PDF of linear transformations

- The joint PDF of Z can be found directly in terms of the joint PDF of X by finding the equivalent events of infinitesimal rectangles.

$$\left.\begin{aligned} V &= aX + bY \\ W &= cX + eY \end{aligned}\right\} \Rightarrow \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \triangleq A \begin{bmatrix} X \\ Y \end{bmatrix}$$

A^{-1} exists

$$\forall (v,w) \exists \text{unique} (x,y) : \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} v \\ w \end{bmatrix}$$

PDF of linear transformations

$$f_{V,W}(v,w) = \frac{f_{X,Y}(x,y)}{\left|\frac{dP}{dxdy}\right|} = \frac{f_{X,Y}(x,y)}{|A|}$$

In general: $\mathbf{Z} = A\mathbf{X}$

$$f_{\mathbf{Z}}(\mathbf{z}) \triangleq f_{Z_1,\cdots,Z_n}(z_1,\cdots,z_n) = \left.\frac{f_{X_1,\cdots,X_n}(x_1,\cdots,x_n)}{|A|}\right|_{\mathbf{x}=A^{-1}\mathbf{z}} = \frac{f_{\mathbf{x}}(A^{-1}\mathbf{z})}{|A|}$$

Ex 4.36 linear trafo of jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(x^2 - 2\rho xy + y^2\right) / 2\left(1-\rho^2\right)}$$

$$\begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \triangleq A \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow |A| = 1$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1} \begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} V - W \\ V + W \end{bmatrix}$$

$$f_{V,W}(v,w) = f_{X,Y}\left(\frac{v-w}{\sqrt{2}}, \frac{v+w}{\sqrt{2}}\right)$$

Ex 4.36 linear trafo of jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(x^2-2\rho xy+y^2\right) / 2\left(1-\rho^2\right)}$$

$$\frac{x^2-2\rho xy+y^2}{2\left(1-\rho^2\right)} \rightarrow \frac{\frac{(v-w)^2}{2} - 2\rho\frac{(v-w)(v+w)}{2} + \frac{(v+w)^2}{2}}{2\left(1-\rho^2\right)}$$

$$= \frac{v^2(1-\rho)+w^2(1+\rho)}{2(1-\rho)(1+\rho)}$$

$$f_{V,W}(v,w) = f_{X,Y}\left(\frac{v-w}{\sqrt{2}}, \frac{v+w}{\sqrt{2}}\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left\{\frac{v^2}{2(1+\rho)}+\frac{w^2}{2(1-\rho)}\right\}}$$

V and *W* are independent, Gaussian, zero-mean, r.v.'s with variance 1+ρ and 1-ρ respectively

PDF of general transformations

$$V = g_1(X,Y) \text{ and } W = g_2(X,Y)$$

assume ⇓ invertibility

$$x = h_1(v,w) \text{ and } \mathrm{y} = h_2(v,w)$$

find equivalent event of infinitesimal rectangles

$$f_{V,W}(v,w) = \frac{f_{X,Y}\left(h_1(v,w),h_2(v,w)\right)}{\left|\dfrac{dP}{dxdy}\right|} = \frac{f_{X,Y}\left(h_1(v,w),h_2(v,w)\right)}{\underbrace{\left|J(x,y)\right|}_{\text{Jacobian of trafo}}}$$

$$= f_{X,Y}\left(h_1(v,w),h_2(v,w)\right)\left|J(v,w)\right|$$

Jacobian definition & property

$$J(x,y) = \det\begin{bmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix} \qquad J(v,w) = \det\begin{bmatrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \end{bmatrix}$$

$$|J(v,w)| = \frac{1}{|J(x,y)|}$$

$$f_{V,W}(v,w) = \frac{f_{X,Y}\left(h_1(v,w), h_2(v,w)\right)}{|J(x,y)|}$$

$$= f_{X,Y}\left(h_1(v,w), h_2(v,w)\right)|J(v,w)|$$

sum of such terms when there are multiple solutions

Ex 4.37 Rayleigh r.v.

- Let *X* and *Y* be zero-mean, unit-variance, independent Gaussian r.v.'s. Find joint PDF of *V* and *W*, when $V = \left(X^2 + Y^2\right)^{\frac{1}{2}}; W = \angle(X,Y)$

inverse trafo

Cartesian→polar

$$x = v\cos w \qquad y = v\sin w$$

$$J(v,w) = \det\begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix} = \begin{vmatrix} \cos w & -v\sin w \\ \sin w & v\cos w \end{vmatrix} = v$$

$$f_{V,W}(v,w) = f_{X,Y}\left(h_1(v,w), h_2(v,w)\right)\left|J(v,w)\right|$$

$$= \frac{v}{2\pi} e^{-\left[v^2\cos^2(w) + v^2\sin^2(w)\right]/2} = \underbrace{\frac{1}{2\pi}}_{U[0,2\pi)}\underbrace{v e^{-v^2/2}}_{Rayleigh} ; v \geq 0, 0 \leq w < 2\pi$$

independent

$$\mathbf{X} = \left(X_1, \cdots, X_n\right)$$

Finding joint PDF for *n* functions of *n* r.v.'s

$$Z_1 = g_1\left(\mathbf{X}\right), Z_2 = g_2\left(\mathbf{X}\right), \cdots, Z_n = g_n\left(\mathbf{X}\right)$$

$$z_1 = g_1\left(\mathbf{x}\right), z_2 = g_2\left(\mathbf{x}\right), \cdots, z_n = g_n\left(\mathbf{x}\right)$$

unique ⇓ inverse

$$x_1 = h_1\left(\mathbf{z}\right), x_2 = h_2\left(\mathbf{z}\right), \cdots, x_n = h_n\left(\mathbf{z}\right)$$

$$f_{Z_1,\cdots,Z_n}\left(z_1,\cdots,z_n\right) = \frac{f_{X_1,\cdots,X_n}\left(h_1\left(\mathbf{z}\right),\cdots,h_n\left(\mathbf{z}\right)\right)}{\left|J\left(x_1,\cdots,x_n\right)\right|}$$

$$= f_{X_1,\cdots,X_n}\left(h_1\left(\mathbf{z}\right),\cdots,h_n\left(\mathbf{z}\right)\right)\left|J\left(z_1,\cdots,z_n\right)\right|$$

$$J\left(x_1,\cdots,x_n\right) = \det\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}; J\left(z_1,\cdots,z_n\right) = \det\begin{bmatrix} \frac{\partial h_1}{\partial z_1} & \cdots & \frac{\partial h_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial z_1} & \cdots & \frac{\partial h_n}{\partial z_n} \end{bmatrix}$$

Expected value of functions of r.v.'s

similar to expected value of a function of one r.v.

$$E[Z]=\begin{cases}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)\,dxdy & X,Y \text{ jointly continuous}\\ \sum\limits_{i}\sum\limits_{n} g(x_i,y_n)\,p_{X,Y}(x_i,y_n) & X,Y \text{ discrete}\end{cases}$$

above generalizes to function of *n* r.v.'s

in the obvious way: add more integrals, sums, and arguments

Ex 4.39 expected value of sum of r.v.'s

$$Z = X + Y$$

$$E[Z] = E[X+Y]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x'+y')f_{X,Y}(x',y')\,dx'dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x'f_{X,Y}(x',y')\,dx'dy' + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y'f_{X,Y}(x',y')\,dx'dy'$$

$$= \int_{-\infty}^{\infty}x'f_X(x')\,dx' + \int_{-\infty}^{\infty}y'f_Y(y')\,dy' = E[X]+E[Y]$$

generalized: *E* sum of r.v.'s is sum of *E*'s

no independence needed

Ex 4.40

X and Y are independent r.v.'s

$g(X,Y) = g_1(X)\,g_2(Y)$

2 assumptions!

$$E\left[g(X,Y)\right] = E\left[g_1(X)\,g_2(Y)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x')\,g_2(y')\,f_X(x')\,f_Y(y')\,dx'\,dy'$$

$$= \left\{\int_{-\infty}^{\infty} g_1(x')\,f_X(x')\,dx'\right\}\left\{\int_{-\infty}^{\infty} g_2(y')\,f_Y(y')\,dy'\right\}$$

$$= E\left[g_1(X)\right]E\left[g_2(Y)\right]$$

generalizes directly

Correlation of 2 r.v.'s

jk^{th} joint moment of X and Y

$$E\left[X^jY^k\right]=\begin{cases}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}x^jy^kf_{X,Y}\left(x,y\right)dxdy & X,Y \text{ jointly continuous}\\ \sum\limits_i\sum\limits_n x_i^jy_n^kp_{X,Y}\left(x_i,y_n\right) & X,Y \text{ discrete}\end{cases}$$

j=0→ moments of Y k=0→ moments of X

$j=1,k=1\rightarrow E\left[XY\right]$: correlation of X and Y

if $E\left[XY\right]=0$ we say that X and Y are orthogonal

Covariance of 2 r.v.'s

*jk*th central moment of *X* and *Y*: $E\left[\underbrace{\left(X-E[X]\right)}_{}^{j}\underbrace{\left(Y-E[Y]\right)}_{}^{k}\right]$

centered r.v.'s

j=2, *k*=0 → VAR(*X*) *j*=0, *k*=2 → VAR(*Y*)

$$j=1, k=1 \rightarrow E\left[\left(X-E[X]\right)\left(Y-E[Y]\right)\right] \triangleq COV(X,Y)$$

$$\begin{aligned} COV(X,Y) &= E\left[XY - XE[Y] - YE[X] + E[X]E[Y]\right] \\ &= E[XY] - 2E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \\ &= E[XY] \text{ if } E[X]=0 \text{ \&/or } E[Y]=0 \end{aligned}$$

Ex 4.41 COV of independent r.v.'s

$$
\begin{aligned}
COV(X,Y) &= E\Big[\big(X-E[X]\big)\big(Y-E[Y]\big)\Big] \\
&= E\Big[\big(X-E[X]\big)\Big]E\Big[\big(Y-E[Y]\big)\Big] \\
&= 0
\end{aligned}
$$

independence

pairs of independent r.v.'s have COV zero

Correlation coefficient of *X* and *Y*

$$\rho_{X,Y} \triangleq \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{E\left[\left(X - E[X]\right)\left(Y - E[Y]\right)\right]}{\sigma_X \sigma_Y}$$

$$= E\left[\left(\frac{X - E[X]}{\sigma_X}\right)\left(\frac{Y - E[Y]}{\sigma_Y}\right)\right]$$

correlation of zero-mean, unit-variance normalized r.v.'s

X and *Y* are said to be uncorrelated if $\rho_{X,Y} = 0$

X and *Y* independent $\Rightarrow$ COV(*X*,*Y*)=0 $\Rightarrow$ $\rho_{X,Y}$=0 $\Rightarrow$ uncorrelated

Ex 4.18

$\rho_{X,Y}$=0, with *X* and *Y* jointly Gaussian $\Rightarrow$ *X* and *Y* independent

Keep them straight

$E[XY] = 0: \ X \text{ and } Y \text{ are orthogonal}$

correlation is zero: orthogonal

$\rho_{X,Y} = 0: \ X \text{ and } Y \text{ are uncorrelated}$

covariance is zero: uncorrelated

independence $\Rightarrow$ uncorrelated

Gaussian + uncorrelated $\Rightarrow$ independence

correlation ~ linear dependence

dependence can be other than linear

Jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) =$$

$$= \frac{\exp\left\{\frac{-1}{2\left(1-\rho_{X,Y}^2\right)}\left[\left(\frac{x-m_x}{\sigma_x}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right)+\left(\frac{y-m_y}{\sigma_y}\right)^2\right]\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{X,Y}^2}}$$

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

PDF is centered at (m_x, m_y)

PDF bell-shape depends on σ_x, σ_y, $\rho_{X,Y}$

Sums of r.v.'s

Let $X_1, X_2, \cdots, X_n$ be a sequence of r.v.'s

$$S_n = \sum_{i=1}^{n} X_i \quad \Longrightarrow \quad E[S_n] = \sum_{i=1}^{n} E[X_i]$$

Ex 5.1 $Z = X + Y \rightarrow E[Z] = E[X] + E[Y]$

$$VAR(Z) = E\left[(Z - E[Z])^2\right] = E\left[(X + Y - E[X] - E[Y])^2\right]$$

$$= E\left[\{(X - E[X]) + (Y - E[Y])\}^2\right]$$

$$= E\left[(X - E[X])^2 + (Y - E[Y])^2 + 2(X - E[X])(Y - E[Y])\right]$$

$$= VAR(X) + VAR(Y) + 2COV(X, Y)$$ not sum of individual *VARs*

Variance of sum of r.v.'s

variance of sum of r.v.'s, in general:

$$VAR\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} VAR(X_i) + \sum_{i=1}^{n}\sum_{j=1,\neq i}^{n} COV(X_i, X_j)$$

variance of sum $\neq$ sum of variances

special case X_i, X_j independent r.v.'s $\Rightarrow COV(X_i, X_j) = 0$

$$\Rightarrow VAR\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} VAR(X_i) \text{ if } X_i \text{ are independent r.v.'s}$$

Ex 5.2 sum of *n* i.i.d. r.v's

$$S_n = \sum_{i=1}^{n} X_i \text{ ; with } X_i \text{ independent, identically distributed r.v.'s}$$

$$X_i \sim \left(\mu, \sigma^2\right)$$

$$E\left[S_n\right] = \sum_{i=1}^{n} E\left[X_i\right] = n\mu$$

$$VAR\left(S_n\right) = \sum_{i=1}^{n} VAR\left(X_i\right) = nVAR\left(X_i\right) = n\sigma^2$$

Central Limit Theorem

Let $X_1, X_2, \cdots$ be a sequence of iid r.v.'s with finite mean $E[X] = \mu$

and finite variance σ^2, and let $S_n = \sum_{j=1}^{n} X_j$; this $\Rightarrow S_n \sim \left(n\mu, n\sigma^2\right)$

- as *n* becomes large…the CDF of a properly normalized S_n approaches the CDF of a Gaussian r.v. (Central Limit Theorem)

CLT

Let S_n be the sum of n iid r.v.'s with finite mean $E[X] = \mu$ and finite variance σ^2, and let Z_n be the zero-mean, unit variance r.v.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \text{ then } \lim_{n\to\infty} P[Z_n \le z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx$$

$S_n = \sum_{j=1}^{n} X_j$ the X_j can have any distribution

Ex 5.11

- Orders in a restaurant are iid r.v.'s ~(μ=\$8,σ=\$2). Estimate the probability that the first 100 customers spend a total of more than \$840. Estimate the probability that they spend a total of between \$780 and \$820.

Let X_k be the expenditure of the k^{th} customer,

$$\text{then } S_{100} = \sum_{j=1}^{100} X_j \sim \left(n\mu = 800, n\sigma^2 = 400\right)$$

$$\Longrightarrow \quad Z_{100} = \frac{S_{100} - n\mu}{\sigma\sqrt{n}} = \frac{S_{100} - 800}{2(10)}$$

Ex 5.11 cont'd

$$Z_{100} = \frac{S_{100} - n\mu}{\sigma\sqrt{n}} = \frac{S_{100} - 800}{2(10)}$$

$$P[S_{100} > 840] = P\left[Z_{100} > \frac{840-800}{2(10)}\right] = Q(2) \simeq 2.28(10^{-2})$$

$$P[780 \le S_{100} \le 820] = P\left[\frac{780-800}{2(10)} \le Z_{100} \le \frac{820-800}{2(10)}\right]$$

$$= 1 - 2Q(1) \simeq 1 - 3.18(10^{-1}) = 0.682$$

Note: these are estimated probabilities