Random or stochastic processes

Definition

- Random experiment specified by $\zeta \in S$, by the events defined on S, and by the probabilities on these events. To each $\zeta \in S$, a function of time is assigned according to some rule.

$$X(t,\zeta) \quad t \in I$$

$$X(t,\zeta)\big|_{\zeta} \quad \text{is a function of time } t$$

realization, sample path, sample function

$$X(t,\zeta)\big|_{t=t_k} \quad \text{is a function of } \zeta, \text{ i.e. an r.v.}$$

$$\{X(t,\zeta), t \in I\}$$

an indexed family of r.v.'s
or random process
or stochastic process

if I is countable: discrete time SP
if I is continuous (non-negative) real line: continuous SP

Ex 6.1 discrete-time SP

$$\zeta \in S = (0,1]$$

$$\forall \zeta \in S \rightarrow \zeta = \sum_{n=1}^{\infty} b_n 2^{-i}$$ binary expansion

discrete-time SP: $X(n,\zeta) = b_n \; ; n = 1, 2, \ldots$

Ex 6.2 continuous-time SP

$$\zeta \in S = [-1,1]$$

$$X(t,\zeta) = \zeta \cos(2\pi t) \; ; -\infty < t < \infty$$

sinusoids with amplitude ζ

$$\zeta \in S = [-\pi, \pi]$$

time-shifted versions of cos

$$Y(t,\zeta) = \cos(2\pi t + \zeta) \; ; -\infty < t < \infty$$

Specifying a SP

sample the SP $X(t,\zeta)\big|_{t=t_i} \rightarrow \text{r.v.'s } X_1, X_2, \cdots, X_k$

joint behavior of the SP at these *k* time instants is specified by the joint CDF of the vector r.v. $\mathbf{X} = (X_1, X_2, \cdots, X_k)$

we can now compute probabilities as before

a SP is specified by the collection of k^{th}-order joint CDFs:

$$F_{X_1,\cdots,X_k}(x_1,\cdots,x_k) = P[X_1 \le x_1, X_2 \le x_2, \cdots, X_k \le x_k]$$

for any k and any choice of sampling instants $t_1,\cdots t_k$

any any

$$p_{X_1,\cdots,X_k}(x_1,\cdots,x_k) = P[X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k]$$

for discrete-time SP

$$f_{X_1,\cdots,X_k}(x_1,\cdots,x_k)$$ for continuous-time SP

Mean, autocorrelation, autocovariance

$$m_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

$$R_X(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1)X(t_2)}(x,y)\,dxdy$$

$$C_X(t_1,t_2) = E\left[\left\{X(t_1)-m_X(t_1)\right\}\left\{X(t_2)-m_X(t_2)\right\}\right]$$
$$= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

$$VAR\left[X(t)\right] = E\left[\left\{X(t)-m_X(t)\right\}^2\right] = C_X(t,t)$$

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)C_X(t_2,t_2)}} \Rightarrow \left|\rho_X(t_1,t_2)\right| \le 1$$

~predicting one r.v. as a linear function from another

Ex 6.6 random amplitude sinusoid

$$X(t) = A\cos 2\pi t \qquad m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t$$

mean varies with *t*

process is always =0 for *t*-values for which cos2π*t*=0

$$R_X(t_1,t_2) = E[A\cos(2\pi t_1)A\cos(2\pi t_2)]$$
$$= E[A^2]\cos(2\pi t_1)\cos(2\pi t_2)$$

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$
$$= \{E[A^2] - E[A]E[A]\}\cos(2\pi t_1)\cos(2\pi t_2)$$
$$= VAR(A)\cos(2\pi t_1)\cos(2\pi t_2)$$

Ex 6.7 random phase sinusoid

$$X(t)=\cos(\omega t+\Theta)\,;\,\Theta\sim U(-\pi,\pi)$$

$$m_X(t)=E\left[\cos(\omega t+\Theta)\right]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+\theta)\,d\theta=0$$

mean is constant

$$C_X(t_1,t_2)=R_X(t_1,t_2)=E\left[\cos(\omega t_1+\Theta)\cos(\omega t_2+\Theta)\right]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left\{\cos\left(\omega(t_1-t_2)\right)+\cos\left(\omega(t_1+t_2)+2\theta\right)\right\}d\theta$$

$$=\frac{1}{2}\cos\left(\omega(t_1-t_2)\right)$$

covariance depends on absolute time difference only

Gaussian SP

A random process $X(t)$ is a Gaussian SP

if for any k and any choice of sampling instants $t_1 < t_2 < \ldots < t_k$

the r.v.'s $X_1 = X(t_1), X_2 = X(t_2), \ldots, X_k = X(t_k)$

are jointly Gaussian r.v.'s

discrete & continuous

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k)$$

$$= \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T K^{-1}(\mathbf{x}-\mathbf{m})\right\}}{(2\pi)^{k/2} |K|^{\frac{1}{2}}}$$

k jointly Gaussian r.v.'s

r.v.'s $X_1, X_2, \cdots, X_k$ are said to be jointly Gaussian if their joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k)$$

$$= \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T K^{-1}(\mathbf{x}-\mathbf{m})\right\}}{(2\pi)^{k/2}|K|^{\frac{1}{2}}}$$

$$\mathbf{m} = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_k) \end{bmatrix} \qquad K = \left\{C_X(t_i, t_j)\right\}_{i,j}$$

Multiple SPs

- Joint behavior of two or more SPs is specified by the collection of joint distributions for all possible choices of time samples of the SPs
- e.g. for $X(t)$ and $Y(t)$ specify all possible joint density functions of $X(t_1),\ldots, X(t_k)$ and $Y(t_1'),\ldots,Y(t_j')$ for all k,j and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

Multiple SPs

- The SPs $X(t)$ and $Y(t)$ are said to be independent if the vector r.v.'s $(X(t_1),\ldots, X(t_k))$ and $(Y(t_1'),\ldots,Y(t_j'))$ are independent for all k,j and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

cross-correlation $R_{XY}(t_1,t_2)=E\left[X(t_1)Y(t_2)\right]$

SPs $X(t)$ and $Y(t)$ are said to be orthogonal if $R_{XY}(t_1,t_2)=0\,\forall t_1,t_2$

cross-covariance

$$C_{XY}(t_1,t_2)=E\left[\{X(t_1)-m_X(t_1)\}\{Y(t_2)-m_Y(t_2)\}\right]$$

$$=R_{XY}(t_1,t_2)-m_X(t_1)m_Y(t_2)$$

SPs $X(t)$ and $Y(t)$ are said to be uncorrelated if $C_{XY}(t_1,t_2)=0\,\forall t_1,t_2$

Ex 6.10 signal + noise

$$Y(t) = X(t) + N(t)$$

cross-correlation ↓

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$

$$= E\left[X(t_1)\left\{X(t_2)+N(t_2)\right\}\right]$$

$$= E\left[X(t_1)X(t_2)\right] + E\left[X(t_1)N(t_2)\right]$$

independence

$$= R_X(t_1,t_2) + E\left[X(t_1)\right]E\left[N(t_2)\right]$$

$$= R_X(t_1,t_2) + m_X(t_1)m_N(t_2)$$

Stationary random processes

- Essentially…the nature of the randomness does not change with time
- Assuming the process started at $t=-\infty$…
  A discrete-time or continuous-time SP $X(t)$ is stationary if the joint distribution of any set of samples does not depend on the placement of the time origin

$$F_{X(t_1),\cdots,X(t_k)}\left(x_1,\cdots,x_k\right)=F_{X(t_1+\tau),\cdots,X(t_k+\tau)}\left(x_1,\cdots,x_k\right);\forall\tau,\forall k,\forall t_1,\cdots,t_k$$

if SP starts at definite time, eg. $t=0$ or $n=0$, joint CDF does not change under time shifts to the right

Jointly stationary SPs

- Two processes *X*(*t*) and *Y*(*t*) are said to be jointly stationary if

$$F_{X(t_1),\cdots,X(t_k),Y(t_1^{'}),\cdots,Y(t_j^{'})}\left(x_1,\cdots,x_k,y_1,\cdots,y_j\right)=$$

$$=F_{X(t_1+\tau),\cdots,X(t_k+\tau),Y(t_1^{'}+\tau),\cdots,Y(t_j^{'}+\tau)}\left(x_1,\cdots,x_k,y_1,\cdots,y_j\right)$$

$$\forall\tau,\forall k,j,\forall t_1,\cdots,t_k,\forall t_1^{'},\cdots,t_k^{'}$$

First-order CDF of stationary SP

$$F_{X(t)}(x) = F_{X(t+\tau)}(x)\,;\forall\tau,\forall t$$

$$= F_X(x)$$

$$\Downarrow$$

$$m_X(t) = E\left[X(t)\right] = m\,;\forall t$$

$$VAR\left(X(t)\right) = E\left[\left(X(t) - m\right)^2\right] = \sigma^2\,;\forall t$$

mean and variance are constant

Second-order CDF of stationary SP

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2)$$
$$= F_{X(0),X(t_2-t_1)}(x_1,x_2)\,;\ \forall t_1,t_2$$

$$\Downarrow$$

$$R_X(t_1,t_2) = R_X(t_2-t_1)\,;\ \forall t_1,t_2$$

$$C_X(t_1,t_2) = C_X(t_2-t_1)\,;\ \forall t_1,t_2$$

autocorrelation and covariance depend only on time difference

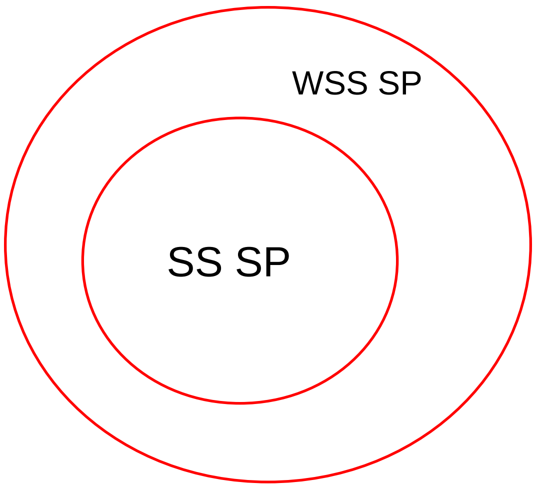
Wide-sense stationary (WSS) SP

- Often we cannot determine whether a process is stationary, but we can determine whether
  - *the mean is constant* $m_X(t) = m\,;\forall t$
  - *the autocovariance (equivalently the autocorrelation) is a function of time difference only* $C_X(t_1,t_2) = C_X(t_1 - t_2)\,;\forall t_1,t_2$
- SP is WSS if both conditions hold

Jointly WSS SPs

- $X(t)$ and $Y(t)$ are jointly WSS SPs if
  - $X(t)$ is WSS
  - $Y(t)$ is WSS
  - $C_{XY}(t_1,t_2)=C_{XY}(t_1-t_2)=C_{XY}(\tau)$

- All stationary SPs are WSS



Autocorrelation function of WSS SP

- Plays a crucial role in the design of linear signal processing algorithms

i $R_X(0) = E\left[X^2(t)\right]$ average power of the process

ii $R_X(\tau) = E\left[X(t+\tau)X(t)\right] = E\left[X(t)X(t+\tau)\right] = R_X(-\tau)$

an even function of τ

iii $$E^2[XY] \le E\left[X^2\right]E\left[Y^2\right] \forall X, Y$$

$$\Downarrow$$

$$R_X^2(\tau) = E^2\left[X(t+\tau)X(t)\right] \le E\left[X^2(t+\tau)\right]E\left[X^2(t)\right] = R_X^2(0)$$

$$\Downarrow$$

$$\left|R_X(\tau)\right| \le R_X(0)$$

autocorrelation is max at τ=0

- If $R_x(d)=R_x(0)$ → $R_x(\tau)$ is periodic with period equal to d.
- Let $X(t) = m + N(t)$

  $N(t)$ is a zero-mean process $\ni R_N(\tau) \underset{\tau\to\infty}{\to} 0$

$$R_X(\tau) = E\left[\{m+N(t+\tau)\}\{m+N(t)\}\right]$$
$$= m^2 + 2mE\left[N(t)\right] + R_N(\tau)$$
$$= m^2 + R_N(\tau) \underset{\tau\to\infty}{\to} m^2$$

$R_X(\tau)$ approaches m^2 (the square of the mean) as $\tau \to \infty$

Autocorrelation function of WSS SP

- Can have three types of component:
- A component that vanishes as $\tau \to \infty$
- A periodic component
- A component due to a nonzero mean

Ex 6.30

$$R_X(\tau) = e^{-2\alpha|\tau|}$$

$X(t)$ is zero-mean and $R_X(\tau) \underset{\tau \to \infty}{\to} 0$

$$R_Y(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau)$$ random phase sinusoid

$Y(t)$ is zero-mean and $R_X(\tau)$ has period f_0^{-1}

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + m^2$$

$Z(t) = X(t) + Y(t) + m$; $X(t)$ and $Y(t)$ independent

Note that it is not possible to determine which fraction of the mean is contributed by the individual components

WSS Gaussian SPs

- Mean is not a function of *t*
- Autocovariance is a function of time difference only

↓

- Joint PDF depends on mean and autocovariance only

↓

- A WSS Gaussian SP is stationary

WSS --> SS

all we need is m and $C_X(\tau)$

Ex 6.31

- X_n is iid sequence of $N(0,\sigma^2)$ r.v.'s. Y_n is the average of two consecutive values

$$Y_n = \frac{X_n + X_{n-1}}{2} \Longrightarrow m_Y = 0.5E\left[X_n + X_{n-1}\right] = 0$$

$$C_Y(i,j) = E\left[Y_i Y_j\right] = 0.25E\left[\left(X_i + X_{i-1}\right)\left(X_j + X_{j-1}\right)\right]$$

$$= 0.25\left\{E\left[X_i X_j\right] + E\left[X_i X_{j-1}\right] + E\left[X_{i-1} X_j\right] + E\left[X_{i-1} X_{j-1}\right]\right\}$$

$$= \frac{1}{2}\sigma^2\delta_{i-j} + \frac{1}{4}\sigma^2\delta_{i-j+1} + \frac{1}{4}\sigma^2\delta_{i-j-1}$$

Y_n is WSS

Y_n is a linear trafo of Gaussian r.v.'s, so Y_n is Gaussian

joint PDF is Gaussian, specified by m_Y and $C_Y(i,j)$

Time averages & ergodicity

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i)$$

ensemble averaging
repeating the experiment
from many realizations

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\zeta)\,dt$$

time averaging
based on single realization

Ergodic ⟶ Stationary but not Vice versa

an ergodic theorem states conditions under which a time average converges when the observation interval becomes large

we're interested in ergodic theorems that state when time averages converge to the ensemble average or expected value

$$X(t) = A\cos(2\pi f_c t + \vartheta)$$

ϑ is U. RV from $0 \longrightarrow 2\pi$

Is X(t) mean and autocorrelation ergodic?