**EXAMPLE 2.37 Total Solution Computation of an LTI System for a Constant Input**

Let us determine the total solution for $n \geq 0$ of a discrete-time system characterized by the following difference equation:

$$y[n] + y[n-1] - 6y[n-2] = x[n], \tag{2.103}$$

for a step input $x[n] = 8\mu[n]$ and with initial conditions $y[-1] = 1$ and $y[-2] = -1$.

We first determine the form of the complementary solution. Setting $x[n] = 0$ and $y[n] = \lambda^n$ in Eq. (2.103), we arrive at

$$\begin{aligned}\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} &= \lambda^{n-2}(\lambda^2 + \lambda - 6)\\ &= \lambda^{n-2}(\lambda + 3)(\lambda - 2) = 0,\end{aligned}$$

and hence the roots of the characteristic polynomial $\lambda^2 + \lambda - 6$ are $\lambda_1 = -3, \lambda_2 = 2$. Therefore, the complementary solution is of the form

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n. \tag{2.104}$$

For the particular solution, we assume

$$y_p[n] = \beta.$$

Substituting the above in Eq. (2.103), we get

$$\beta + \beta - 6\beta = 8\mu[n],$$

which for $n \geq 0$ yields $\beta = -2$.

The total solution is therefore of the form

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0. \tag{2.105}$$

The constants $\alpha_1$ and $\alpha_2$ are chosen to satisfy the specified initial conditions. From Eqs. (2.103) and (2.105), we get

$$\begin{aligned}y[-2] &= \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 2 = -1,\\ y[-1] &= \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 2 = 1.\end{aligned}$$

Solving these two equations, we arrive at

$$\alpha_1 = -1.8, \ \alpha_2 = 4.8.$$

Thus, the total solution is given by

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0. \tag{2.106}$$

**EXAMPLE 2.38** **Total Solution Computation of an LTI System for an Exponential Input**

We determine the total solution for $n \geq 0$ of the difference equation of Eq. (2.103) for an input $\boxed{x[n] = 2^n \mu[n]}$ with the same initial conditions as in Example 2.37.

As indicated in Example 2.37, the complementary solution contains a term $\alpha_2(2)^n$, which is of the same form as the specified input. Hence, we need to select a form for the particular solution that is distinct and does not contain any terms similar to those contained in the complementary solution. We assume

$$\boxed{y_p[n] = \beta n(2)^n.}$$

Substituting the above in Eq. (2.103), we get

$$\beta n(2)^n + \beta(n-1)(2)^{n-1} - 6\beta(n-2)(2)^{n-2} = (2)^n \mu[n].$$

For $n \geq 0$, we obtain from the above equation $\beta = 0.4$. The total solution is now of the form

$$\boxed{y[n] = \alpha_1(-3)^n + \alpha_2(2)^n + 0.4n(2)^n, \qquad n \geq 0.} \tag{2.107}$$

To determine the values of $\alpha_1$ and $\alpha_2$, we make use of the specified initial conditions. From Eqs. (2.103) and (2.107), we arrive at

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} + 0.4(-2)(2)^{-2} = -1,$$
$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} + 0.4(-1)(2)^{-1} = 1,$$

which when solved yields $\alpha_1 = -5.04$, $\alpha_2 = -0.96$. Therefore, the total solution is given by

$$\boxed{y[n] = -5.04(-3)^n - 0.96(2)^n + 0.4n(2)^n, \qquad n \geq 0.}$$

### 2.7.2 Zero-Input Response and Zero-State Response

An alternate approach to determining the total solution $y[n]$ of the difference equation of Eq. (2.90) is by computing its *zero-input response*, $y_{zi}[n]$, and *zero-state response*, $y_{zs}[n]$. The component $y_{zi}[n]$ is obtained by solving Eq. (2.90) by setting the input $x[n] = 0$, and the component $y_{zs}[n]$ is obtained by solving Eq. (2.90) by applying the specified input with all initial conditions set to zero. The total solution is then given by $y_{zi}[n] + y_{zs}[n]$.

This approach is illustrated in Example 2.39.

**EXAMPLE 2.39 Total Solution Computation from Zero-Input and Zero-State Responses**

We determine the total solution of the discrete-time system of Example 2.37 by computing the zero-input response and the zero-state response.

$$y[n] + y[n-1] - 6y[n-2] = x[n],$$

The zero-input response, $y_{zi}[n]$, of Eq. (2.103) is given by the complementary solution of Eq. (2.104), where the constants $\alpha_1$ and $\alpha_2$ are chosen to satisfy the specified initial conditions. Now, from Eq. (2.103), we get

$$y[0] = -y[-1] + 6y[-2] = -1 - 6 = -7, \qquad y[1] = -y[0] + 6y[-1] = 7 + 6 = 13.$$

Next, from Eq. (2.104), we get

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n.$$

$$y[0] = \alpha_1 + \alpha_2, \qquad y[1] = -3\alpha_1 + 2\alpha_2.$$

Solving these two sets of equations, we arrive at $\alpha_1 = -5.4$, $\alpha_2 = -1.6$. Therefore,

$$y_{zi}[n] = -5.4(-3)^n - 1.6(2)^n, \qquad n \geq 0.$$

The zero-state response is determined from Eq. (2.105) by evaluating the constants $\alpha_1$ and $\alpha_2$ to satisfy the zero initial conditions. From Eq. (2.103), we get

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0.$$

$$y[0] = x[0] = 8, \qquad y[1] = x[1] - y[0] = 0.$$

Next, from Eq. (2.105) and the above set of equations, we arrive at $\alpha_1 = 3.6$, $\alpha_2 = 6.4$. Thus, the zero-state response for $n \geq 0$ with initial conditions $y_{zs}[-2] = y_{zs}[-1] = 0$ is given by

$$y_{zs}[n] = 3.6(-3)^n + 6.4(2)^n - 2.$$

Hence, the total solution $y[n]$ is given by the sum $y_{zi}[n] + y_{zs}[n]$, resulting in

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0,$$

which is identical to that derived in Example 2.37, as expected.

# Total solution computation from zero input and zero state response

Example

$$\boxed{y_n = \frac{5}{6} y_{n-1} - \frac{1}{6} y_{n-2} + x_n + \frac{1}{2} x_{n-1}}$$

$$\left.\begin{aligned} y_{-1} &= 6 \\ y_{-2} &= 6 \\ x_{-1} &= 1 \end{aligned}\right\} \text{actually in SFG shown!}$$

$x_n = 2^n, \; n \geq 0$

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{3}\right)$$

$$\Downarrow$$

$$\lambda_1 = \frac{1}{2}$$

$$\lambda_2 = \frac{1}{3}$$

$$\Downarrow$$

$$\boxed{y_{h,n} = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n} \quad \text{FORM}$$

Home work example (from A.A.Beex lecture notes)

## zero input response

$y_{zi,n}$ ? $\rightarrow$ assumes zero input $\Rightarrow$ $y_{p,n} = c$

$\Downarrow$

$c - \frac{5}{6}c + \frac{1}{6}c = 0 + 0$

$\Downarrow$

$c = 0$

$\Downarrow$

$y_{p,n} = 0$

$$\boxed{y_{zi,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n \qquad \text{form of } y_{h,n}}$$

$n=0: \; y_0 = \frac{5}{6}\overset{6}{\cancel{y_{-1}}} - \frac{1}{6}\overset{6}{\cancel{y_{-2}}} + \overset{0}{\cancel{x_0}} + \frac{1}{2}\overset{1}{\cancel{x_{-1}}} = 5 - 1 + \frac{1}{2} = 4\frac{1}{2} = \left(\tfrac{1}{2}\right)^0 c_1 + \left(\tfrac{1}{3}\right)^0 c_2$

$n=1: \; y_1 = \frac{5}{6}\overset{4\frac{1}{2}}{\cancel{y_0}} - \frac{1}{6}\overset{6}{\cancel{y_{-1}}} + \overset{0}{\cancel{x_1}} + \frac{1}{2}\overset{0}{\cancel{x_0}} = \frac{22.5}{6} - 1 = \left(\tfrac{1}{2}\right)^1 c_1 + \left(\tfrac{1}{3}\right) c_2$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \boxed{y_{zi,n} = 7.5\left(\tfrac{1}{2}\right)^n - 3\left(\tfrac{1}{3}\right)^n \; ; \; n \geq 0}$$

Home work example (from A.A.Beex lecture notes)

## zero state response

$y_{ZS,n}$ ?

from before : $y_{p,n} = 2 \cdot 2^n$ ; $n \geq 0$

like a $y_{TOT,n}$ but for $\underline{i.c.'s \equiv 0}$

$$\boxed{y_{ZS,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n + 2\cdot 2^n}$$

$n=0$: $\frac{5}{6}\cancel{y_{-1}}^{0} + \frac{1}{6}\cancel{y_{-2}}^{0} + \cancel{x_0}^{1} + \frac{1}{2}\cancel{x_{-1}}^{0} = 1 = c_1 + c_2 + 2$

$n=1$: $\frac{5}{6}\cancel{y_0}^{1} - \frac{1}{6}\cancel{y_{-1}}^{0} + \cancel{x_1}^{2} + \frac{1}{2}\cancel{x_0}^{1} = \frac{5}{6} + 2.5 = c_1\cdot\frac{1}{2} + c_2\cdot\frac{1}{3} + 4$

$y_2 = \frac{5}{6}\cancel{y_1}^{1} - \frac{1}{6}\cancel{y_0}^{1} + \cancel{x_2}^{4} + \frac{1}{2}\cancel{x_1}^{2} = \cdots = c_1\left(\tfrac{1}{2}\right)^2 + c_2\left(\tfrac{1}{3}\right)^2 + 8$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{y_{ZS,n} = (-2)\left(\tfrac{1}{2}\right)^n + 1\cdot\left(\tfrac{1}{3}\right)^n + 2\cdot 2^n \;;\; n \geq 0}$$

$y_{SS,n}$

↑ steady state

Note: $y_{ZS,n} + y_{ZI,n} = \underbrace{(-2+7.5)}_{5.5}\left(\tfrac{1}{2}\right)^n + \underbrace{(1-3)}_{-2}\left(\tfrac{1}{3}\right)^n + 2\cdot 2^n$; $n \geq 0$

$= y_n = y_{TOT,n}$ !

the rest is $y_{transient,n}$

$\left(= 5.5\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{3}\right)^n ; n \geq 0\right)$

Home work example (from A.A.Beex lecture notes)

### 2.7.3 Impulse Response Calculation

The impulse response $h[n]$ of a causal LTI discrete-time system is the output observed with input $x[n] = \delta[n]$. Thus, it is simply the zero-state response with $x[n] = \delta[n]$. Now for such an input, $x[n] = 0$ for $n > 0$, and thus, the particular solution is zero, that is, $y_p[n] = 0$. Hence, the impulse response can be computed from the complementary solution of Eq. (2.101) in the case of simple roots of the characteristic equation by determining the constants $\alpha_i$ to satisfy the zero initial conditions. A similar procedure can be followed in the case of multiple roots of the characteristic equation. A system with all zero initial conditions is often called a *relaxed* system.

We illustrate the impulse response computation in Examples 2.40 and 2.41.

$$\boxed{h_n? \;=\; y_{TOT,n} \text{ for i.c.'s} \equiv 0 \text{ and } x_n = \delta_n}$$

$$y_{h,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n \quad \underline{\text{as before}} \quad \text{(check it out)}$$

$$\Rightarrow \quad h_n = y_{TOT,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n + K\delta_n \;;\; n \geq 0$$

$n=0$: $h_0 = \frac{5}{6}(0) - \frac{1}{6}(0) + \cancel{x_0}^{1} + \frac{1}{2}\cancel{x_{-1}}^{0} = 1 = c_1 + c_2 + K$

$n=1$: $h_1 = \frac{5}{6}\cancel{h_0}^{1} - \frac{1}{6}(0) + \cancel{x_1}^{0} + \frac{1}{2}\cancel{x_0}^{1} = \frac{5}{6} + \frac{1}{2} = c_1 \cdot \frac{1}{2} + c_2 \cdot \frac{1}{3} + 0$

$n=2$: $h_2 = \frac{5}{6}\underbrace{\left(\frac{5}{6}+\frac{1}{2}\right)}_{h_1} - \frac{1}{6}\underbrace{(1)}_{h_0} + \cancel{x_2}^{0} + \frac{1}{2}\cancel{x_1}^{0} = \quad = c_1\left(\frac{1}{2}\right)^2 + c_2\left(\frac{1}{3}\right)^2 + 0$

$$\boxed{y_n = \frac{5}{6}y_{n-1} - \frac{1}{6}y_{n-2} + x_n + \frac{1}{2}x_{n-1}}$$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ K \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix}$$

$$\boxed{h_n = 6\left(\tfrac{1}{2}\right)^n - 5\left(\tfrac{1}{3}\right)^n \;;\; n \geq 0}$$

it's causal

$$\underline{\underline{h_n = \left[6\left(\tfrac{1}{2}\right)^n - 5\left(\tfrac{1}{3}\right)^n\right] u_n}}$$

lasts forever $\Rightarrow$ $\underline{\underline{\text{IIR}}}$

$\sum_n |h_n| < \infty \iff$ BIBO

## EXAMPLE 2.40 Impulse Response Computation from Zero-State Response

In this example, we determine the impulse response $h[n]$ of the causal discrete-time system of Example 2.37. From Eq. (2.104), we get

$$\boxed{h[n] = \alpha_1(-3)^n + \alpha_2(2)^n, \qquad n \geq 0.}$$

From the above, we arrive at

$$h[0] = \alpha_1 + \alpha_2, \qquad h[1] = -3\alpha_1 + 2\alpha_2.$$

Next, from Eq. (2.103) with $x[n] = \delta[n]$, we get $\boxed{y[n] + y[n-1] - 6y[n-2] = x[n],}$

$$h[0] = 1, \qquad h[1] + h[0] = 0.$$

Solution of the above two sets of equations yields $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$.

Thus, the impulse response is given by

$$\boxed{h[n] = 0.6(-3)^n + 0.4(2)^n, \qquad n \geq 0.}$$

## 2.7.4 Output Computation Using MATLAB

The causal LTI system of the form of Eq. (2.91) can be simulated in MATLAB using the function `filter` already made use of in Program 2_4. The function implements Eq. (2.91) in the form of a set of equations as indicated below:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k],$$

$$\begin{aligned}
y[n] &= \frac{p_0}{d_0} x[n] + s_1[n-1], \\
s_1[n] &= \frac{p_1}{d_0} x[n] - \frac{d_1}{d_0} y[n] + s_2[n-1], \\
&\vdots \\
s_{N-1}[n] &= \frac{p_{N-1}}{d_0} x[n] - \frac{d_{N-1}}{d_0} y[n] + s_{N-2}[n-1], \\
s_N[n] &= \frac{p_N}{d_0} x[n] - \frac{d_N}{d_0} y[n],
\end{aligned} \tag{2.113}$$

where $s_i[n]$, $1 \le i \le N$, are $N$ internal variables. By back substitution, it can be shown that the above set of equations indeed reduces to Eq. (2.91). The values of the internal variables $s_i[n]$ at the starting instant are called the *initial conditions*.

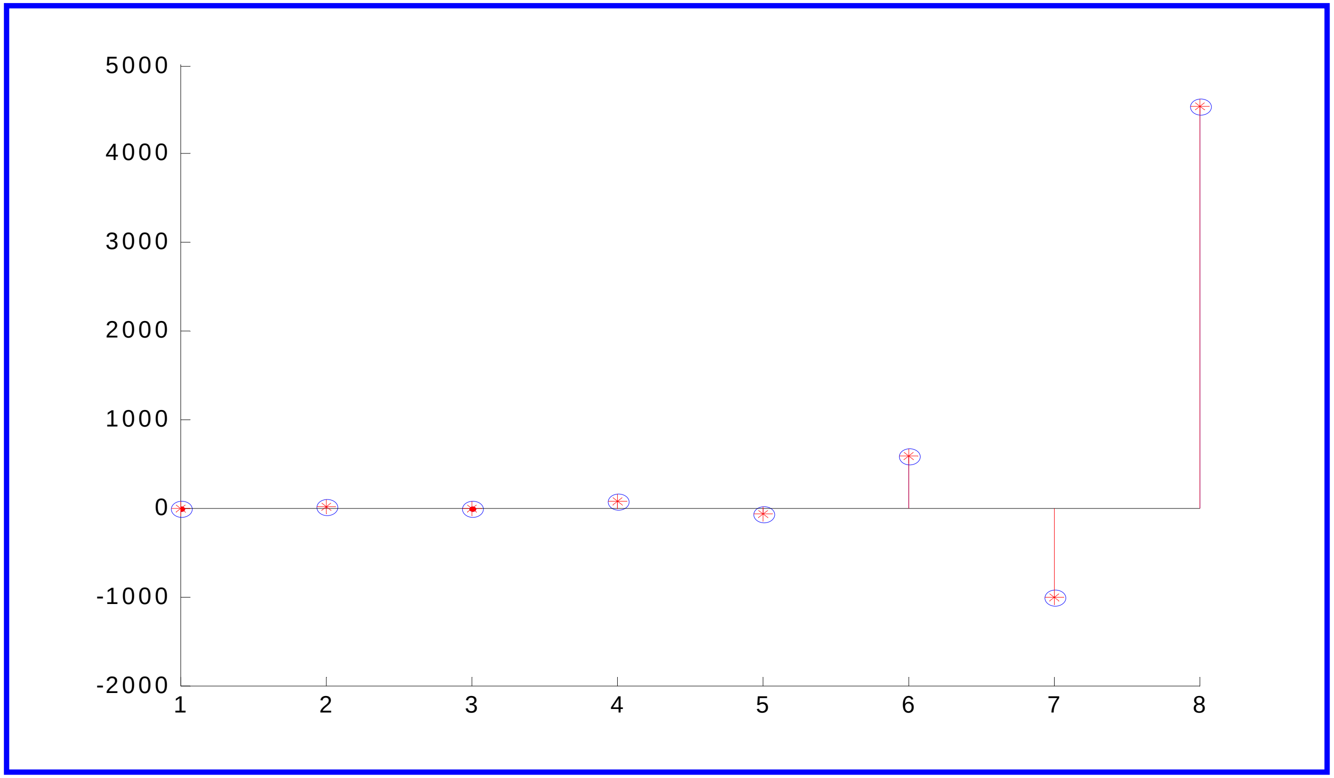
The basic forms of the function `filter` are as follows:

```
y = filter(p,d,x)
[y, sf] = filter(p,d,x, si]
```

In the first form, the input data vector `x` is processed by the system characterized by the coefficient vectors `p` and `d` to generate the output vector `y`, assuming zero initial conditions. The length of `y` is the same as the length of `x`. The second form permits the inclusion of nonzero initial conditions of the internal variables $s_i[n]$ in the vector `si` and provides an output that includes the vector `sf` as the final values of $s_i[n]$. Since the function implements Eq. (2.91), the coefficient $d_0$ must be nonzero.

Example 2.43 illustrates the use of the function `filter` in the computation of the total solution.

```
[y1,sf]=filter(1,[1,1,-6],8*ones(1,8),[-7,6]);
stem(y1)
hold on;
n=0:7;
y2=-1.8*(-3).^n+4.8*(2).^n-2;
stem(y2,'r*')
```



### 2.7.5 Impulse and Step Response Computation Using MATLAB

The impulse and step responses of a causal LTI discrete-time system can be computed using the MATLAB M-files `impz` and `stepz`, respectively. Each function is available with several options. We illustrate the use of these two functions in Example 2.44.

**EXAMPLE 2.44** **Impulse and Step Response Computations Using MATLAB**

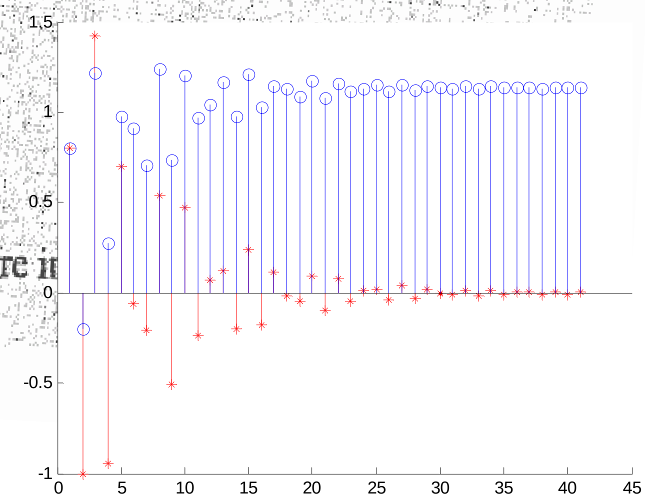
Determine the first 41 samples of the impulse and response samples of the causal LTI system defined by

$$\begin{aligned} y[n] + 0.7y[n-1] - 0.45y[n-2] - 0.6y[n-3] \\ = 0.8x[n] - 0.44x[n-1] + 0.36x[n-2] + 0.02x[n-3]. \end{aligned} \tag{2.114}$$

The code fragments that can be used to compute the impulse and step response samples are as follows:

```
p = [0.8 -0.44 0.36 0.02];
d = [1 0.7 -0.45 -0.6];
[h,m] = impz(p,d,41);
[s,m] = stepz(p,d,41);
```

The computed first 41 samples of the impulse and step response samples are i
respectively.



### 2.7.6 Location of Roots of Characteristic Equation for BIBO Stability

It should be noted that the impulse response samples of a stable LTI system decay to zero values as the time index $n$ becomes very large. Likewise, the step response samples of a stable LTI system approach a constant value as $n$ becomes very large. From the plots of Figure 2.36(a) and (b), we can conclude that most likely the LTI system of Eq. (2.114) is BIBO stable. However, it is impossible to check the stability of a system just by examining only a finite segment of its impulse or step response as in these figures.

The BIBO stability of a causal LTI system characterized by a constant coefficient difference equation of the form of Eq. (2.90) can be inferred from the values of the roots $\lambda_i$ of its characteristic polynomial. To establish the stability conditions, recall that the form of the impulse response is the same as that of the complementary solution. From Eq. (2.101), assuming all the roots to be distinct, we have

$$h[n] = \sum_{i=1}^{N} \alpha_i \lambda_i^n \mu[n]. \tag{2.115}$$

The constants $\alpha_i$ in the above expression are determined to satisfy zero initial conditions. From Eq. (2.115) we get

$$\boxed{\sum_{n=0}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left| \sum_{i=1}^{N} \alpha_i (\lambda_i)^n \right| \leq \sum_{i=1}^{N} |\alpha_i| \sum_{n=0}^{\infty} |\lambda_i|^n.} \tag{2.116}$$

It follows from the above equation that if $|\lambda_i| < 1$ for all values of $i$, then $\sum_{n=0}^{\infty} |\lambda_i|^n < \infty$, and as a result, $\sum_{n=0}^{\infty} |h[n]| < \infty$; that is, the impulse response is absolutely summable, implying BIBO stability of the causal LTI discrete-time system. However, the impulse response sequence is not absolutely summable if one or more of the roots $\lambda_i$ has a magnitude greater than or equal to one. It should be noted that the discrete-time system of Example 2.37 described in Eq. (2.103) is clearly an unstable system as both roots of the characteristic equation have magnitudes greater than one.

In the case of multiple roots of the characteristic equation, the impulse response will contain terms of the form $n^K \lambda_i^n$. As a result, the expression for $\sum_{n=0}^{\infty} |h[n]|$ will contain the term

$$\sum_{n=0}^{\infty} |n^K (\lambda_i)^n|,$$

which converges if $|\lambda_i| < 1$ (Problem 2.89), and as a result, here also the impulse response is absolutely summable.

Summarizing, a causal LTI system characterized by a linear constant coefficient difference equation of the form of Eq. (2.90) is BIBO stable if the magnitude of each of the roots of its characteristic equation is less than one. This condition is both necessary and sufficient.

## 2.8 Classification of LTI Discrete-Time Systems

Linear time-invariant (LTI) discrete-time systems are usually classified either according to the length of their impulse response sequences or according to the method of calculation employed to determine the output samples.

### 2.8.1 Classification Based on Impulse Response Length

If $h[n]$ is of finite length, that is,

$$h[n] = 0 \qquad \text{for} \quad n < N_1 \qquad \text{and} \quad n > N_2 \qquad \text{with} \quad N_1 < N_2, \tag{2.117}$$

then it is known as a *finite impulse response* (FIR) discrete-time system, for which the convolution sum reduces to

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]. \tag{2.118}$$

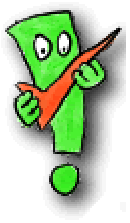
Note that the above convolution sum, being a finite sum, can be used to calculate $y[n]$ directly. The basic operations involved are simply multiplication and addition. Note that the calculation of the present value of the output sequence involves the value of the input sample at $n = N_1$ and $N_2 - N_1$ previous values of the input sequence along with the $N_2 - N_1 + 1$ impulse response samples describing the FIR discrete-time system.

Examples of FIR discrete-time systems are the moving-average system of Eq. (2.61) and the linear interpolators of Eqs. (2.65) and (2.66).

If $h[n]$ is of infinite length, then it is known as an *infinite impulse response* (IIR) discrete-time system. For a causal IIR discrete-time system with a causal input $x[n]$, the convolution sum can be expressed in the form

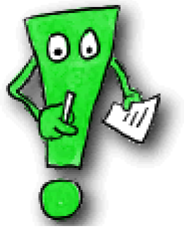
$$y[n] = \sum_{k=0}^{n} x[k]h[n-k],$$

which can be used to compute the output samples. However, for increasing $n$, the computational complexity to compute the output sample increases as the number of products to be summed also increases.

# Chapter (2) - Checklist 

- 2.1 Discrete time signal √
- 2.2 Typical sequences and sequence representation√
- 2.3 The sampling Process√
- 2.4 Discrete Time systems√
- 2.5 Time Domain characterization of LTI Discrete-Time systems √
- 2.6 Simple interconnection schemes√
- 2.7 Finite-Dimensional LTI Discrete time systems√
- 2.8 classification of LTI Discrete time systems √

# Home work # (3)



- Reading section correlation of signals:
  - Only 2.9.1 and 3 , examples 2.46 and 2.47
- Solve proplems of chapter 2 page 107-115:
  - # 1, 5, 6, 7(a and c), 8, 17, 25, 38, 50, 64,83,90
- Implement matlab exercise page 115:
  - # 9

Notes:

- Some final answers will be posted on the course web page
- Submit it as a hardcopy.
- Due date is 07 Feb ,