing the existence of $X(e^{j\omega})$. Thus, Eq. (3.43) is a sufficient condition for the existence of the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$. Moreover, it can be shown that for an absolutely summable sequence, the infinite series of Eq. (3.12) defining the Fourier transform converges uniformly for all values of ω.

A large class of sequences encountered in practice are of finite length with finite sample values. These sequences are absolutely summable, and hence, their Fourier transforms converge uniformly. On the other hand, infinite length sequences may or may not converge uniformly. The sequence $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$, of Example 3.5 is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n|\mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty,$$

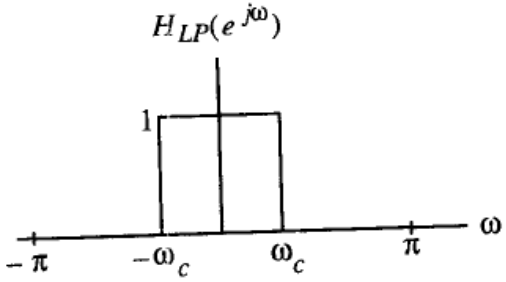


Figure 3.5: Plot of the Fourier transform of Eq. (3.45).

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0, \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0. \end{cases}$$

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad (-\infty < n < \infty)$$

The mean-square convergence property of the sequence $h_{LP}[n]$ discussed in Example 3.8 can be further illustrated by examining the plot of the function

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n},$$

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}, \tag{3.50}$$

for various values of K, as shown in Figure 3.6. It can be seen from this figure that, independent of the number of terms in the above sum, there are ripples in the plot of $H_{LP}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$. The number of ripples increases as K increases, with the height of the largest ripple remaining the same for all values of K. As K goes to infinity, the condition of Eq. (3.44) holds, indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$. The oscillatory behavior in the plot of $H_{LP,K}(e^{j\omega})$ approximating a Fourier transform $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity, as indicated in Figure 3.6, is commonly known as the *Gibbs phenomenon*. We shall return to this phenomenon in the design of FIR filters based on the windowed Fourier series discussed in Section 10.2.3.

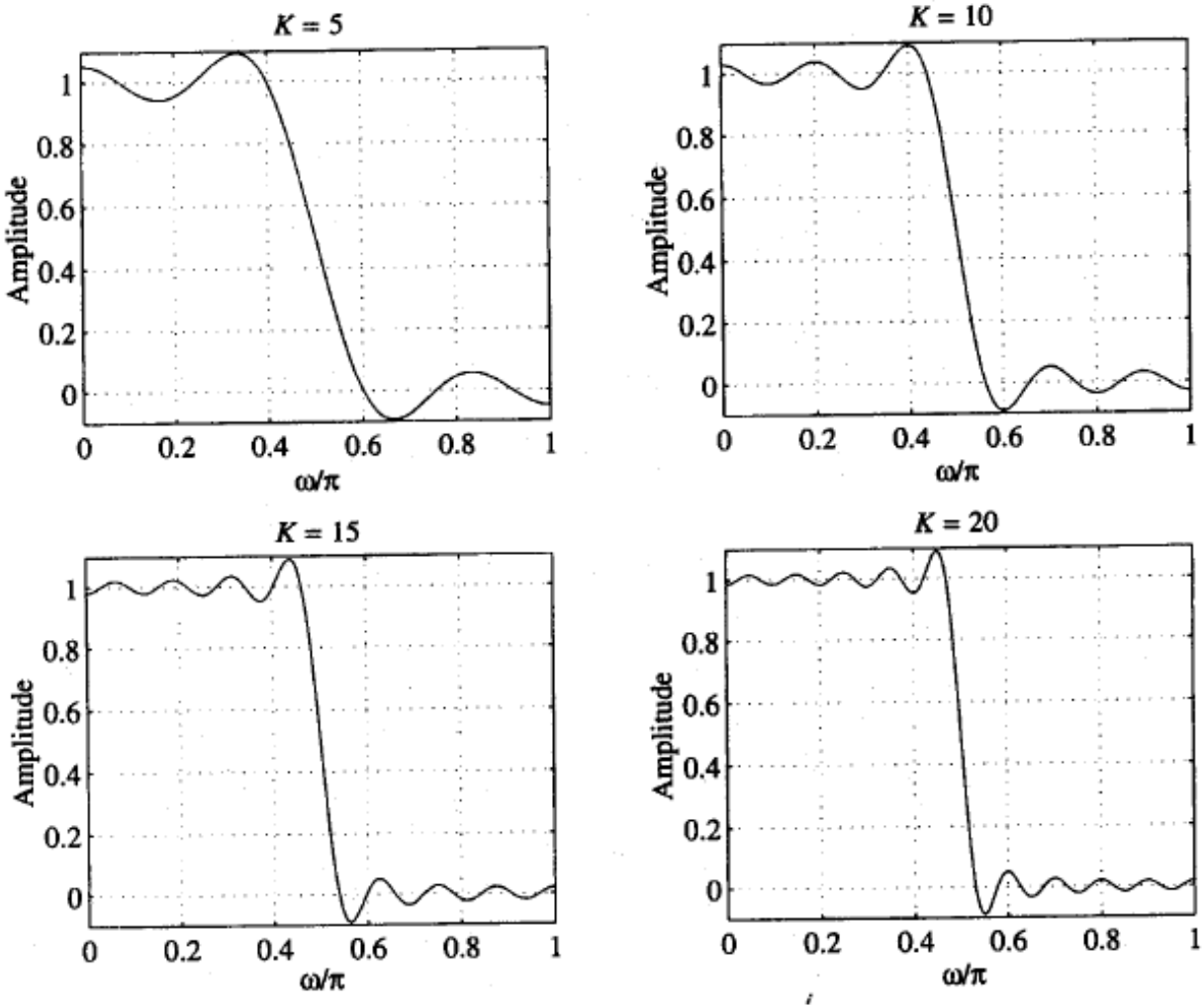


Figure 3.6: Frequency response plots of Eq. (3.51) for various values of K.

Existence of the Fourier Representations

Under what conditions do the Fourier representations exist?

- The DTFS and its inverse always exist for any discrete-time periodic signal (after all, they are just finite sums).
- However, the other Fourier representations involve infinite sums or integrals, so we need to be careful with *convergence*.
- As an example, consider the convergence properties of the DTFT.
- The DTFT converges uniformly if the signal is *absolutely summable*:

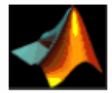
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

- On the other hand, the DTFT only converges in mean square if the signal is only *square summable*:

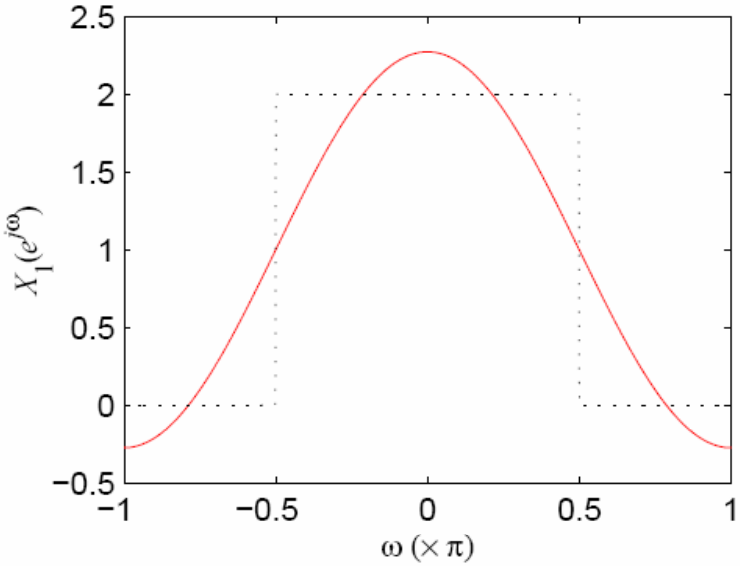
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

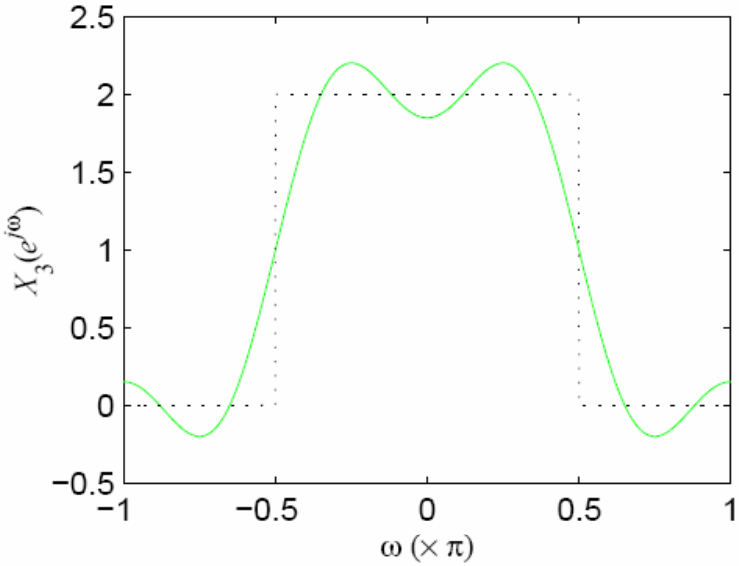
- This leads to Gibbs' effect, now demonstrated for a sinc pulse.

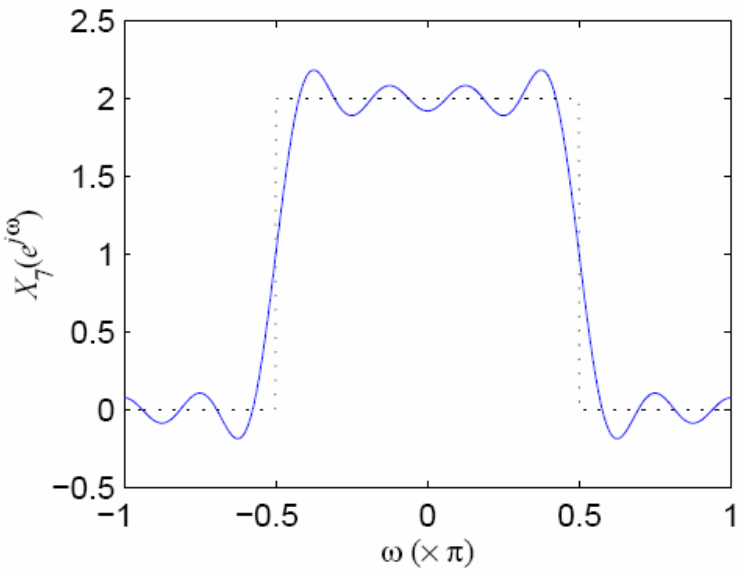
Gibbs' Effect



gibbs.m, gibbs2.m







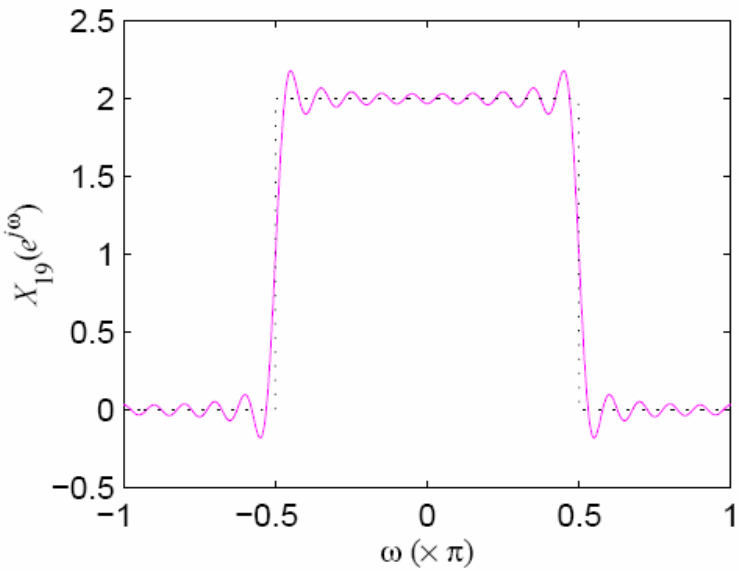


Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1-\alpha e^{-j\omega}}$
$(n+1)\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$

3.3 Discrete-Time Fourier Transform Theorems

There are a number of important theorems of the discrete-time Fourier transform that are useful in digital signal processing applications. These theorems can be used to determine the Fourier transforms of sequences obtained by combining sequences with known transforms. We review these theorems in this section. For compactness, the theorems are stated using the operator notation introduced in Eq. (3.17), and we shall make use of the following Fourier transform pairs:

$$g[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega}), \tag{3.56a}$$

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}). \tag{3.56b}$$

The proofs of most of the theorems given here are quite straightforward and are left as exercises.

Convolution Theorem

The Fourier transform $Y(e^{j\omega})$ of the convolution sum of two sequences, $y[n] = g[n] \circledast h[n]$, is given by the product of their Fourier transforms $G(e^{j\omega})H(e^{j\omega})$; that is,

$$g[n] \circledast h[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega})H(e^{j\omega}). \tag{3.65}$$

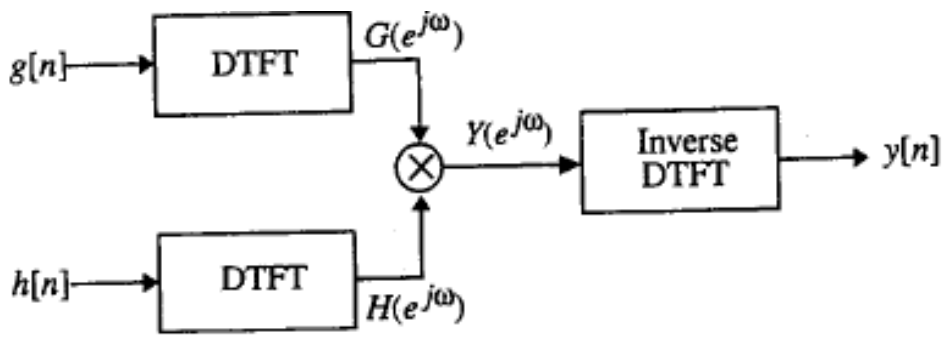


Figure 3.9: Linear convolution using the discrete-time Fourier transform.

Modulation Theorem

The Fourier transform $Y(e^{j\omega})$ of the product of two sequences given by $y[n] = g[n]h[n]$ is given by the convolution integral of their Fourier transforms $\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$; that is,

$$g[n]h[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta. \tag{3.66}$$

The modulation theorem plays a key role in the amplitude modulation scheme used in digital communications. The modulation theorem is also known as the *windowing theorem*. One application of this theorem, to be considered later in Section 10.2, is in the design of linear-phase FIR filter based on the windowing of the impulse response of an ideal linear-phase filter with a doubly-infinite impulse response.

Parseval's Relation

This theorem expresses the sum of sample-by-sample product of two complex sequences in terms of an integral of the product of their Fourier transforms. Specifically, the most general form of this theorem is given by

$$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega. \tag{3.67}$$

Linearity Theorem

Consider a sequence $x[n] = \alpha g[n] + \beta h[n]$ obtained by a linear combination of $g[n]$ and $h[n]$, where α and β are arbitrary constants. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is then given by $\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$; that is,

$$\alpha\, g[n] + \beta\, h[n] \overset{\mathcal{F}}{\longleftrightarrow} \alpha\, G(e^{j\omega}) + \beta H(e^{j\omega}). \tag{3.57}$$

It should be noted that the Fourier transforms of the conjugate-symmetric and conjugate-antisymmetric parts of a sequence derived in Eqs. (3.34) and (3.35) made use of the linearity theorem.

Time-Reversal Theorem

The Fourier transform of the time-reversed sequence $g[-n]$ is given by $G(e^{-j\omega})$; that is,

$$g[-n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{-j\omega}). \tag{3.58}$$

Time-Shifting Theorem

The Fourier transform of the delayed sequence $x[n] = g[n - n_o]$, with n_o an integer, is given by $X(e^{j\omega}) = e^{-j\omega n_o} G(e^{j\omega})$; that is,

$$g[n - n_o] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_o} G(e^{j\omega}). \tag{3.59}$$

It follows from Equation 3.59 that since $|e^{-j\omega n_o}| = 1$, $|G(e^{j\omega})| = |X(e^{j\omega})|$; that is, the magnitude spectrum is unchanged by shifting a signal in time.

Frequency-Shifting Theorem

The Fourier transform of a sequence $x[n] = e^{j\omega_o n} g[n]$ is given by $X(e^{j\omega}) = G\left(e^{j(\omega-\omega_o)}\right)$; that is,

$$e^{j\omega_o n} g[n] \overset{\mathcal{F}}{\longleftrightarrow} G\left(e^{j(\omega-\omega_o)}\right). \tag{3.63}$$

Differentiation-in-Frequency Theorem

The Fourier transform of a sequence $x[n] = n\, g[n]$ is given by $X(e^{j\omega}) = j\frac{dG(e^{j\omega})}{d\omega}$; that is,

$$n\, g[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dG(e^{j\omega})}{d\omega}. \tag{3.64}$$

Table 3.4: Discrete-time Fourier transform theorems.

Theorem	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-reversal	$g[-n]$	$G(e^{-j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation-in-frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's Relation	$\displaystyle\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

Properties of the Discrete-Time Fourier Transform

Property	Time domain	Frequency domain
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Differentiation (frequency)	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Time-shift	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-shift	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$
Time-reversal	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Symmetry (real)	$\Im\mathfrak{m}\{x[n]\} = 0$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
Symmetry (imag)	$\Re\mathfrak{e}\{x[n]\} = 0$	$X(e^{j\omega}) = -X^*(e^{-j\omega})$
Parseval	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\lvert X(e^{j\omega})\right\rvert^2 d\omega$	

Four Fourier Representations

Time Domain	Periodic	Non-periodic	Freq. Domain
Discrete	**Discrete-Time Fourier Series** $\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi kn/N}$ $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k]e^{j2\pi kn/N}$	**Discrete-Time Fourier Transform** $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$	Periodic
Continuous	**Fourier Series** $X[k] = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-j2\pi kt/T}\,dt$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi kt/T}$	**Fourier Transform** $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$	Non-periodic
	Discrete	Continuous	

From http://www.itee.uq.edu.au/~elec3600/

3.6 DTFT Computation Using MATLAB

The *Signal Processing Toolbox* in MATLAB includes a number of M-files to aid in the DTFT-based analysis of discrete-time signals. Specifically, the functions that can be used are `freqz`, `abs`, `angle`, and `unwrap`. In addition, the built-in MATLAB functions `real` and `imag` are also useful in some applications.

FREQZ Digital filter frequency response.

[H,W] = FREQZ(B,A,N) returns the N-point complex frequency response vector H and the N-point frequency vector W in radians/sample of the filter:

$$H(e^{jw}) = \frac{B(e^{jw})}{A(e^{jw})} = \frac{b(1) + b(2)e^{-jw} + \ldots + b(m+1)e^{-jmw}}{a(1) + a(2)e^{-jw} + \ldots + a(n+1)e^{-jnw}}$$

given numerator and denominator coefficients in vectors B and A.

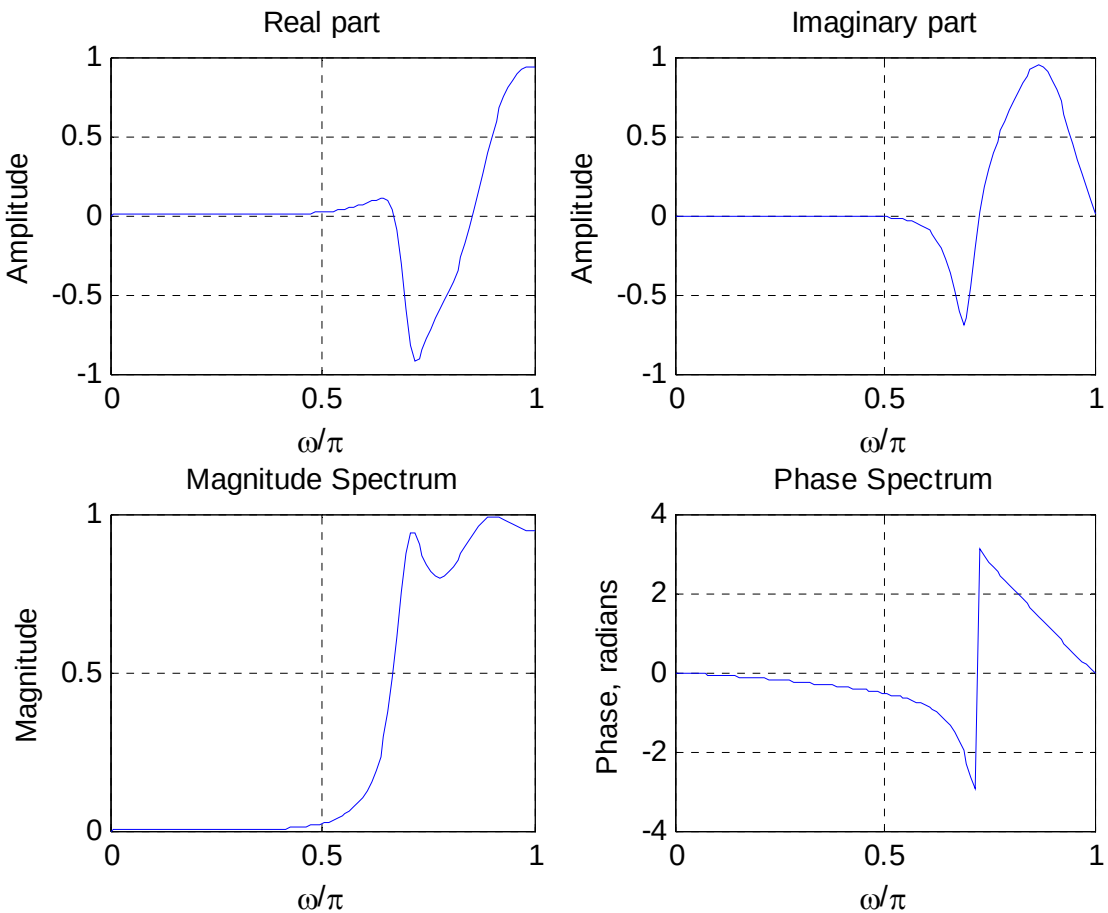
The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

In this book, many of the Fourier transforms we shall encounter are rational functions in $e^{-j\omega}$, that is, ratios of polynomials in $e^{-j\omega}$, and are of the form

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{D(e^{j\omega})} = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}. \tag{3.55}$$

```
% Program 3_1
% Discrete-Time Fourier Transform Computation
%
% Read in the desired number of frequency samples
k = input('Number of frequency points = ');
% Read in the numerator and denominator coefficients
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
% Compute the frequency response
w = 0:pi/(k-1):pi;
h = freqz(num, den, w);

% Plot the frequency response
subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')
```



Number of frequency points = 100
Numerator coefficients = [0.008 -0.033 0.05 -0.033 0.008]
Denominator coefficients = [1 2.37 2.7 1.6 0.41]