

Chapter 6

Synchronization of Spread Spectrum Signals

6.1 Introduction

All digital communication systems must perform synchronization. Synchronization is the process of aligning any locally generated reference signals with the incoming waveform[1]. Synchronization must be accomplished for symbol timing, frame timing, carrier frequency and possibly carrier phase. The latter two are typically termed *frequency tracking* and *phase tracking*, while the former two are typically called *synchronization*. In spread spectrum systems, we have the additional burden of synchronizing the spreading waveform. Recall that in order for despreading to take place at the receiver, the locally generated spreading waveform must be aligned with the incoming spreading waveform. Due to the low SNR prior to despreading this synchronization is a priority in the overall process and includes synchronization at the chip level and at the sequence level. In this section we are primarily interested in this added synchronization burden, but we will have to also consider the impact of frequency synchronization. Symbol synchronization is essentially known once chip and sequence synchronization has occurred. We will ignore frame synchronization in this discussion.

When the receiver first attempts to detect the incoming signal, it does not know the relative phase and frequency of the carrier, nor the timing of the spreading waveform. This initial misalignment prevents proper detection. Misalignment of the spreading waveform (in the case of DS/SS) is illustrated in Figure 6.1. In the example, the locally generated waveform is offset from the incoming waveform by $T_c/2$ or one half of the chip period. In practice the initial offset is much greater. In fact if no information is available, on average the initial misalignment will be $\frac{NT_c}{2}$ where N is the length of the spreading sequence. The range of possible delay values is what we call the *uncertainty region*. The impact of timing error on the despreading process can be seen by examining the decision statistic for a DS/SS BPSK system. Recall that the

received signal can be written in complex baseband notation as

$$r(t) = s(t) + n(t) \quad (6.1)$$

$$= \sqrt{P}b(t-\tau)a(t-\tau) + n(t) \quad (6.2)$$

where P is the received signal power, $b(t)$ is the data waveform, $a(t)$ is the spreading waveform, and $n(t)$ is a Gaussian random process representing AWGN. Note that we will ignore the impact of phase/frequency offset at this point (i.e., we will initially assume that the carrier frequency has been obtained exactly), but will remove this assumption shortly. The receiver attempts to despread the signal by correlating with a local copy of the spreading waveform using an assumed delay:

$$\begin{aligned} Z &= \int_0^{NT_c} r(t)a(t)dt \\ &= \int_0^{NT_c} \left(\sqrt{P}b(t-\tau)a(t-\tau) + n(t) \right) a(t)dt \\ &= \int_0^{NT_c} \left(\sqrt{P}b(t-\tau)a(t-\tau) \right) a(t)dt + \int_0^{T_p} n(t)a(t)dt \\ &\approx \sqrt{P}R_a(\tau) + n \end{aligned} \quad (6.3)$$

where τ is the difference in time between the assumed delay and the actual delay, $R_a(\tau)$ is the autocorrelation function of the spreading waveform, NT_c is the integration time (assumed to be equal to one code period but it need not be), we have assumed the data to be unity due to the presence of a pilot signal for synchronization purposes, and n is a term due to thermal noise. Now, if we assume that timing is exact, i.e., $\tau = 0$, the resulting signal-to-noise ratio for the decision statistic is

$$\begin{aligned} \frac{S}{N} &= \frac{\mathbb{E}\{Z\}^2}{\text{var}\{Z|b\}} \\ &= \frac{P}{\sigma_o^2} \end{aligned} \quad (6.4)$$

where σ_o^2 is the variance of the noise after integration. Now, if the timing is not exact we have

$$\frac{S}{N} = \frac{R_a^2(\tau)P}{\sigma_o^2}$$

since the noise properties are not affected while there is a loss in desired correlation energy. An example of the resulting SNR (in linear) is plotted in Figure 6.2 for an output SNR of 10dB (with perfect timing), an m -sequence of length $N=31$ for spreading and square pulses. We can see that the SNR loss can be substantial. At only one quarter of a chip offset ($T_c/4$), the SNR is reduced from 10dB to approximately 7.5dB (5.6 linear). In fact, if the timing error is greater than one chip, the SNR is

$$\frac{S}{N} = \frac{P}{N^2\sigma_o^2}$$

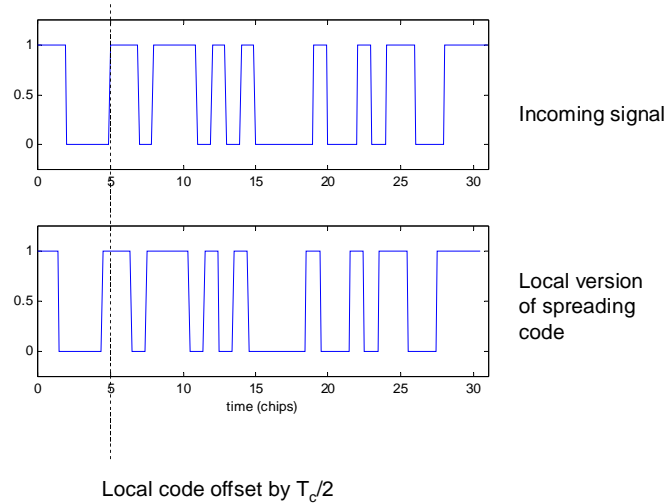


Figure 6.1: Illustration of Code Timing Offset

which is -20dB in this case and prohibitively low. Thus, in order for proper detection to take place we must have very accurate timing information. The role of synchronization is to achieve and maintain this accurate timing information.

6.2 Maximum Likelihood Parameter Estimation

Before we discuss practical synchronization procedures, we will first examine the maximum likelihood parameter estimator [2]. This will then guide us to more practical estimators. Consider a signal

$$r(t) = s(t; \Psi) + n(t)$$

where $s(t; \Psi)$ is the signal of interest, Ψ is the vector of parameters to be estimated and $n(t)$ is AWGN with variance σ^2 . Using the concept of the signal space [1], the received and desired signals can be represented in terms of basis functions

$$r(t) = \sum_{i=1}^N r_i f_i(t)$$

$$s(t; \Psi) = \sum_{i=1}^N s_i(\Psi) f_i(t)$$

where $f_i(t)$ are the orthonormal basis functions. Using this decomposition we can state our problem as finding Ψ which maximizes the a posteriori probability $p(\Psi | \mathbf{r})$ where \mathbf{r} is the vector representing the received signal in our

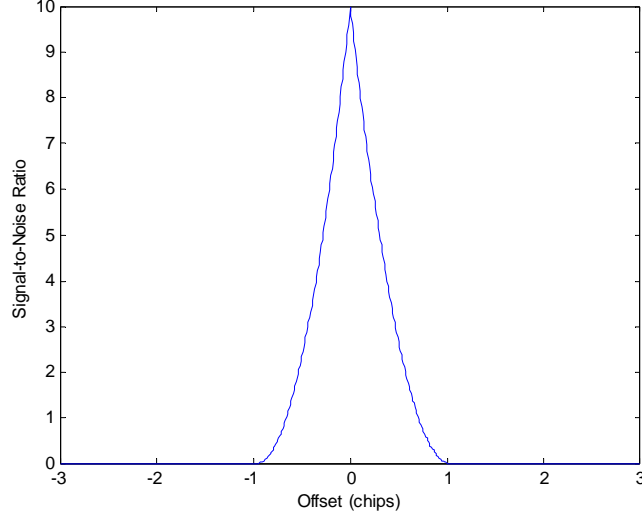


Figure 6.2: Illustration of SNR Loss Due to Code Offset (m -sequence of length 31, SNR is linear and assumed to be 10 with perfect synchronization)

N -dimensional signal space. Using Bayes' rule, we can re-write the probability as

$$p(\Psi|\mathbf{r}) = \frac{p(\mathbf{r}|\Psi)p(\Psi)}{p(\mathbf{r})}$$

Assuming that all values of Ψ are equally likely, maximizing $p(\Psi|\mathbf{r})$ is equivalent to maximizing $p(\mathbf{r}|\Psi)$ since $p(\mathbf{r})$ is independent of Ψ . Since $n(t)$ is a Gaussian random process, \mathbf{r} is a Gaussian random vector with mean $\mathbf{s}(\Psi)$ and covariance $\sigma^2\mathbf{I}$ where \mathbf{I} is the identity matrix. The independence of the noise samples is due to the use of orthonormal bases. Since the individual Gaussian random variables are independent we can write

$$\begin{aligned} p(\mathbf{r}|\Psi) &= \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r_i - s_i(\Psi))^2}{2\sigma^2}\right) \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\sum_{i=1}^N \frac{(r_i - s_i(\Psi))^2}{2\sigma^2}\right) \end{aligned}$$

Now it can be shown [1] that

$$\sum_{i=1}^N \frac{(r_i - s_i(\Psi))^2}{2\sigma^2} = \int_T [r(t) - s(t; \Psi)]^2 dt$$

where T is the signal period. This substitution gives us

$$\begin{aligned} p(\mathbf{r}|\Psi) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\int_T [r(t) - s(t; \Psi)]^2 dt\right) \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\int_T [r^2(t) - 2r(t)s(t; \Psi) + s^2(t; \Psi)] dt\right) \end{aligned}$$

Now, since $r^2(t)$ is a constant for all values of Ψ , and assuming that the signal energy is not dependent on Ψ , maximizing $p(\mathbf{r}|\Psi)$ is equivalent to maximizing

$$\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(2\int_T r(t)s(t; \Psi) dt\right)$$

Since $\exp(x)$ is a monotonically increasing function, our maximum likelihood estimate of the parameter vector Ψ is

$$\Psi_{ML} = \arg \max_{\Psi} \int_T r(t)s(t; \Psi) dt$$

In other words, the maximum likelihood estimator is one which correlates the received signal with the signal corresponding to each of the possible parameter vectors Ψ and choosing the one which provides the maximum result. In general $\Psi = \{f_c, \theta, \tau\}$. However, if we ignore the frequency parameter, and assume that the phase parameter is uniformly distributed over $[0, 2\pi)$, the non-coherent maximum likelihood estimator can be written as [3]

$$\tau_{ML} = \arg \max_{\tau} \left[\left(\int_T r(t)a(t-\tau)\cos(2\pi f_c t) dt \right)^2 + \left(\int_T r(t)a(t-\tau)\sin(2\pi f_c t) dt \right)^2 \right]$$

Thus, the non-coherent maximum likelihood estimator for the signal delay, correlates (non-coherently) the received signal with the desired signal spreading waveform at every possible value of the delay τ and chooses the one which produces the maximum energy. The complexity of this estimator is high since it must examine every possible delay value. In practice we will simplify the synchronization process by quantizing the delay parameter and through various search strategies. One main way of simplifying the synchronization process is to divide the process into two steps: acquisition (which determines a coarse delay estimate) and tracking (which fine-tunes the delay estimate).

6.3 Acquisition and Tracking

As stated, in practice the synchronization process is generally divided into two phases: acquisition and tracking [4, 5]. The acquisition process achieves coarse

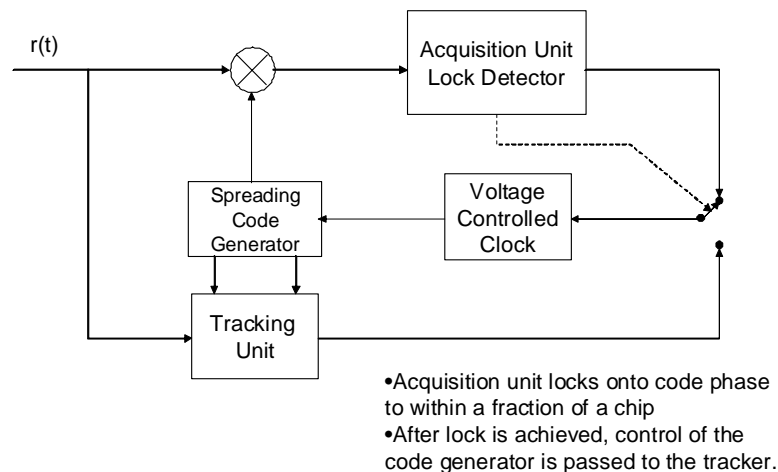


Figure 6.3: General Code Timing Acquisition Circuit

synchronization (to within approximately $1/2$ chip) whereas the tracking process achieves fine synchronization. This separation greatly simplifies the search procedure since it dramatically reduces the number of delay values which need to be evaluated. Additionally, acquisition (as the name implies) is a part of the synchronization process that occurs (ideally) once during initial synchronization. Conversely, tracking is an on-going process which continually adjusts the receiver code timing to account for movement by the transmitter or receiver or for relative drifts in the two symbol clocks. A block diagram of the general procedure is given in Figure 6.3. The figure provides a logical demonstration of the procedure, although current systems may no longer explicitly use such circuitry. Instead, many systems simply sample the incoming signal and perform acquisition (sometimes also called "searching") and tracking via digital signal processing. Thus, the block diagram presents a logical representation of what occurs rather than a representation of the actual circuit. In this chapter we will focus on the acquisition process while in Chapter 7 we will discuss tracking techniques.

6.4 Search Strategies

In the acquisition process, we do not examine every possible candidate delay in the uncertainty region. Instead, we quantize the uncertainty region into cells and search only over these values. The tracking process will refine the delay estimate within the cell. The approximate maximum likelihood approach is to examine the delay associated with each cell, attempt to despread the signal (i.e., correlate with the desired waveform) using each candidate delay and choose

the delay which produces the maximum output. This search can either be done serially or in parallel. That is, we can either attempt to despread the signal with each delay simultaneously (parallel search), or we can despread the signal using each delay candidate serially. The former approach would require Nt correlators where Nt is the number of delay cells as compared to a single correlator in the serial approach as shown in Figures 6.5 and 6.6. The serial approach while being less complex in terms of needed circuitry would however take a substantially longer time (Nt times longer). In practice it is impractical to use a fully parallel search. Thus, some form of serial processing is required. Thus, we will focus on this approach during the remainder of the chapter.

In addition to quantizing the search space, another simplification to the serial search technique which can speed up the acquisition process is to avoid searching over all cells. The maximum likelihood estimation process requires that we examine each candidate delay and determine the maximum correlation (i.e., most likely delay). A simplification would be to allow the search procedure to end if a candidate delay is with high probability the true delay. In this technique we set a threshold and end the search as soon as a correlation exceeds the threshold. The threshold is chosen such that the probability of a correlation using the incorrect delay exceeding the threshold is small. This form of the serial search procedure is very common and will be examined in detail in the remainder of this chapter.

6.5 Serial Search Technique

To restate the general synchronization process, we have a local version of the spreading code $a(t - \tau_L)e^{-j2\pi\Delta f_L t + \phi_L}$ which we are using to despread the incoming signal $r(t) = \sqrt{P}b(t - \tau)e^{j2\pi\Delta f_r t + \phi_r} + n(t)$ where $\Delta f_L, \phi_L$ are the differences in between the nominal carrier frequency and phase and the locally generated frequency and phase and $\Delta f_r, \phi_r$ are the differences in between the nominal carrier frequency and phase and the received frequency and phase. Our goal is to estimate τ_r and Δf_r and set $\tau_L = \tau_r$ and $\Delta f_L = \Delta f_r$. This is a classic estimation problem. However, it is not typically treated as such. Instead we divide the delay/frequency uncertainty region into discrete cells with each cell associated with a specific value of $\{\tau, \Delta f\}$ as shown in Figure 6.4. The approximate maximum likelihood estimate would then be to test all C cells and find the cell which produces the maximum despread output energy. Alternatively, we pose the problem as a hypothesis test. For each cell evaluated we test two hypotheses H_0 and H_1 . The hypothesis H_0 is the hypothesis that the current cell is not the cell containing $\{\tau_r, \Delta f_r\}$. The hypothesis H_1 is the hypothesis that the current cell does contain the correct delay/frequency pair. We then simply test each cell in succession and determine which hypothesis H_0 or H_1 is more likely. As soon as hypothesis H_1 is found to be more likely, the search is terminated regardless of the number of cells examined. This approach is termed a *serial search* technique [6, 7]. A block diagram of this technique is presented in Figure 6.5. As mentioned previously, a common technique is

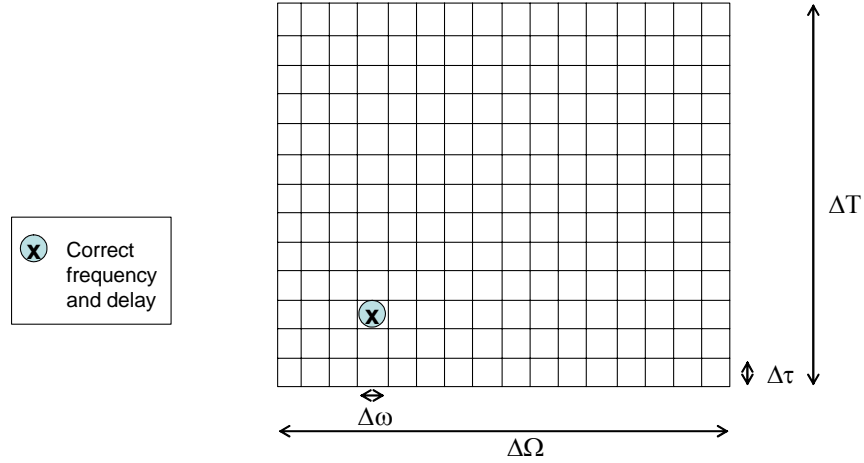


Figure 6.4: Search Space for Time-Frequency Uncertainty

to ignore frequency offset and search only over τ . Frequency estimation then occurs after symbol/chip timing is completed.

Several definitions should be provided before we continue. When we correctly make a determination that we have found the cell containing the true delay/frequency pair, we label this as a "hit". Further, when this occurs we state that the receiver has achieved "lock". If we evaluate the cell containing the true delay/frequency pair but do not declare a lock, this is termed a "miss". The probability of these events are called the probability of detection P_d and the probability of miss P_m [5][8]. Additionally, if the circuit incorrectly declares a lock (i.e., believes that it has found the correct delay when in fact it has not), we label this as a false alarm. Correspondingly, its event probability is called the probability of false alarm P_{fa} .

6.6 Acquisition Performance

When determining the performance of acquisition circuitry, there are two predominant measures: (1) the probability of detection (P_d) vs. probability of false alarm (P_{fa}) curve and (2) the mean acquisition time. In packet radio systems where messages are short and traffic is bursty, the P_d vs. P_{fa} curve is more appropriate since it tells the probability of properly detecting a transmission. However, in systems employing circuit-based connections the mean acquisition time is more useful. As we will show, however, the mean acquisition time is dependent on P_d and P_{fa} so it is instructive to examine those first.

The probability of false alarm is the probability that a hit is declared when the currently tested code phase is not the true code phase. This is simply the

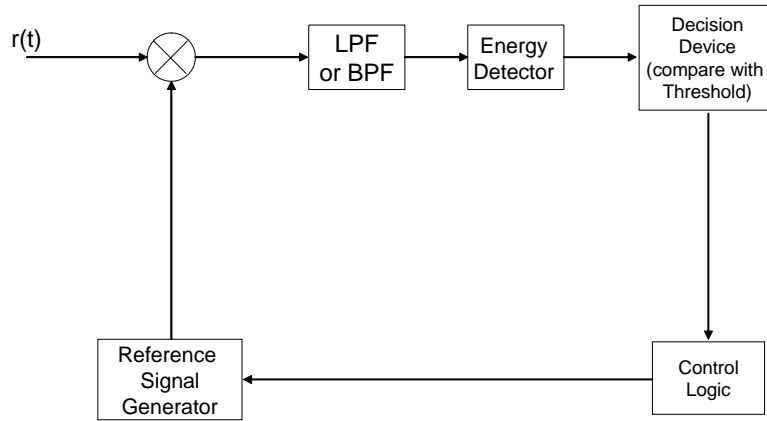


Figure 6.5: Serial Search

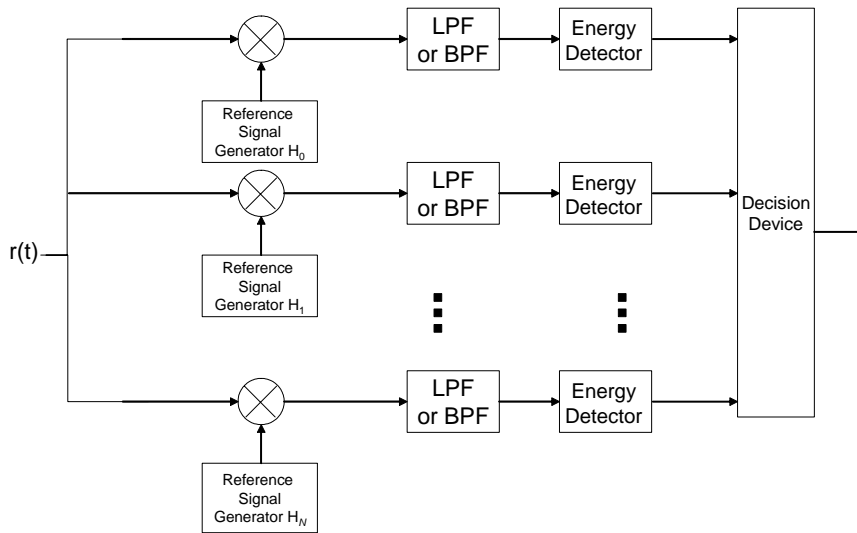


Figure 6.6: Parallel Search

probability that the decision statistic Z exceeds the chosen threshold γ when the code phase is incorrect. Let us examine the decision statistic in this case. Again, the decision statistic is simply the received signal correlated with the spreading waveform $a(t)$ assuming the current delay estimate $\hat{\tau}$ given that the true estimate is τ . That is

$$\begin{aligned} Z &= \left| \frac{1}{T} \int_0^T r(t) a(t - \hat{\tau}) \right|^2 \\ &= \left| \frac{1}{T} \int_0^T \left(\sqrt{P} a(t - \tau) e^{j\phi} + n(t) \right) a(t - \hat{\tau}) \right|^2 \end{aligned} \quad (6.5)$$

where T is the integration period, $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_o}{2}$ and we have assumed that the frequency offset is such that the phase ϕ is constant over the integration period. Simplifying the above equation

$$Z = \left| \sqrt{P} e^{j\phi} R_a(\tau - \hat{\tau}) + n \right|^2 \quad (6.6)$$

Now, assuming that the spreading waveform is based on an m -sequence and that the integration time is over an entire code phase, the autocorrelation function $R_a(\tau - \hat{\tau})$ can be approximated

$$R_a(\tau - \hat{\tau}) = \begin{cases} 1 & \tau - \hat{\tau} = 0 \\ 0 & \tau - \hat{\tau} \neq 0 \end{cases} \quad (6.7)$$

Thus, when hypothesis H_0 is the correct hypothesis,

$$Z = |n|^2. \quad (6.8)$$

The random variable n is the result of the integration of a complex Gaussian random process $n(t)$ and is thus a complex Gaussian random variable. Specifically, n is a zero mean complex Gaussian variable with power

$$\sigma_n^2 = \frac{1}{T^2} N_o 2E_a \quad (6.9)$$

where E_a is the energy in the spreading waveform. More specifically, $E_a = 1^2 * T$. Thus,

$$\begin{aligned} \sigma_n^2 &= \frac{1}{T^2} N_o 2T \\ &= \frac{N_o 2}{NT_c} \end{aligned} \quad (6.10)$$

where we have substituted $T = NT_c$. Thus, Z is a central chi-square random variable with two degrees of freedom. The probability of false alarm is then

$$\begin{aligned} P_{fa} &= \Pr(Z > \gamma | H_0) \\ &= 1 - F_{Z|H_0}(\gamma) \end{aligned} \quad (6.11)$$

where $F_{Z|H_0}(z)$ is the cumulative distribution function of Z given H_0 . The cumulative distribution function for a central chi-square random variable with two degrees of freedom is

$$F_Z(z) = 1 - e^{-\frac{z}{2\sigma^2}} \quad (6.12)$$

where σ^2 is the variance of the underlying Gaussian random variables. Thus, the probability of false alarm is

$$\begin{aligned} P_{fa} &= 1 - \left(1 - e^{-\frac{\gamma}{2\sigma^2}}\right) \\ &= e^{-\frac{\gamma NT_c}{N_o}} \end{aligned} \quad (6.13)$$

Now let us find P_d . The probability of detection is the probability that the correct code phase is determined when it is present. When the code phase is correct the decision variable is

$$Z = \left| \sqrt{P}e^{j\phi} + n \right|^2. \quad (6.14)$$

In this case Z is a central chi-square random variable with two degrees of freedom and centrality parameter

$$\begin{aligned} m^2 &= m_1^2 + m_2^2 \\ &= P\cos^2(\phi) + P\sin^2(\phi) \\ &= P \end{aligned} \quad (6.15)$$

The cumulative distribution function of a central chi-square random variable is

$$F_Z(z) = 1 - Q_1\left(\frac{m}{\sigma}, \frac{\sqrt{z}}{\sigma}\right) \quad (6.16)$$

where σ^2 is the variance of the underlying Gaussian random variables and $Q_m(a, b)$ is the generalized Marcum's Q function defined as

$$Q_m(a, b) = \int_a^b x \left(\frac{x}{a}\right)^{m-1} e^{-(x^2+a^2)/2} I_{m-1}(ax) dx \quad (6.17)$$

and $I_m(x)$ is the m th order modified Bessel function of the first kind. Thus, the probability of detection is

$$\begin{aligned} P_d &= \Pr(Z > \gamma | H_1) \\ &= 1 - F_{Z|H_1}(\gamma) \\ &= 1 - \left(1 - Q_1\left(\frac{m}{\sigma}, \frac{\sqrt{\gamma}}{\sigma}\right)\right) \\ &= Q_1\left(\frac{\sqrt{P}}{\sqrt{\frac{N_o}{2NT_c}}}, \frac{\sqrt{\gamma}}{\sqrt{\frac{N_o}{2NT_c}}}\right) \\ &= Q_1\left(\sqrt{\frac{2PNT_c}{N_o}}, \sqrt{\gamma \frac{2NT_c}{N_o}}\right) \end{aligned} \quad (6.18)$$

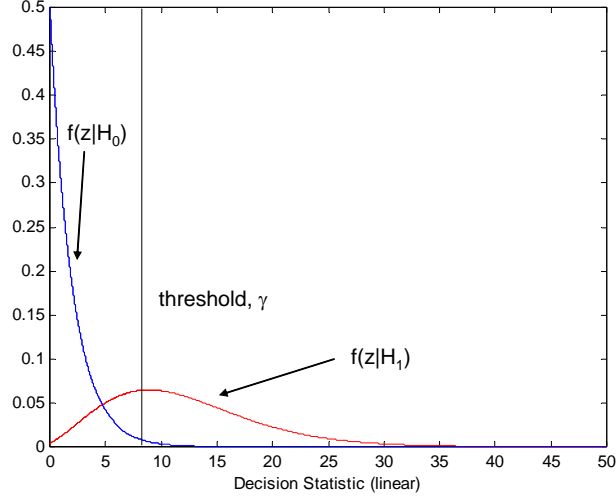


Figure 6.7: Example Probability Density Functions for Two Hypotheses H_0 and H_1

Examples plots of the density functions of H_0 and H_1 are given in Figure 6.7. From the plot we can see that the choice of threshold γ impacts both the probability of detection and the probability of false alarm. Additionally, P_d and P_{fa} are impacted by the received signal SNR $\frac{2P}{N_o}$ and the integration time T . However, we can see that changing γ we can either increase P_d or decrease P_{fa} , but we cannot do both. Clearly, P_d and P_{fa} are not independent. A more direct relationship between P_d and P_{fa} can be found by solving for the threshold γ in terms of P_{fa} . Specifically, we can show that

$$\gamma = -\frac{N_o}{NT_c} \ln \{P_{fa}\} \quad (6.19)$$

Substituting this into the probability of detection results in

$$P_d = Q_1 \left(\sqrt{\frac{2PNT_c}{N_o}}, \sqrt{-2 \ln \{P_{fa}\}} \right) \quad (6.20)$$

Using this relationship we can plot P_d versus P_{fa} . The influence of SNR can be seen in Figure 6.8 where P_d is plotted versus P_{fa} for SNR values of 0dB, 3dB, 6dB and 9dB and $N = 1$. We can see that as SNR increases the curve becomes more skewed toward the upper left hand corner improving the relationship. That is, for the same probability of detection, probability of false alarm is reduced

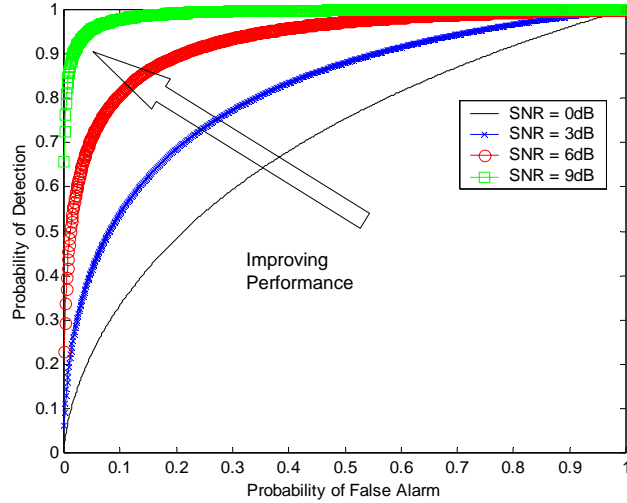


Figure 6.8: Probability of Detection vs. Probability of False Alarm for Various SNR Values

as SNR increases. This is intuitive. Increasing SNR reduces the variance of the H_0 and H_1 curves (see Figure 6.7) decreasing the area of their overlap.

The impact of integration time can be seen in Figure 6.9 where P_d is plotted versus P_{fa} for an SNR of 0dB and $N = 1, N = 10$, and $N = 100$. Increasing N clearly improves performance, decreasing P_{fa} for a constant P_d . Increasing integration time also decreases the variance of Z and thus improves performance in a manner similar to an increase in SNR. Since SNR is difficult to increase arbitrarily, it would seem that the key to improving performance is to simply increase integration time. However, two limitations prevent us from doing this. First, in most practical implementations we do not attempt to determine the frequency offset and the code time simultaneously, but rather attempt to acquire the code timing alone. As a result, the integration time must be limited to the duration over which the incoming signal phase is approximately a constant. We will examine the impact of this later. Additionally, acquisition time is directly related to integration time (also called "dwell time" since it is the time spent evaluating a single delay estimate), thus increasing the integration time may increase acquisition time even if P_d increases and P_{fa} decreases. We will examine this in the next section.

6.6.1 Non-coherent Integration

As mentioned in the last section, one factor which limits integration time is the uncorrected frequency offset. However, we should qualify this statement.

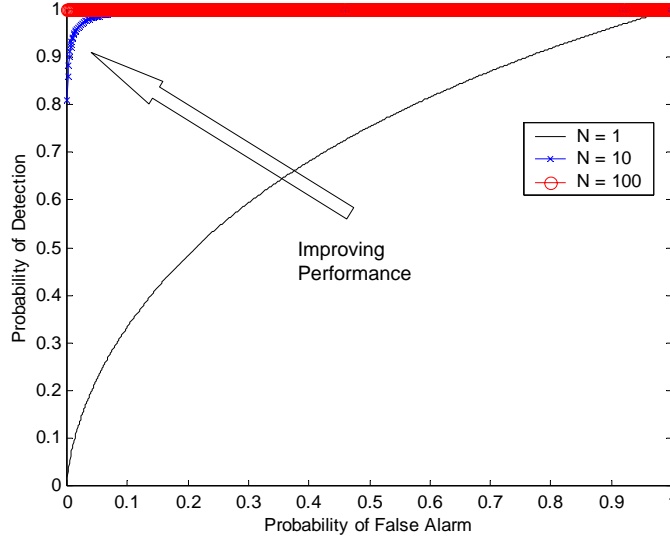


Figure 6.9: The Impact of Coherent Integration Time on the Probability of False Alarm / Detection Curve (SNR = 0dB)

Specifically, frequency offset limits *coherent* integration time. Even in the face of frequency offset, we may increase integration time if we do so *non-coherently*. In other words we may define the decision statistic as

$$Z = \sum_{i=1}^L \left| \frac{1}{T} \int_{(i-1)T}^{iT} r(t) a(t - \hat{\tau}) \right|^2 \quad (6.21)$$

The new decision statistic is the sum of L Chi-Square random variables each with 2 degrees of freedom. This results in a Chi-Square random variable with $2L$ degrees of freedom. Specifically, when non-coherently combining L coherent integrations, each of period $T = NT_c$ hypothesis H_0 results in a central Chi-Square random variable with $2L$ degrees of freedom and where the variance of the underlying Gaussian random variables is

$$\sigma^2 = \frac{N_o}{2NT_c} \quad (6.22)$$

Similarly, hypothesis H_1 results in a non-central Chi-Square random variable with $2L$ degrees of freedom a non-centrality parameter

$$m^2 = LP \quad (6.23)$$

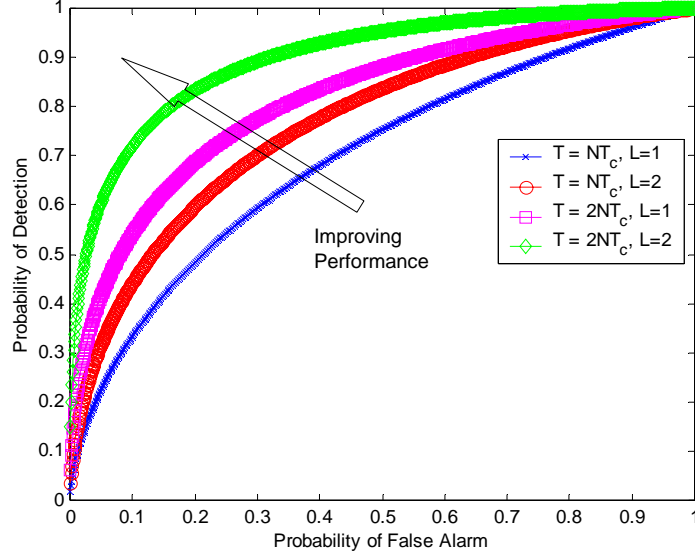


Figure 6.10: The Impact of Non-Coherent Integration Time on Probability of False Alarm / Detection Curve (SNR = 0dB)

and the variance of the underlying Gaussian random variables is the same as H_0 . Thus, using the same steps as before the probability of false alarm is found as

$$P_{fa} = e^{-\gamma \frac{NT_c}{N_o}} \sum_{k=0}^{L-1} \frac{1}{k!} \left(\gamma \frac{NT_c}{N_o} \right)^k \quad (6.24)$$

while the probability of detection is found to be

$$Q_L \left(\sqrt{\frac{2LPNT_c}{N_o}}, \sqrt{\gamma \frac{2NT_c}{N_o}} \right) \quad (6.25)$$

As an example consider the acquisition case examined previously. Figure 6.10 plots P_d vs. P_{fa} SNR = 0dB for $N=1$ and $L=1$. Additionally, the plot shows the improvement achieved when increasing L to 2 and doubling coherent integration time. It can be seen that doubling the integration time either coherently or non-coherently improves acquisition. However, it should be noticed that increasing integration time *coherently* provides better improvement than increasing integration via non-coherent combining of coherent integrations. However, as mentioned non-coherent integration may be necessary when frequency offset exists.

6.6.2 Impact of Frequency Error

As we have discussed previously, the receiver must ultimately obtain proper estimates of the incoming signal timing (*i.e.*, code timing), carrier frequency and possibly carrier phase. Typically, the former will occur before the latter two are estimated. This limits the coherent integration time possible. In this section we would like to investigate this limitation more closely. The acquisition decision statistic for DS/SS can be written as

$$Z = \left| \frac{1}{T} \int_0^T \sqrt{P} a(t - \tau) e^{j2\pi\Delta f t + \phi} a(t - \hat{\tau}) dt \right|^2 \quad (6.26)$$

It can be easily shown that when $\hat{\tau} \neq \tau$, Z is not effected by the value of Δf . In other words, H_0 is relatively insensitive to frequency offset. However, H_1 can be impacted dramatically. This can be seen by expanding Z when $\hat{\tau} = \tau$:

$$\begin{aligned} Z &= \left| \frac{1}{T} \int_0^T \sqrt{P} e^{j2\pi\Delta f t + \phi} dt \right|^2 \\ &= \left(\frac{1}{T} \int_0^T \sqrt{P} \cos(2\pi\Delta f t + \phi) dt \right)^2 + \left(\frac{1}{T} \int_0^T \sqrt{P} \sin(2\pi\Delta f t + \phi) dt \right)^2 \end{aligned}$$

Now examining the first term:

$$\begin{aligned} I^2 &= \left(\frac{1}{T} \int_0^T \sqrt{P} \cos(2\pi\Delta f t + \phi) dt \right)^2 \\ &= \left(\frac{1}{T} \int_0^T \sqrt{P} \cos \phi \cos(2\pi\Delta f t) - \sin \phi \sin(2\pi\Delta f t) dt \right)^2 \\ &= \left(\frac{\cos \phi}{T} \int_0^T \sqrt{P} \cos(2\pi\Delta f t) dt - \frac{\sin \phi}{T} \int_0^T \sin(2\pi\Delta f t) dt \right)^2 \\ &= \left(\frac{\sqrt{P} \cos \phi \sin(2\pi\Delta f t)}{T \cdot 2\pi\Delta f} - \frac{\sqrt{P} \sin \phi (1 - \cos(2\pi\Delta f t))}{T \cdot 2\pi\Delta f} \right)^2 \end{aligned} \quad (6.28)$$

Similarly, we can show that the second term is equal to

$$Q^2 = \left(\frac{\sqrt{P} \cos \phi (1 - \cos(2\pi\Delta f t))}{T \cdot 2\pi\Delta f} + \frac{\sqrt{P} \sin \phi \sin(2\pi\Delta f t)}{T \cdot 2\pi\Delta f} \right)^2 \quad (6.29)$$

Combining the two terms we find

$$\begin{aligned}
Z &= I^2 + Q^2 \\
&= \left(\frac{\sqrt{P} \cos \phi \sin(2\pi\Delta ft)}{T} - \frac{\sqrt{P} \sin \phi (1 - \cos(2\pi\Delta ft))}{2\pi\Delta f} \right)^2 + \dots \\
&\quad \left(\frac{\sqrt{P} \cos \phi (1 - \cos(2\pi\Delta ft))}{2\pi\Delta f} + \frac{\sqrt{P} \sin \phi \sin(2\pi\Delta ft)}{T} \right)^2 \\
&= P \left(\frac{1 - \cos(2\pi\Delta ft)}{2\pi\Delta f NT_c} \right)^2 + P \left(\frac{\sin(2\pi\Delta ft)}{2\pi\Delta f NT_c} \right)^2 \\
&= P \frac{1 - \cos(2\pi\Delta f NT_c) + \cos^2(2\pi\Delta f NT_c) + \sin^2(2\pi\Delta f NT_c)}{(2\pi\Delta f NT_c)^2} \\
&= P \frac{2 - 2\cos(2\pi\Delta f NT_c)}{(2\pi\Delta f NT_c)^2} \\
&= P \frac{\sin^2(\pi\Delta f NT_c)}{(\pi\Delta f NT_c)^2} \\
&= P \operatorname{sinc}^2(\pi\Delta f NT_c) \tag{6.30}
\end{aligned}$$

Thus, we can see that in order to avoid drastic reduction in correlation energy, we require $T = NT_c \ll \Delta f$. Since Δf will in general be a property of cost restrictions on the transmit and receive designs, the integration time will in general be limited by the frequency tolerance allowed.

We can also incorporate this frequency error into the probability of detection and probability of false alarm as follows. As discussed previously we can increase overall integration time by performing non-coherent integration. This is particularly effective in the presence of a frequency offset. Returning to the definitions above, but now including noise:

$$I_k = \frac{1}{T} \int_{(k-1)T}^{kT} \sqrt{P} \cos(2\pi\Delta ft + \phi) dt + n_I \tag{6.31}$$

$$Q_k = \frac{1}{T} \int_{(k-1)T}^{kT} \sqrt{P} \sin(2\pi\Delta ft + \phi) dt + n_Q \tag{6.32}$$

where n_I and n_Q are AWGN samples with zero mean and variance $\sigma_{IQ}^2 = \frac{N_c T}{2}$. If we then non-coherently sum L such terms:

$$Z = \sum_{k=1}^L (I_k^2 + Q_k^2) \tag{6.33}$$

where Z is a non-central Chi-Square random variable with $2L$ degrees of freedom and pdf given by

$$f_Z(z) = \frac{1}{2\sigma_{IQ}^2} \left(\frac{z}{\lambda}\right)^{\frac{N-1}{2}} \exp\left(-\frac{z+\lambda}{2\sigma_{IQ}^2}\right) I_{L-1}\left(\frac{\sqrt{\lambda z}}{\sigma_{IQ}^2}\right) \tag{6.34}$$

for $z \geq 0$ and $I_{L-1}(\cdot)$ is the modified Bessel function of the first kind of order $L - 1$. The normalized non-centrality parameter is

$$\frac{\lambda}{\sigma_{IQ}^2} = \frac{\frac{2L^2TP}{N_o}}{(2\pi\Delta fT)} \left[1 - \cos\left(\frac{2\pi\Delta fT}{L}\right) \right] \quad (6.35)$$

Note that the probability of false alarm is the same as in (6.24) while the probability of detection is similar to that given in (6.25) except that the non-centrality parameter is reduced by the frequency offset:

$$Q_L\left(\sqrt{\frac{\lambda}{\sigma_{IQ}^2}}, \sqrt{\frac{\gamma}{2\sigma_{IQ}^2}}\right) \quad (6.36)$$

It can be ascertained from the preceding expression that there exists an optimal coherent integration time and non-coherent integration time [9].

6.6.3 Impact of Finite Number of Delay Values

Up until this point we have assumed that there is a delay estimate among the C cells that is identical to the true delay. In reality since there are a finite number of cells, it is unlikely that one of the delays will fall exactly on the true delay. Specifically, if we sample the delay uncertainty region once per chip period, the true delay could be as much as $T_c/2$ away from one of the C delay estimates. This is equivalent to sampling the auto-correlation function at a rate of $1/T_c$ as shown in Figure 6.11. As can be seen in Figure 6.11 the worst case results in correlator outputs that $1/2$ of the maximum correlator output which is a 6dB reduction in output power. This can be equated to a 6dB reduction in SNR which drastically impacts the relationship between P_d and P_{fa} as shown in Figure 6.8. This can be mitigated by increasing the number of possible delay estimates at the expense of acquisition time. By increasing the number of estimates to two per chip (i.e., half-chip sampling) decreases the worst case degradation as shown in Figure 6.12. In this example the worst case offset is one quarter of a chip period. This results in 3dB degradation in the correlation energy.

Another factor which can impact the performance of acquisition performance is partial integration. We have assumed that the correlator integrates over a full code period. However, this may not always be the case, particularly in systems with long PN sequences. In such cases, the output of H_1 is unaffected by partial integration as it will still result in the same output. However, the output of H_0 will be affected. This is because, in general, $R_a(\tau - \hat{\tau}) \neq 0$. Rather the partial correlations will result in additional output energy which will increase the probability of false alarm.

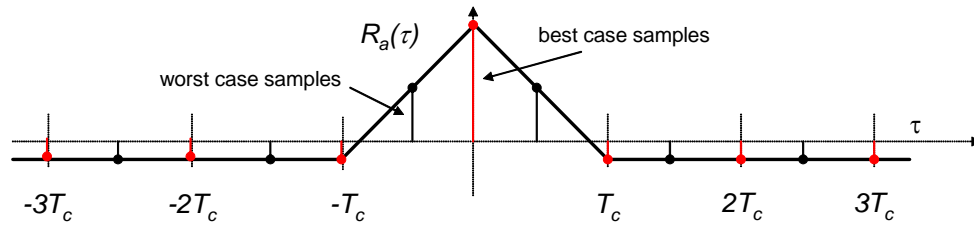


Figure 6.11: The Impact of Sampling Once Per Chip on Acquisition

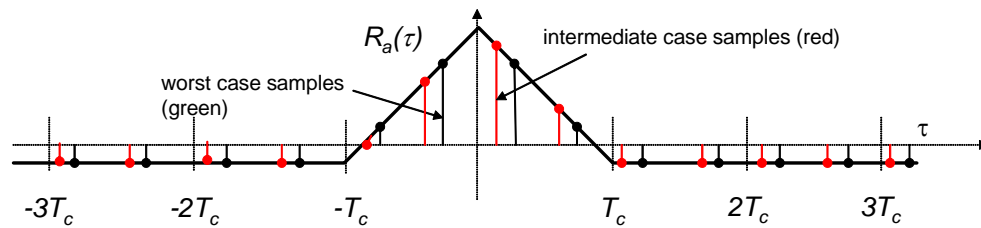


Figure 6.12: The Impact of Sampling Twice Per Chip on Acquisition

6.7 Acquisition Time

As stated earlier investigating P_d vs. P_{fa} curves may be useful for some applications more than others. In other applications we are more concerned with acquisition time. Thus, in this section we investigate the acquisition time of serial search techniques.

The time to acquire a code lock depends on the integration time per cell (or dwell time T), the number of cells being searched C , the position n of the correct cell among the C possible cells, the number of false alarms k , the number of missed detections j and the penalty time associated with a false alarm T_{fa} . Specifically,

$$T_{acq}(n, j, k) = nT + jCT + kT_{fa} \quad (6.37)$$

We are interested in finding the mean acquisition time $\overline{T_{acq}}$:

$$\overline{T_{acq}} = \sum_{n=1}^C \sum_{j=0}^{\infty} \sum_{k=0}^K T_{acq}(n, j, k) P(n, j, k) \quad (6.38)$$

where $K = n + jC - j - 1$ is the number of incorrect code phases tested and $P(n, j, k)$ is the probability of the correct cell being the n th cell, j missed detections, and k false alarms. The joint probability can be determined as

$$P(n, j, k) = P(k|n, j)P(j|n)P(n) \quad (6.39)$$

Further, the probability of the n th cell being the correct cell is simply $\frac{1}{C}$ since we assume that the true delay is uniformly distributed in the uncertainty region. The conditional probability $P(j|n)$ is actually independent of n and is simply equal to the probability of not detecting the correct code time j times and detecting the correct code phase once: $P(j|n) = (1 - P_d)^j P_d$. Further, the probability of k false alarms given n and j is simply equal to the probability of experiencing k false alarms of K incorrect cells: $P(k|n, j) = \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k}$. Now, substituting these values into equation (6.38)

$$\begin{aligned} \overline{T_{acq}} &= \sum_{n=1}^C \sum_{j=0}^{\infty} \sum_{k=0}^K T_{acq}(n, j, k) P(n, j, k) \\ &= \sum_{n=1}^C \sum_{j=0}^{\infty} \sum_{k=0}^K (nT + jCT + kT_{fa}) P(n, j, k) \\ &= \sum_{n=1}^C \sum_{j=0}^{\infty} \sum_{k=0}^K (nT + jCT + kT_{fa}) \frac{1}{C} (1 - P_d)^j P_d \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k} \\ &= \sum_{n=1}^C \sum_{j=0}^{\infty} \frac{1}{C} (1 - P_d)^j P_d \sum_{k=0}^K (nT + jCT + kT_{fa}) \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k} \end{aligned} \quad (6.40)$$

Now we can use the following two properties of the binomial distribution:

$$\sum_{k=0}^K \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k} = 1 \quad (6.41)$$

and

$$\sum_{k=0}^K k \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k} = K P_{fa} \quad (6.42)$$

we arrive at

$$\begin{aligned} \bar{T}_{acq} &= \sum_{n=1}^C \sum_{j=0}^{\infty} \frac{1}{C} (1 - P_d)^j P_d \sum_{k=0}^K (nT + jCT + kT_{fa}) \binom{K}{k} P_{fa}^k (1 - P_{fa})^{K-k} \\ &= \sum_{n=1}^C \sum_{j=0}^{\infty} \frac{1}{C} (1 - P_d)^j P_d (nT + jCT + K P_{fa} T_{fa}) \\ &= \sum_{n=1}^C \frac{1}{C} \sum_{j=0}^{\infty} (1 - P_d)^j P_d (nT + jCT + (n + jC - j - 1) P_{fa} T_{fa}) \\ &= \sum_{n=1}^C \frac{1}{C} \sum_{j=0}^{\infty} (1 - P_d)^j P_d [n(T + P_{fa} T_{fa}) + j(CT + (C - 1) P_{fa} T_{fa}) - P_{fa} T_{fa}] \end{aligned} \quad (6.43)$$

Now, using the following two properties of the geometric distribution:

$$\sum_{j=0}^{\infty} (1 - P_d)^j P_d = 1 \quad (6.44)$$

and

$$\sum_{j=0}^{\infty} j (1 - P_d)^j P_d = \frac{1 - P_d}{P_d} \quad (6.45)$$

we can simplify the above equation as

$$\begin{aligned} \bar{T}_{acq} &= \sum_{n=1}^C \frac{1}{C} \sum_{j=0}^{\infty} (1 - P_d)^j P_d [n(T + P_{fa} T_{fa}) + j(CT + (C - 1) P_{fa} T_{fa}) - P_{fa} T_{fa}] \\ &= \sum_{n=1}^C \frac{1}{C} \left[n(T + P_{fa} T_{fa}) - P_{fa} T_{fa} + \frac{1 - P_d}{P_d} (CT + (C - 1) P_{fa} T_{fa}) \right] \\ &= (C - 1)(T + P_{fa} T_{fa}) \left(\frac{1}{P_d} - \frac{1}{2} \right) + \frac{T}{P_d} \end{aligned} \quad (6.46)$$

Now examining the final equation for mean acquisition time we can see it increases with P_{fa} or T_{fa} and decreases with P_d as expected. However, it also increases with integration time and the number of cells C . Thus, while decreasing P_{fa} and increasing P_d is desirable, if we increase T to accomplish this, we

are not guaranteed to reduce mean acquisition time. Additionally, although increasing the sampling rate improves the worst case SNR of the acquisition loop, we also directly increase the acquisition time by increasing the search space.

Let's examine a few special cases for instructional purposes. First, let us assume that we decrease the threshold so that $P_d = 1$. In this case

$$\begin{aligned}\bar{T}_{acq} &= (C - 1)(T + P_{fa}T_{fa}) \left(\frac{1}{P_d} - \frac{1}{2} \right) + \frac{T}{P_d} \\ &= \frac{(C - 1)}{2}(T + P_{fa}T_{fa}) + T \\ &= \frac{C + 1}{2}T + \frac{1}{2}(C - 1)P_{fa}T_{fa}\end{aligned}\quad (6.47)$$

Thus, we are guaranteed to find the correct timing when we examine it. Since on average it will be in the middle of the timing uncertainty region we get a factor of $\frac{C+1}{2}T$ in the mean acquisition time. Further, since there are on average $\frac{C-1}{2}$ incorrect cells examined, they will cause on average $\frac{C-1}{2}P_{fa}$ false alarms.

If on the other hand we increase the threshold to ensure that $P_{fa} = 0$

$$\begin{aligned}\bar{T}_{acq} &= (C - 1)(T + P_{fa}T_{fa}) \left(\frac{1}{P_d} - \frac{1}{2} \right) + \frac{T}{P_d} \\ &= (C - 1)T \left(\frac{1}{P_d} - \frac{1}{2} \right) + \frac{T}{P_d} \\ &= T \left(\frac{C}{P_d} - \frac{C - 1}{2} \right)\end{aligned}\quad (6.48)$$

If we increase integration time such that $P_d = 1$ and $P_{fa} = 0$

$$\bar{T}_{acq} = \frac{C - 1}{2}T \quad (6.49)$$

Thus, since we have increased T , the mean integration time can still be large even with ideal P_d and P_{fa} .

As an example, let us consider a DS/SS system where there is a 100 chip uncertainty region, an operating SNR = 0dB and a penalty time of 250 integration periods. How do we choose the integration time and the threshold to minimize the mean acquisition time?

First, let us examine three threshold values $\gamma = 1$, $\gamma = 10$, and $\gamma = 50$, over a range of 0-200 chips for an integration period. Ignoring the impact of finite delays and partial correlation the probability of false alarm is shown in Figure 6.13. We can see that a low threshold leads to extremely high integration time requirements in order to reduce P_{fa} . However, with moderate thresholds of 10 or 50, P_{fa} can be made as low as 1% by relatively small integration times. Figure 6.14 plots the probability of detection over the same range of parameters. In this case we can obtain a 99% probability of detection regardless of threshold provided that the integration time is above 40 chips. The resulting mean acquisition time is plotted in Figure 6.15. It is clear that increasing

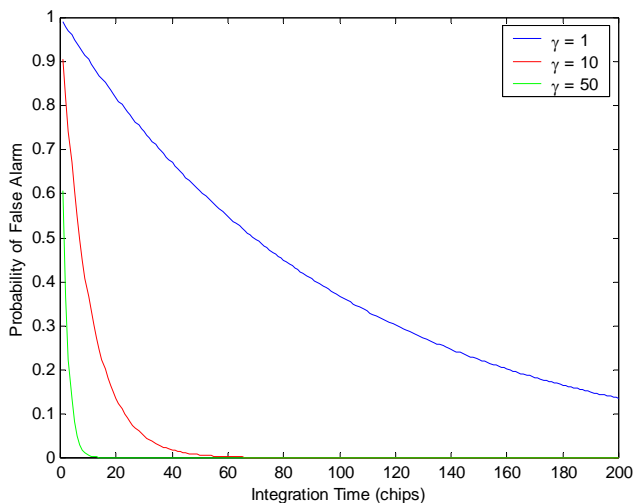


Figure 6.13: Probability of False Alarm versus Integration Time for Various Thresholds

integration time can lead to higher mean acquisition times. This is most clearly seen for $\gamma = 50$. Initially, \bar{T}_{acq} decreases with T as P_d and P_{fa} dominate. However, after approximate $T = 18T_c$, the acquisition time increases. This is because P_d and P_{fa} no longer impact the acquisition time. The value at which this point occurs is different for different values of γ . Additionally, it is found that $\gamma = 50$ provides the minimum acquisition time for those examined. Clearly, we would need to perform a more exhaustive search to find the absolute minimum.

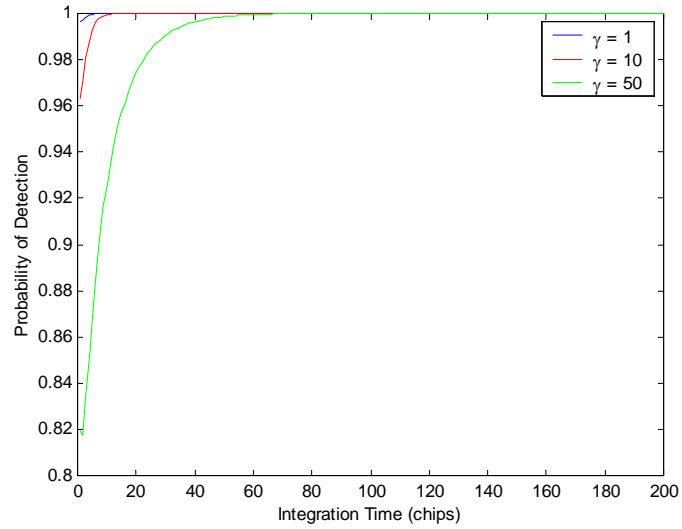


Figure 6.14: Probability of Detection versus Integration Time for Various Thresholds

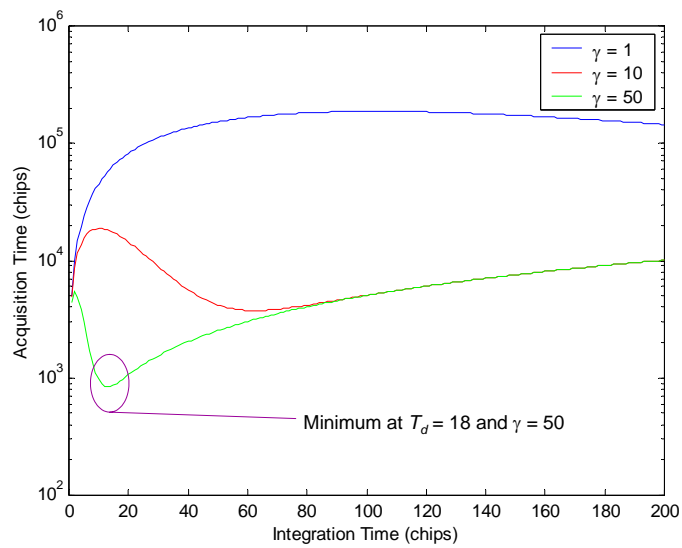


Figure 6.15: Mean Acquisition Time versus Integration Time for Various Thresholds

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