Multiple r.v.'s

- Several r.v.'s at a time
  - *Measuring different quantities simultaneously*
    - Engine oil pressure, RPM, generator voltage
  - *Repeated measurement of the same quantity*
    - Sampling a waveform, such as EEG, speech

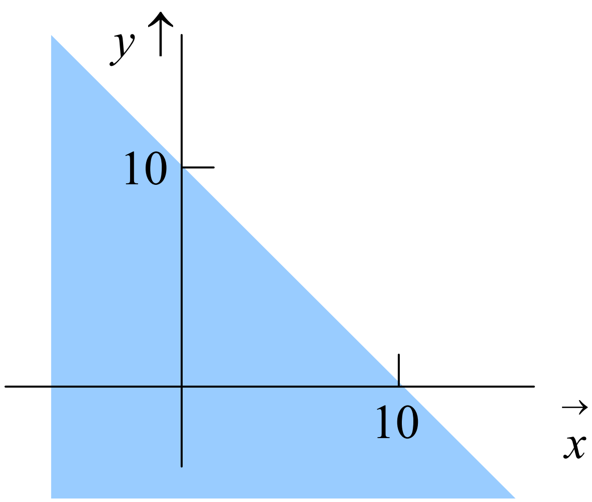
↓

- Joint behavior of two or more r.v.'s
  - *Independence of sets of r.v.'s*
  - *Correlation if not independent*

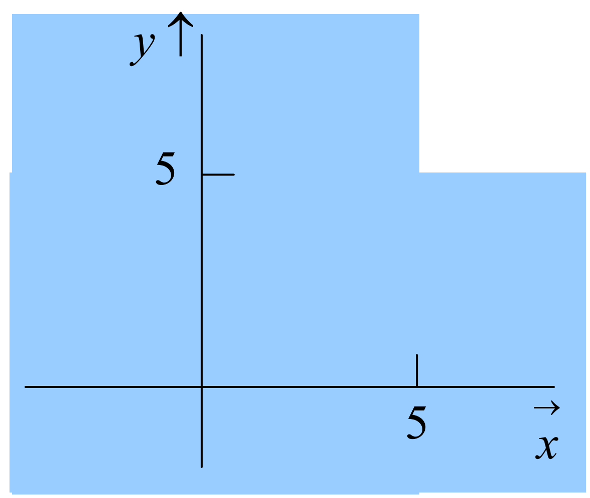
Events & probabilities

- Each event involving an *n*-dimensional r.v. **X**=(X_1, X_2,…, X_n) has a corresponding region in an *n*-dimensional real space
  - *E.g. 2-D r.v.* **X**=(X,Y)

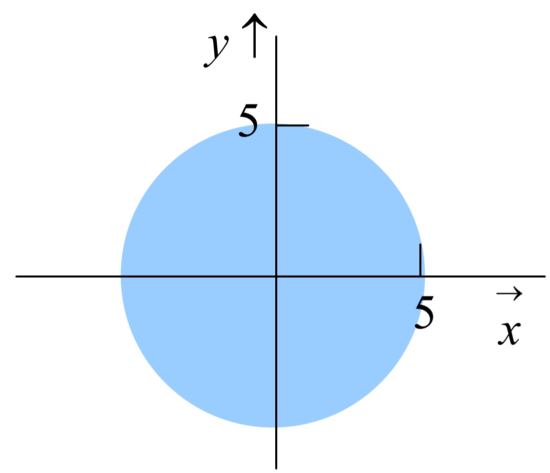
$$A = \{X + Y \le 10\}$$



$$B = \{\min(X, Y) \le 5\}$$



$$C = \{X^2 + Y^2 \le 25\}$$



Independence

- Intuitively, if r.v.'s X and Y are "independent," then events that involve only X should be independent of events that involve only Y. In other words, if A_1 is any event that involves X only and A_2 is any event that involves Y only, then

$$P[X \in A_1, Y \in A_2] = P[X \in A_1]P[Y \in A_2]$$

- In general, n r.v.'s are independent if

$$P[X_1 \in A_1, \cdots, X_n \in A_n] = P[X_1 \in A_1] \cdots P[X_n \in A_n]$$
$$\text{where } A_k \text{ is an event that involves } X_k \text{ only}$$

if r.v.'s are independent, knowing the probabilities of the r.v.'s in isolation suffices to specify probabilities of joint events

Pairs of discrete r.v.'s

vector r.v. $\mathbf{X} = (X, Y)$

$$S = \{(x_j, y_k),\ j = 1, 2, \cdots, k = 1, 2, \cdots\}$$

joint probability mass function:

$$p_{X,Y}(x_j, y_k) = P\left[\{X = x_j\} \cap \{Y = y_k\}\right]$$

$$\triangleq P\left[X = x_j, Y = y_k\right] \quad j = 1, 2, \cdots \quad k = 1, 2, \cdots$$

gives probability of occurrence of pairs (x_j, y_k)

For any event A: $P[\mathbf{X} \in A] = \sum_{(x_j, y_k)} \sum_{\in A} p_{X,Y}(x_j, y_k)$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k) = P[S] = 1$$

Marginal PMF's

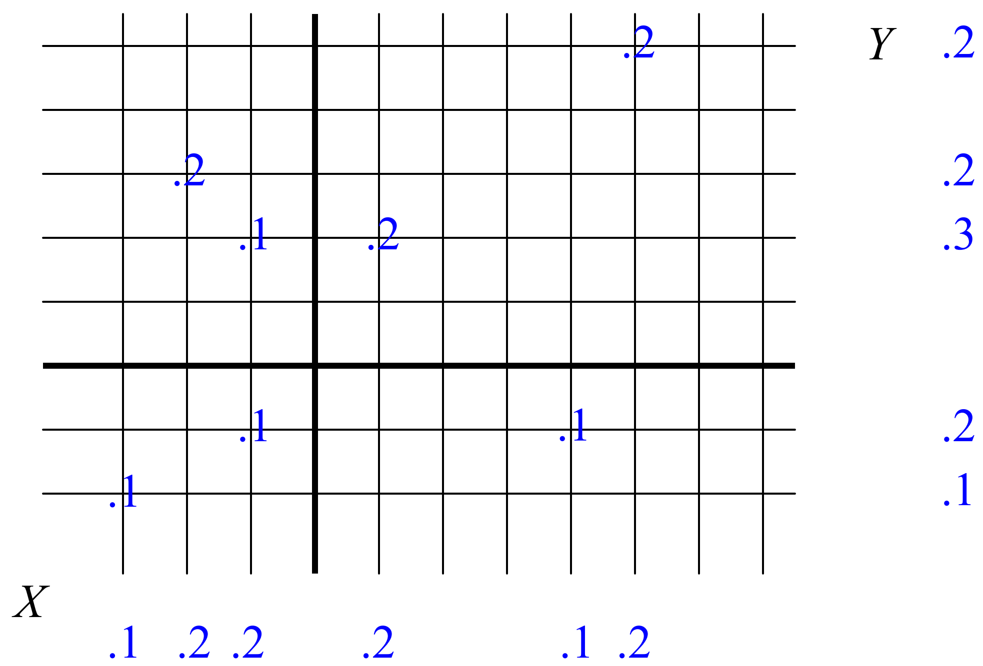
$$p_X\left(x_j\right) = p_{X,Y}\left(x_j, \text{any } y_k\right)$$

$$= P\left[\left\{\left\{X = x_j\right\} \cap \left\{Y = y_1\right\}\right\} \cup \left\{\left\{X = x_j\right\} \cap \left\{Y = y_2\right\}\right\} \cup \cdots\right]$$

$$= \sum_{k=1}^{\infty} p_{X,Y}\left(x_j, y_k\right)$$

$$p_Y\left(y_k\right) = \sum_{j=1}^{\infty} p_{X,Y}\left(x_j, y_k\right)$$

marginal PMF's are insufficient – in general - for specifying joint PMF



Ex 4.6 tossing loaded dice

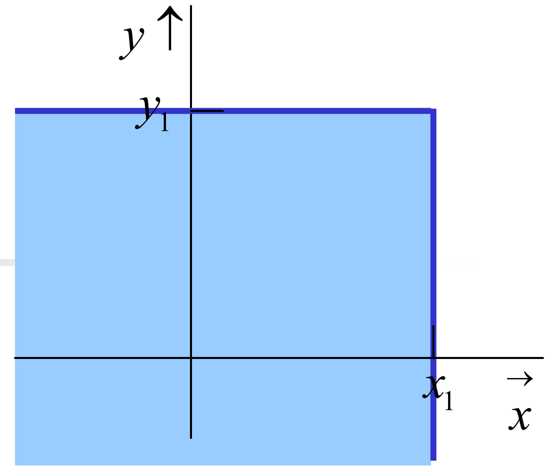
joint PMF in units of $\frac{1}{42}$

$j \backslash k$	1	2	3	4	5	6		
1	2	1	1	1	1	1	7	1
2	1	2	1	1	1	1	7	1
3	1	1	2	1	1	1	7	1
4	1	1	1	2	1	1	7	1
5	1	1	1	1	2	1	7	1
6	1	1	1	1	1	2	7	1
	7	7	7	7	7	7	$/42$	
	1	1	1	1	1	1		$/6$

marginals give no clue

Joint CDF of *X* and *Y*

- Basic building block:



semi-infinite rectangle: $\left\{(x,y):\{x \le x_1\} \cap \{y \le y_1\}\right\}$

joint cumulative distribution function of *X* and *Y*:

$$F_{X,Y}(x_1,y_1) = P\left[\{x \le x_1\} \cap \{y \le y_1\}\right] = P\left[X \le x_1, Y \le y_1\right]$$

product-form event

long-term proportion of times in which the outcome (*x*,*y*) falls in the blue rectangle, or the probability mass in it

properties of the joint CDF

i. $F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2) \quad x_1 \le x_2,\, y_1 \le y_2$

non-decreasing in NE direction

ii. $F_{X,Y}(-\infty,y_1) = F_{X,Y}(x_2,-\infty) = 0$

neither *X* nor *Y* takes on values <-∞

iii. $F_{X,Y}(\infty,\infty) = 1$

X and *Y* can take on only values <∞

iv. $F_X(x) = F_{X,Y}(x,\infty) = P[X \le x, Y < \infty] = P[X \le x]$

$F_Y(y) = F_{X,Y}(\infty,y) = P[X < \infty, Y \le y] = P[Y \le y]$

marginal cumulative distribution functions

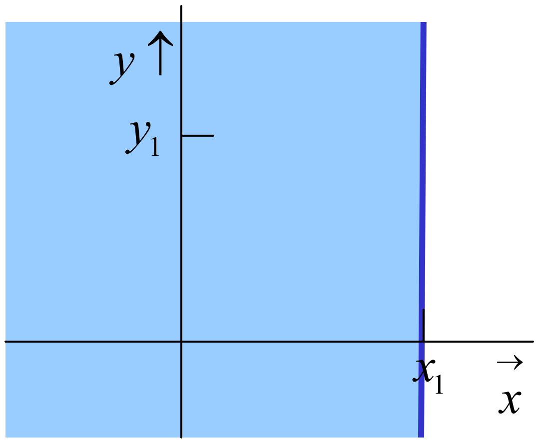
v. $\lim_{x \to a^+} F_{X,Y}(x,y) = F_{X,Y}(a,y)$

$\lim_{y \to b^+} F_{X,Y}(x,y) = F_{X,Y}(x,b)$

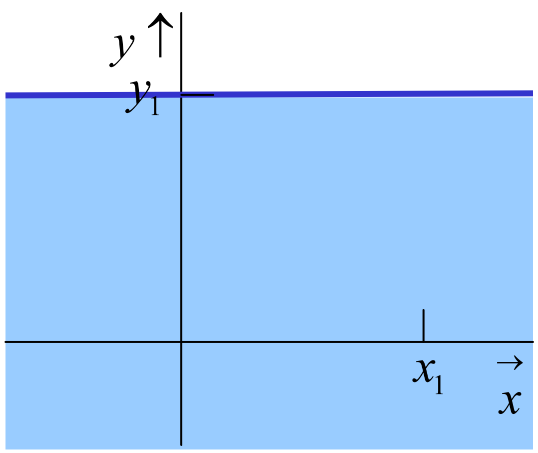
continuous from N & E

Ex 4.8 marginals for continuous joint r.v.'s

$$\text{Given } F_{X,Y}(x,y) = \left(1-e^{-\alpha x}\right)\left(1-e^{-\beta y}\right)u(x)u(y)$$



$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \left(1-e^{-\alpha x}\right)u(x)$$



$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = \left(1-e^{-\beta y}\right)u(y)$$

marginal CDFs are exponential distributions with parameters α and β respectively

Joint CDF in terms of joint PDF & v.v.

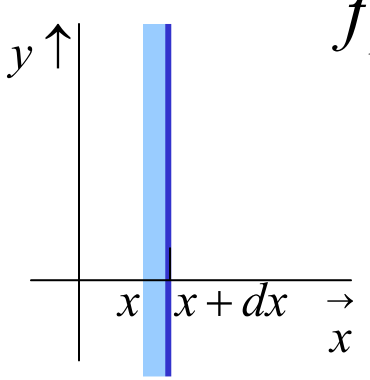
$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x',y')\,dx'\,dy'$$

if jointly continuous r.v.'s

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

if the CDF is discontinuous
- or the partial derivatives are discontinuous -
then the joint PDF does not exist

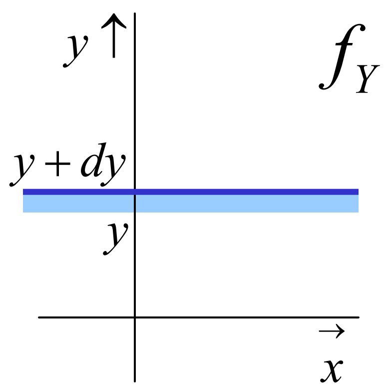
Marginal PDF's from joint PDF



$$f_X(x) = \frac{d}{dx} F_{X,Y}(x,\infty) = \frac{d}{dx} \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{X,Y}(x', y') dy' \right\} dx'$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x, y') dy'$$

concentrating mass on x-axis



$$f_Y(y) = \frac{d}{dy} F_{X,Y}(\infty, y) = \int_{-\infty}^{\infty} f_{X,Y}(x', y) dx'$$

concentrating mass on y-axis

marginal PDF's are obtained by integrating out the other variable(s)

$$f_{X,Y}(x,y) = \begin{cases} 1 & (x,y) \in [0,1]\times[0,1] \\ 0 & o.w. \end{cases}$$

Ex 4.10 CDF for jointly uniform r.v.'s

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x',y')dx'dy'$$

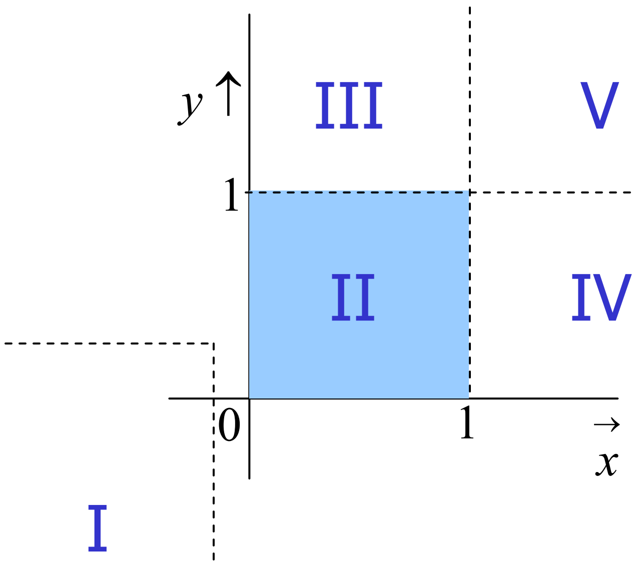
$$= 0 \quad \text{for } (x,y) \in I$$

$$= \int_0^x\int_0^y 1dx'dy' = xy \quad \text{for } (x,y) \in II$$

$$= \int_0^x\int_0^1 1dx'dy' = x \quad \text{for } (x,y) \in III$$

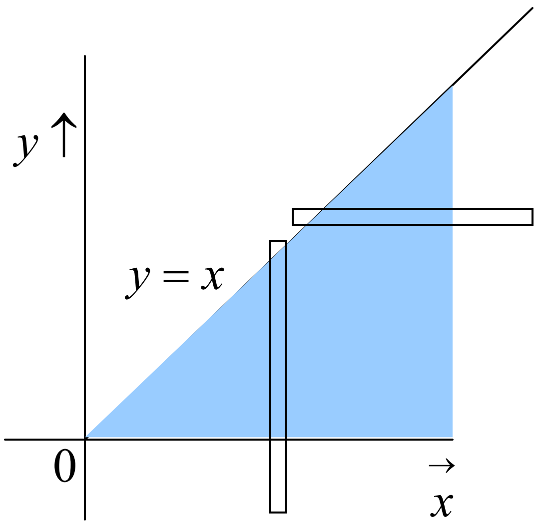
$$= \int_0^1\int_0^y 1dx'dy' = y \quad \text{for } (x,y) \in IV$$

$$= \int_0^1\int_0^1 1dx'dy' = 1 \quad \text{for } (x,y) \in V$$



Ex 4.11 finding marginal PDF's

$$f_{X,Y}(x,y)=\begin{cases} ce^{-x}e^{-y} & 0\le y\le x<\infty \\ 0 & o.w. \end{cases}$$



$$1=\int_0^\infty\int_0^x ce^{-x}e^{-y}\,dy\,dx=\int_0^\infty ce^{-x}\int_0^x e^{-y}\,dy\,dx$$

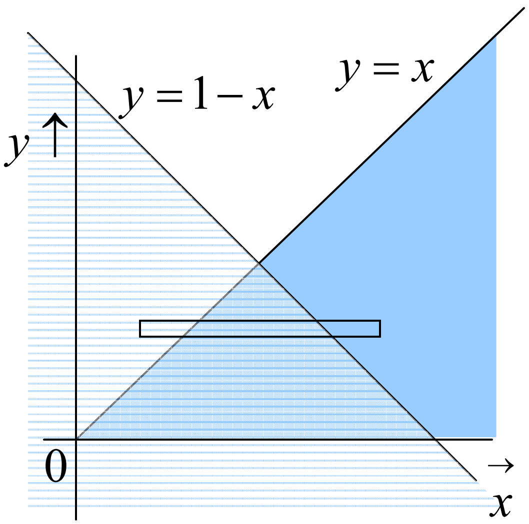
$$=\int_0^\infty ce^{-x}\frac{e^{-x}-1}{-1}\,dx=\frac{c}{2}\Rightarrow c=2$$

$$f_X(x)=\int_0^x 2e^{-x}e^{-y}\,dy=2e^{-x}\left(1-e^{-x}\right)\quad 0\le x<\infty$$

$$f_Y(y)=\int_y^\infty 2e^{-x}e^{-y}\,dx=2e^{-y}\left(e^{-y}\right)=2e^{-2y}\quad 0\le y<\infty$$

are these PDF's?

Ex 4.12 Find P[X+ Y≤1] in EX 4.11



$$P[X+Y\le 1]=\int_0^{0.5}\int_y^{1-y} 2e^{-x}e^{-y}\,dxdy$$

$$=\int_0^{0.5} 2e^{-y}\int_y^{1-y} e^{-x}\,dxdy$$

$$=\int_0^{0.5} 2e^{-y}\left[e^{-y}-e^{-(1-y)}\right]dy$$

$$=-e^{-1}+1-e^{-1}=1-2e^{-1}$$

Ex 4.13 Find marginal PDFs of jointly Gaussian PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(x^2-2\rho xy+y^2\right)/2\left(1-\rho^2\right)}$$

$$f_X(x) = \frac{e^{-x^2/2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left(-2\rho xy+y^2\right)/2\left(1-\rho^2\right)} dy$$

completing the square

$$= \frac{e^{-x^2\left(1-\rho^2\right)/2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left(\rho^2 x^2-2\rho xy+y^2\right)/2\left(1-\rho^2\right)} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-(y-\rho x)^2/2\left(1-\rho^2\right)} dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = N(0,1)$$

it's a PDF

symmetry

$$f_Y(y) = f_X(x)$$

Independence of two r.v.'s

discrete r.v.'s X and Y are independent

$\Updownarrow$ iff

joint PMF is product of marginal PMFs

$$\forall x_j, y_k$$

Ex 4.15 another look at Ex 4.6 loaded dice

joint PMF in units of $\frac{1}{42}$

marginals give no clue

$j \backslash k$	1	2	3	4	5	6	
1	2	1	1	1	1	1	1
2	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
4	1	1	1	2	1	1	1
5	1	1	1	1	2	1	1
6	1	1	1	1	1	2	1
	1	1	1	1	1	1	

(marginals in units of $/6$)

not independent

$$p_{X,Y}\left(x_j, y_k\right) \neq p_X\left(x_j\right) p_Y\left(y_k\right) \;\; \forall x_j, y_k$$

Independence of two r.v.'s in general

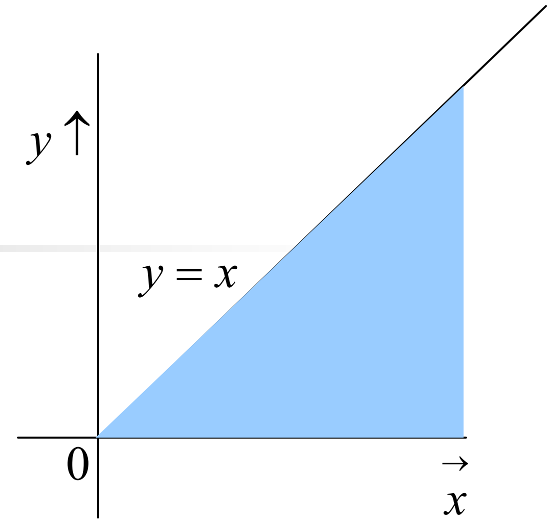
⇕ IFF

$$p_{X,Y}\left(x_j, y_k\right) = p_X\left(x_j\right) p_Y\left(y_k\right) \;\; \forall x_j, y_k$$ discrete

$$F_{X,Y}\left(x, y\right) = F_X\left(x\right) F_Y\left(y\right) \;\; \forall x, y$$

$$f_{X,Y}\left(x, y\right) = f_X\left(x\right) f_Y\left(y\right) \;\; \forall x, y$$ jointly continuous

Ex 4.17 back to Ex 4.11



joint $$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & o.w. \end{cases}$$

marginal PDFs

$$f_X(x) = \int_0^x 2e^{-x}e^{-y}dy = 2e^{-x}\left(1-e^{-x}\right) \quad 0 \le x < \infty$$

$$f_Y(y) = \int_y^\infty 2e^{-x}e^{-y}dx = 2e^{-y}\left(e^{-y}\right) = 2e^{-2y} \quad 0 \le y < \infty$$

product? $2e^{-x}e^{-y} \ne 2e^{-x}\left(1-e^{-x}\right)2e^{-2y}$

X and *Y* are not independent

Ex 4.18 back to Ex 4.13

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\left(x^2-2\rho xy+y^2\right)/2\left(1-\rho^2\right)}$$

$$f_X(x)=\frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad f_Y(y)=\frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$f_X(x)f_Y(y)=\frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}=\frac{e^{-\left(x^2+y^2\right)/2}}{2\pi}$$

$$\equiv \ \text{iff}\ \rho=0$$

Conditional probability

$$P[Y \in A \mid X = x] = \frac{P[Y \in A, X = x]}{P[X = x]}$$

X discrete

$$F_Y(y \mid x_k) = \frac{P[Y \le y, X = x_k]}{P[X = x_k]} \text{ for } P[X = x_k] > 0$$

if derivative exists

$$f_Y(y \mid x_k) = \frac{d}{dy} F_Y(y \mid x_k)$$

$$P[Y \in A \mid X = x_k] = \int_{y \in A} f_Y(y \mid x_k) dy$$

independence

$$F_Y(y \mid x) = F_Y(y)$$

$$f_Y(y \mid x) = f_Y(y)$$

Conditional probability

$$P[Y \in A \mid X = x] = \frac{P[Y \in A, X = x]}{P[X = x]}$$

X and *Y* discrete

$$p_Y(y_j \mid x_k) = P[Y = y_j \mid X = x_k] = \frac{P[X = x_k, Y = y_j]}{P[X = x_k]}$$

δ functions
~ PMF weights

$$= \begin{cases} \dfrac{p_{X,Y}(x_k, y_j)}{p_X(x_k)} \text{ for } P[X = x_k] = p_X(x_k) > 0 \\ 0 \text{ for } P[X = x_k] = p_X(x_k) = 0 \end{cases}$$

in general

$$P[Y \in A \mid X = x_k] = \sum_{y_j \in A} p_Y(y_j \mid x_k) = \sum_{y_j \in A} p_Y(y_j)$$

if also independent

Ex 4.20 back to Ex 4.14

- X is the input and Y the output of a communication channel. $P[Y<0|X=+1]$?

$$f_Y(y|1) \sim U[-1,3] \Longrightarrow P[Y<0 \mid X=+1] = \int_{-1}^{0} \frac{1}{4} dy = \frac{1}{4}$$

integrate over event

Conditional CDF of *Y* given *X*=x

$$F_Y(y \mid x) \triangleq \lim_{h \to 0} F_Y(y \mid x < X \le x+h)$$

$$= \frac{P[Y \le y, x < X \le x+h]}{P[x < X \le x+h]} = \frac{\int_{-\infty}^{y} \int_{x}^{x+h} f_{X,Y}(x', y') dx' dy'}{\int_{x}^{x+h} f_X(x') dx'}$$

$$\simeq \frac{h \int_{-\infty}^{y} f_{X,Y}(x, y') dy'}{h f_X(x)} = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y') dy'}{f_X(x)}$$

if independent

$$= \int_{-\infty}^{y} f_Y(y') dy' = F_Y(y)$$

Conditional PDF of *Y* given *X*=*x*

$$F_Y(y\,|\,x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')\,dy'}{f_X(x)}$$

$$f_Y(y\,|\,x) = \frac{d}{dy} F_Y(y\,|\,x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

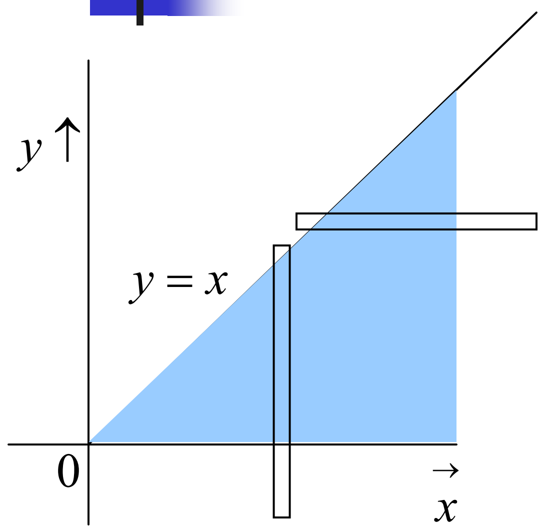
almost Bayes' rule

$$f_Y(y\,|\,x)\,dy = \frac{f_{X,Y}(x,y)\,dxdy}{f_X(x)\,dx}$$

if independent

$$f_Y(y\,|\,x) = f_Y(y)$$

Ex 4.21 back to Ex 4.11



$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & o.w. \end{cases}$$

$$f_X(x) = 2e^{-x}\left(1 - e^{-x}\right) \quad 0 \le x < \infty$$

$$f_Y(y) = 2e^{-2y} \quad 0 \le y < \infty$$

$$f_X(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x}e^{-y}}{2e^{-2y}} = e^{-(x-y)} \;;\; y \le x < \infty$$

$$f_Y(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2e^{-x}\left(1 - e^{-x}\right)} = \frac{e^{-y}}{1 - e^{-x}} \;;\; 0 \le y < x$$