Random signal and noise

Random Process

Autocorrelation

Power spectral density

Ex 6.31

- X_n is iid sequence of $N(0,\sigma^2)$ r.v.'s. Y_n is the average of two consecutive values

$$Y_n = \frac{X_n + X_{n-1}}{2} \quad \Rightarrow \quad m_Y = 0.5E\left[X_n + X_{n-1}\right] = 0$$

$$\Downarrow$$

$$C_Y(i,j) = E\left[Y_i Y_j\right] = 0.25E\left[\left(X_i + X_{i-1}\right)\left(X_j + X_{j-1}\right)\right]$$

$$= 0.25\left\{E\left[X_i X_j\right] + E\left[X_i X_{j-1}\right] + E\left[X_{i-1} X_j\right] + E\left[X_{i-1} X_{j-1}\right]\right\}$$

$$= \frac{1}{2}\sigma^2\delta_{i-j} + \frac{1}{4}\sigma^2\delta_{i-j+1} + \frac{1}{4}\sigma^2\delta_{i-j-1}$$

Y_n is WSS

Y_n is a linear trafo of Gaussian r.v.'s, so Y_n is Gaussian

joint PDF is Gaussian, specified by m_Y and $C_Y(i,j)$

Time averages & ergodicity

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i)$$

ensemble averaging
repeating the experiment
from many realizations

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\zeta)\,dt$$

time averaging
based on single realization

Ergodic ⟶ Stationary but not Vice versa

an ergodic theorem states conditions under which a time average converges when the observation interval becomes large

we're interested in ergodic theorems that state when time averages converge to the ensemble average or expected value

Mean, Autocorrelation of sine wave with random phase

$$X(t) = A\cos(2\pi f_c t + \theta)$$

A and f_c are constants and θ is a RV that is uniformly distributed over range of 0 and 2π

$$m_x(t) = \int_0^{2\pi} A\cos(2\pi f_c t + \theta)\frac{1}{2\pi}d\theta$$

$$m_x(t) = A[\sin(2\pi f_c t + \theta)]_0^{2\pi} = zero$$

$$< X(t)_T >= \int_0^{T} A\cos(2\pi f_c t + \theta)dt$$

$$< X(t)_T >= A\left[\frac{\sin(2\pi f_c t + \theta)}{2\pi f}\right]_0^{T} = zero$$

Ergodic

Mean, Autocorrelation of sine wave with random phase

$$X(t) = A\cos(2\pi f_c t + \theta)$$

A and f_c are constants and θ is a RV that is uniformly distributed over range of 0 and 2π

$$R_x(t_1,t_2) = E[X(t_1)X(t_2)] = A^2 E[Cos(2\pi f_c t_1 + \theta)Cos(2\pi f_c t_2 + \theta)]$$

$$R_x(t_1,t_2) = \frac{A^2}{2} E[Cos(2\pi f_c (t_1 - t_2) + Cos(2\pi f_c (t_1 + t_2) + 2\theta)]$$

$$R_x(t_1,t_2) = \frac{A^2}{2} Cos(2\pi f_c (t_1 - t_2) = R_x(\tau) = \frac{A^2}{2} Cos(2\pi f_c(\tau)) \quad \text{W.S Stationary}$$

$$< X(t)X(t+\tau) >= A^2 \frac{1}{2T} \int_{-T}^{T} Cos(2\pi f_c t + \theta)Cos(2\pi f_c (t+\tau) + \theta)dt$$

$$< X(t)X(t+\tau) >= A^2 \frac{1}{2T} \int_{-T}^{T} [Cos(2\pi f_c \tau) + Cos(2\pi f_c (2t+\tau) + 2\theta)]dt$$

$$< X(t)X(t+\tau) >= \frac{A^2}{2} Cos(2\pi f_c(\tau) = R_x(\tau) \qquad Ergodic$$

Properties of Gaussian RP

1. If a Gaussian process X(t) is applied to a LTIS then the output is also a Gaussian Process.
2. Gaussain Random process defined at a set of time instants is completely defined by the vector mean M and the covariance matrix C
3. If X(t) is a Gaussian WSS RP then it is a SS RP
4. If x(t) is a Gaussian RP with uncorrelated RV then they are also independent.

Cross Correlation function

- Consider two random processes X(t) and Y(t) with autocorrelation functions $R_x(t_1,t_2)$, $R_y(t_1,t_2)$ respectively
- The cross correlation function of X(t) and Y(t) is defined by $R_{xy}(t_1,t_2)=E[X(t_1)Y(t_2)]$ and $R_{yx}(t_1,t_2)=E[Y(t_1)X(t_2)]$.
- If X(t) and Y(t) are WSS then $R_{xy}(t_1,t_2)= R_{xy}(\tau)$
- $R_{xy}(\tau)= R_{yx}(-\tau)$

Power Spectral Density

- From Communication Theory we know that Autocorrelation is the IFT of PSD
- IF X(t) is WSS RP

$$R_X(\tau) \xleftrightarrow{F.T} S_X(f)$$

- Properties of PSD

1. $S_X(0) = \int_{-\infty}^{\infty} R_x(\tau) d\tau$
2. $E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$
3. $S_X(f) \geq 0$ Non Negative
4. $S_X(f) = S_X(-f)$ for real valued random process

Ex. PSD of sine wave with random phase

$$R_x(\tau) = \frac{A^2}{2} Cos(2\pi f_c(\tau)) \quad \text{W.S Stationary}$$

- Using the F.T

$$S_x(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)]$$

Ex.

$$R_x(\tau) = \begin{cases} A^2(1 - \frac{|\tau|}{T}) & |\tau| \le T \\ 0 & Otherwise \end{cases}$$

- Using the F.T

$$S_x(f) = A^2 T \sin c^2(fT)$$

Ex.

$$Y(t) = X(t)\cos(2\pi f_c t + \theta)$$

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(2\pi f_c t + \theta)X(t+\tau)\cos(2\pi f_c (t+\tau) + \theta)]$$

$$R_Y(\tau) = E[X(t)X(t+\tau)]E[\cos(2\pi f_c t + \theta)\cos(2\pi f_c (t+\tau) + \theta)]$$

$$R_Y(\tau) = R_X(\tau)\cos(2\pi f \tau)$$

$$S_Y(f) = \frac{1}{2}[S_X(f - f_c) + S_X(f - f_c)]$$

Relation among PSD of the input and output RP of LTIS

- Suppose RP X(t) is applied to a LTIS with impulse response h(t), producing RP Y(t).
  - What is the PSD of Y(t) w.r.t PSD of X(t) Assuming X(t) is WSS RP.

$$S_Y(f) = S_X(f)\left|H(f)\right|^2$$

Ex.

$$H(f) = 1 - \exp(-j2\pi fT)$$

$$\left|H(f)\right|^2 = H(f)H^*(f)$$

$$\left|H(f)\right|^2 = (1 - \exp(-j2\pi fT))(1 - \exp(+j2\pi fT))$$

$$\left|H(f)\right|^2 = (1 + 1 - \exp(-j2\pi fT) - \exp(+j2\pi fT))$$

$$\left|H(f)\right|^2 = 2(1 - \cos(2\pi fT)) = 4\sin^2(\pi fT)$$

$$S_Y(f) = 4\sin^2(\pi fT)S_X(f)$$

Cross Spectral Density

- Provides a measure of the frequency interrelationship between two random processes.
- X(t) and Y(t) are jointly WSS RP with $R_{XY}(\tau)$ and $R_{YX}(\tau)$
- Thus they have as a FT. $S_{XY}(f)$ and $S_{YX}(f)$
- $S_{XY}(f) = S_{YX}(-f)$

Example

- X(t) and Y(t) have zero mean and they are individually stationary in the wide sense.
- Z(t)=X(t)+Y(t) find $S_Z(f)$

$$R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)]$$

$$= E[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))]$$

$$= E[X(t_1)X(t_2)] + E[X(t_2)Y(t_1)] + E[X(t_1)Y(t_2)] + E[Y(t_1)Y(t_2)]$$

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

$$S_Z(f) = S_X(f) + S_Y(f) + S_{XY}(f) + S_{YX}(f)$$

If X and Y are uncorrelated

$$S_Z(f) = S_X(f) + S_Y(f)$$

Noise

- Unwanted signal that tend to disturb the transmission and processing of signals in communication systems.
- Thermal Noise → random motion of electrons in a conductor.
- Shot noise → arises in electronic devices, sudden change in voltage or current.

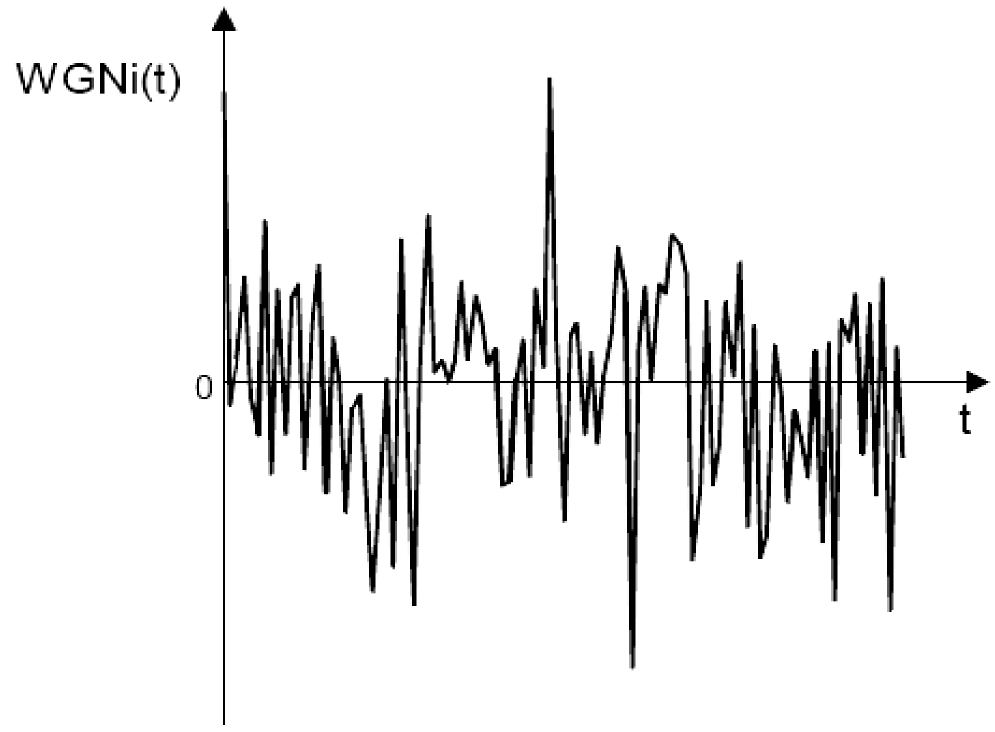
White Noise

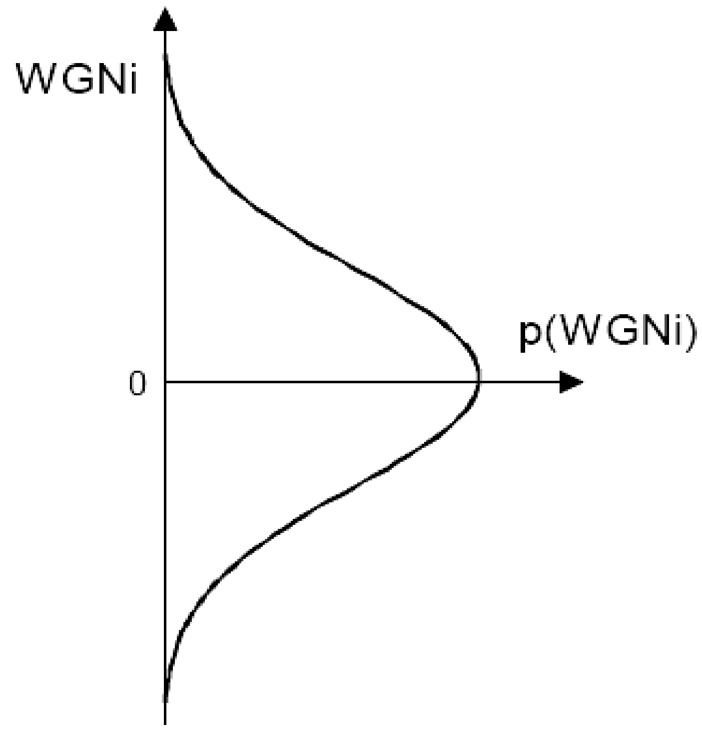
- White → occupies all frequencies→ PSD is independent on the operating frequency
- Dimensions of N_o is watt per Hertz, N_o=KT

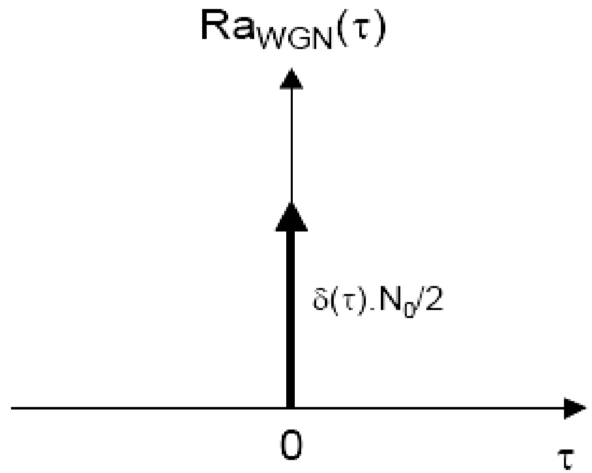
$$S_N(f) = \frac{N_o}{2}$$

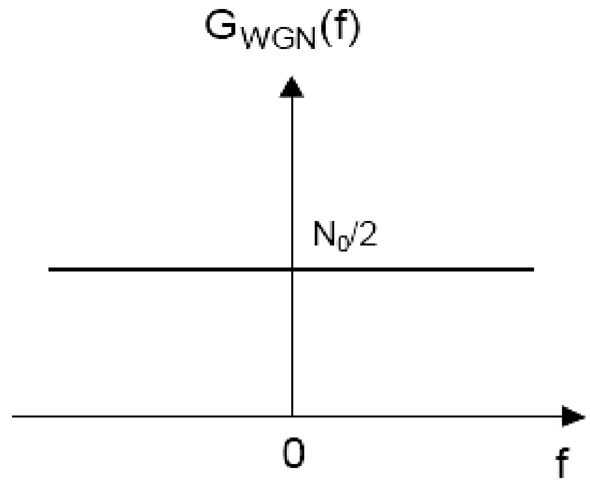
$$R_N(\tau) = \frac{N_o}{2}\delta(\tau)$$

- Any two different samples of white noise no matter how close they are will be uncorrelated.
- If white noise is Gaussian then they will also be independent


WGNi(t)
0
t


WGNi
0
p(WGNi)


Ra_WGN(τ)
δ(τ).N0/2
0
τ


G_WGN(f)
N0/2
0
f

Ideal low pass filtered white noise

- A white Gaussian noise with zero mean and variance $N_0/2$ is applied to an ideal low pass filter of bandwidth B and amplitude response of one.
- PSD of output Y(t) is

$$S_Y(f) = \begin{cases} \frac{N_o}{2} & -B \leq f \leq B \\ 0 & otherwise \end{cases}$$

- The autocorrelation is $R_Y(\tau) = N_0 B \sin c(2B\tau)$
- Autocorrelation maximum at τ equal zero equal $N_o B$ and passes through zero at $\tau = n/2B$ for n= integer values and variance $N_o B$
- If noise is sampled at rate 2B then they are uncorrelated and being Gaussian then statistically independent

RC low pass filtered white noise

- The H(f) of RC filter is $H(f) = \dfrac{1}{1 + j2\pi fRC}$
- The PSD of the o/p is $S_Y(f) = \dfrac{1}{1 + (2\pi fRC)^2}$
- The autocorrelation of the output is $R_Y(\tau) = \dfrac{N_o}{4RC} \exp(-\dfrac{|\tau|}{RC})$
- If noise is samples at rate 0.217/RC then they are uncorrelated and being Gaussian then statistically independent

Ex sine wave plus white noise

$$X(t) = A\cos(2\pi f_c t + \theta) + N(t)$$

θ is Uniformely distributed, N(t) is WGN

$$\mathrm{R_X}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau)$$

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{N_0}{2}$$

Noise Equivalent Bandwidth

- Output average power of ILPF → N_oB
- Output average power of RCLPF → $N_o/4RC$

- Output average power of any filter → $N_oBH^2(0)$
- Equivalent Bandwidth =

$$B = \frac{\int_0^\infty |H(f)|^2 df}{H^2(0)}$$