# Probability Model

- Random experiment
  - *Repeating the experiment produces a different (a priori unknown) outcome each time*
    - Tossing a coin
    - Throwing dice
    - Setting a thermostat
    - Turning on a function generator
    - Powering up a flip-flop circuit
    - Dialing a phone number

# Experiment: select ball from urn

- 3 balls, labeled 0, 1, and 2
  - *Blindly select one ball*
  - *Record label*
  - *Replace ball*
  - *Repeat*

a priori unknown!

… 0 2 2 1 0 2 1 1 0 2 0 0 1 2 1 2 0 1 2 0 ? …

a series of outcomes

outcome is element of the sample space $S = \{0, 1, 2\}$

# Relative frequency of outcome

0 2 2 1 0 2 1 1 0 2 0 0 1 2 1 2 0 1 2 0 ? …

$N_k(n)$ : total # times outcome *k* has occurred after *n* trials

$$N_0(20) = 7 \qquad N_1(20) = 6 \qquad N_2(20) = 7$$

$f_k(n) = \dfrac{N_k(n)}{n}$ : Relative frequency (of occurrence) of outcome *k*

statistical regularity: $\lim_{n \to \infty} f_k(n) = p_k$

constant

probability of the outcome

# Changing the experiment (condition)

- 3 balls, labeled 0, 0, and 2
  - *Blindly select one ball*
  - *Record label*
  - *Replace ball*
  - *Repeat*

a priori unknown!

a series of outcomes

... 0 2 2 0 0 2 0 0 0 2 0 0 0 2 0 2 0 0 2 0 ? ...

each outcome is an element of the sample space $S = \{0, 2\}$

# Relative frequency of outcome

0 2 2 0 0 2 0 0 0 2 0 0 0 2 0 2 0 0 2 0 ? …

$N_0(20) = 13$ $N_2(20) = 7$

changed how?

$$f_k(n) = \frac{N_k(n)}{n}:$$ Relative frequency (of occurrence) of outcome *k*

statistical regularity: $\lim_{n\to\infty} f_k(n) = p_k$ probability of the outcome

constant

changed how?

Conditions under which a random experiment is performed determine probabilities of the outcomes of the experiment

# Axioms

- A random experiment has been defined, and a set *S* of all possible outcomes has been identified
- A class of subsets of *S* – called events – has been specified
- Each event *A* has been assigned a number, *P*[*A*], in such a way that the following axioms are satisfied:

1. $0 \leq P[A] \leq 1$
2. $P[S] = 1$
3. $P[A \; or \; B] = P[A] + P[B]$

   when *A* and *B* are events that cannot occur simultaneously.

do these look familiar?

# Random experiments { experimental procedure, measurement/observation

- $E_1$: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.
- $E_2$: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.
- $E_3$: Toss a coin three times and note the sequence of heads and tails.
- $E_4$: Toss a coin three times and note the number of heads.
- $E_5$: Count the number of voice packets containing only silence produced from a group of *N* speakers in a 10ms period.
- $E_6$: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
- $E_7$: Pick a number at random between zero and one.
- $E_8$: Measure the time between two message arrivals at a message center.

# Sample Space

- The sample space *S* of a random experiment is defined as the set of all possible outcomes.
- An outcome or sample point $\zeta$ of a random experiment is a result that cannot be decomposed into other results. $\zeta \in S$
- One and only one outcome occurs when a random experiment is performed.
  - *Outcomes are mutually exclusive – they cannot occur simultaneously*

# Sample Spaces – example experiments

- $E_1$: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

$$S_1 = \{1, 2, ..., 50\}$$

- $E_2$: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

$$S_2 = \{(1,b),(2,b),(3,w),(4,w)\}$$

- $E_3$: Toss a coin three times and note the sequence of heads and tails.

$$S_3 = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

- $E_4$: Toss a coin three times and note the number of heads.

$$S_4 = \{0, 1, 2, 3\}$$

sample spaces are sets

# Sample Spaces

- $E_5$: Count the number of voice packets containing only silence produced from a group of *N* speakers in a 10ms period.

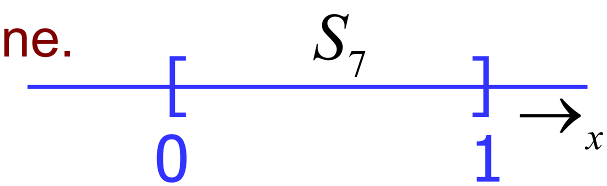
$$S_5 = \{0,1,2,\ldots,N\}$$

- $E_6$: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

$$S_6 = \{1,2,3,\ldots\}$$

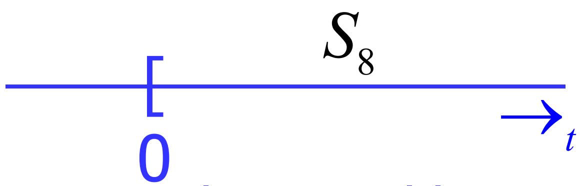
- $E_7$: Pick a number at random between zero and one.

$$S_7 = \{x : 0 \le x \le 1\} = [0,1]$$



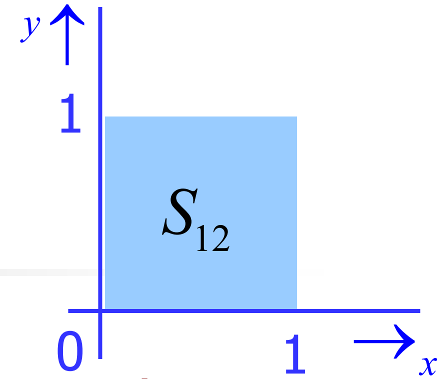

- $E_8$: Measure the time between two message arrivals at a message center.

$$S_8 = \{t : t \ge 0\} = [0,\infty)$$



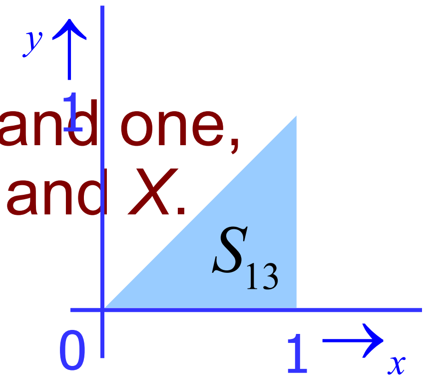

sample spaces visualized as intervals on real line

# Sample Spaces




- $E_{12}$: Pick two numbers at random between zero and one.

$$S_{12} = \{(x, y) : 0 \le x \le 1 \; and \; 0 \le y \le 1\}$$




- $E_{13}$: Pick a number *X* at random between zero and one, then pick a number *Y* at random between zero and *X*.

$$S_{13} = \{(x, y) : 0 \le y \le x \le 1\}$$

- $E_{14}$: A system component is installed at time $t$=0. For $t \geq 0$ let $X(t)$=1 as long as the component is functioning, and let $X(t)$=0 after the component fails.

$$S_{14} = \left\{ X(t) : X(t) = \begin{cases} 1 & 0 \le t < t_0 \\ 0 & t \ge t_0 \end{cases}, \; time \; of \; component \; failure \; t_0 \right\}$$



sample spaces visualized as regions of the plane

# Sample spaces

- Finite, countably infinite, uncountably infinite
- Discrete sample space
  - *If S is countable*
    - Outcomes can be put in 1-to-1 correspondence with the positive integers
- Continuous sample space
  - *If S is not countable*
- Can be multi-dimensional
  - *Can sometimes be written as Cartesian product of other sets*

# Events

- A subset of *S*
  - *Does the outcome satisfy certain conditions?*
    - $E_{10}$: Determine the value of a voltage waveform at time $t_1$.

$$S_{10} = \{v : -\infty < v < \infty\} = (-\infty, \infty)$$

  - *Is the voltage negative?*
- Event *A* occurs iff the outcome of the experiment $\zeta$ is in the subset

iff = if and only if

$$A = \{\zeta : -\infty < \zeta < 0\}$$

# Special events

- Certain event $S$
  - *Consists of all outcomes, hence occurs always*
- Impossible or null event $\varnothing$
  - *Contains no outcomes, hence never occurs*

# Events – example experiments

- $E_1$: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

  "An even-numbered ball is selected" $A_1 = \{2, 4, \ldots, 48, 50\}$

- $E_2$: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

  "The ball is white and even-numbered" $A_2 = \{(4, w)\}$

- $E_3$: Toss a coin three times and note the sequence of heads and tails.

  "The three tosses give the same outcome" $A_3 = \{HHH, TTT\}$

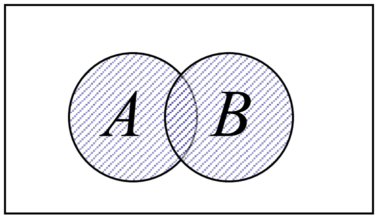
- $E_4$: Toss a coin three times and note the number of heads.

  "The number of heads equals the number of tails" $A_4 = \varnothing$

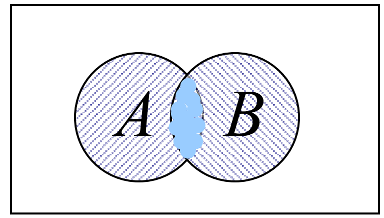
# Set (event) operations

- Union $A \bigcup B$
- Intersection $A \bigcap B$
  - *Mutually exclusive* $A \bigcap B = \varnothing$
- Complement $A^c \; such\, that \; A^c \bigcup A = S \; and \; A^c \bigcap A = \varnothing$
- Implies
  - *all outcomes in A are also outcomes in B* $A \subset B$
- Equal
  - *A and B contain the same outcomes* $A = B$

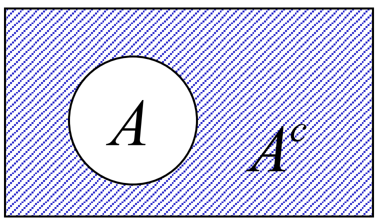
# Venn diagrams

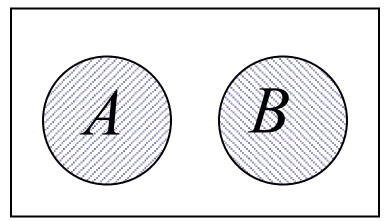



$A \bigcup B$

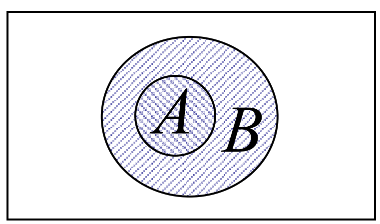



$A \bigcap B$




$A \bigcap B = \varnothing$



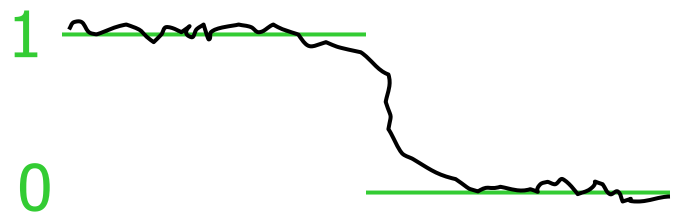

$A \subset B$

# properties

- Commutative $A \bigcup B = B \bigcup A$ $\qquad A \bigcap B = B \bigcap A$
- Associative
$$A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C$$
$$A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$$
- Distributive
$$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$$
$$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$$
- deMorgan's rules
$$(A \bigcap B)^c = A^c \bigcup B^c$$
$$(A \bigcup B)^c = A^c \bigcap B^c$$

# Conditional probability

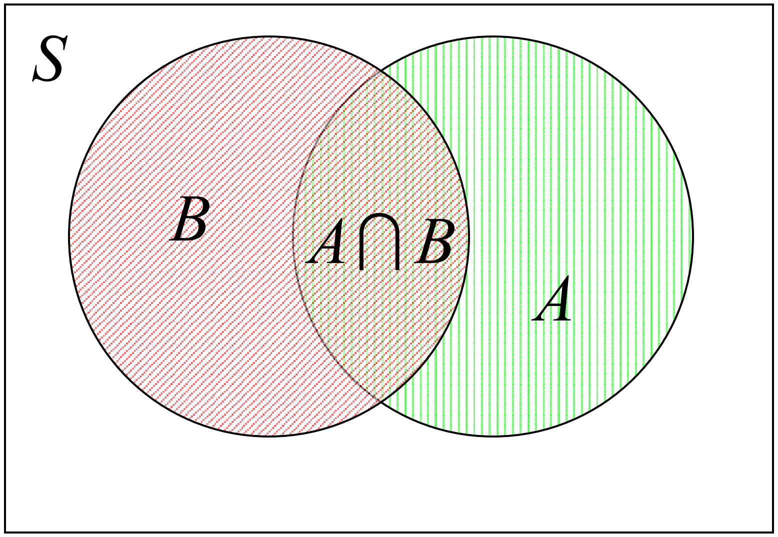
- Are two events *A* and *B* related, in the sense that one tells us something about the other?
  - *We observe/measure something to learn about something else that's not directly measurable*




- Conditional probability of event *A* given that event *B* has occurred

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0$$

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad for\ P[B] > 0$$




event $B$ has occurred $\zeta \in B$

$$S_{\cdot|B} = B$$

$P[A \mid B]$ deals with $\zeta \in A \cap B$

a renormalization of probability for the reduced sample space

# Ex Conditional probability

- Select a ball from an urn containing 2 black balls, labeled 1 and 2, and 2 white balls, labeled 3 and 4

$$S=\{(1,b),(2,b),(3,w),(4,w)\}$$

- Assuming equi-probable outcomes, find *P*[*A*|*B*] and *P*[*A*|*C*], where

  A={(1,b),(2,b)} “black ball selected”

  B={(2,b),(4,w)} “even-numbered ball selected”

  C={(3,w),(4,w)} “number of ball is >2”

$$\left.\begin{aligned} P[A\cap B] &= P[(2,b)] \\ P[A\cap C] &= P[\varnothing]=0 \end{aligned}\right\}$$

$$P[A|B]=\frac{P[A\cap B]}{P[B]}=\frac{.25}{.5}=.5=P[A]$$

$$P[A|C]=\frac{P[A\cap C]}{P[C]}=\frac{0}{.5}=0\neq P[A]$$

knowing B doesn’t help, knowing C does

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]}$$

$$\Downarrow$$

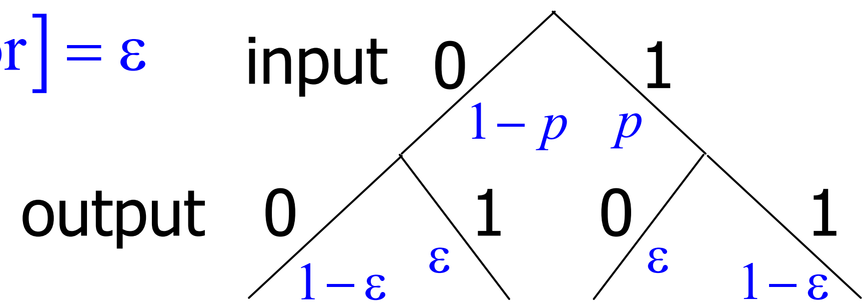
# Ex Conditional probability

$$P[A \cap B] = P[A \mid B]P[B]$$
$$= P[B \mid A]P[A]$$

binary communications channel

$$P[1\,\text{sent}] = p$$

$$P[\text{random decision error}] = \varepsilon$$




$$P[T_0 \cap R_0] = P[R_0 \mid T_0]P[T_0] = (1-\varepsilon)(1-p)$$

$$P[T_0 \cap R_1] = P[R_1 \mid T_0]P[T_0] = \varepsilon(1-p)$$

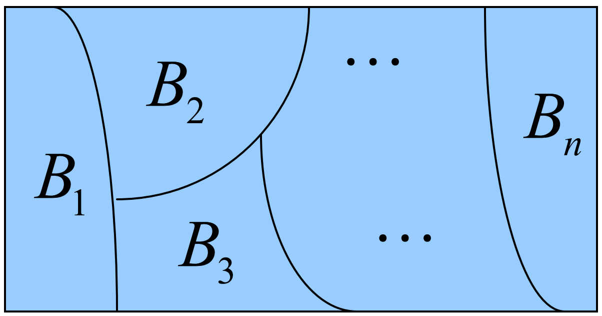
$$P[T_1 \cap R_0] = P[R_0 \mid T_1]P[T_1] = \varepsilon p$$

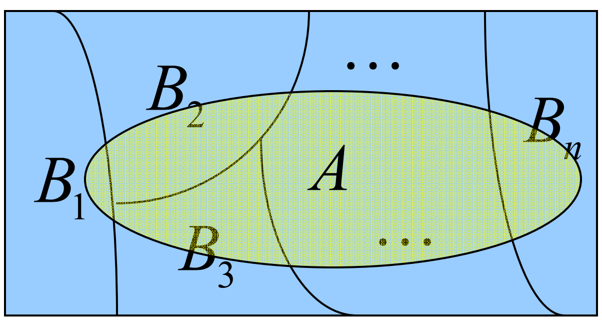
$$P[T_1 \cap R_1] = P[R_1 \mid T_1]P[T_1] = (1-\varepsilon)p$$

# Total probability theorem

a partition of *S:* $\{B_1, B_2, \cdots, B_n\} \ni \bigcup_{i=1}^{n} B_i = S \text{ and } B_i \cap B_j = \varnothing \, \forall i \neq j$

mutually exclusive



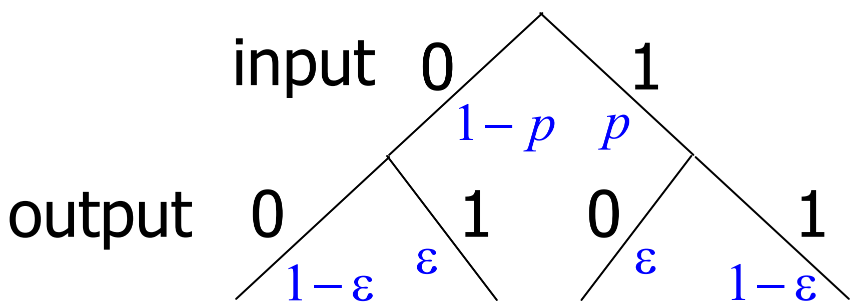

$$A = A \cap S = A \cap \left( \bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

$$P[A] = \sum_{i=1}^{n} P[A \cap B_i] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i]$$

# Ex

$$P[A] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i]$$

Very useful when experiment consists of a sequence of 2 subexperiments




$T_0$ and $T_1$ partition $S$

$$P[R_1] = P[R_1 \mid T_1] P[T_1] + P[R_1 \mid T_0] P[T_0]$$
$$= (1-\varepsilon) p + \varepsilon (1-p)$$

$$P[R_0] = P[R_0 \mid T_1] P[T_1] + P[R_0 \mid T_0] P[T_0]$$
$$= \varepsilon p + (1-\varepsilon)(1-p)$$

# Bayes' Rule

Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of $S$ then

$$P[B_j \mid A] = \frac{P[B_j \cap A]}{P[A]} = \frac{P[A \mid B_j]P[B_j]}{\sum_{k=1}^{n} P[A \mid B_k]P[B_k]}$$

↑

a posteriori probability

the partition corresponds to a priori events (of interest)
*A* corresponds to a measurement/observation

# Ex

- A 1 is received; what is the probability that a 1 was transmitted?

$$P[T_1 \mid R_1] = \frac{P[R_1 \mid T_1]P[T_1]}{P[R_1]} = \frac{(1-\varepsilon)p}{(1-\varepsilon)p+\varepsilon(1-p)} \overset{p=0.5}{=} \frac{(1-\varepsilon)/2}{1/2} = (1-\varepsilon)$$

Bayes' Rule

total probability $P[R_1] = P[R_1 \mid T_1]P[T_1] + P[R_1 \mid T_0]P[T_0]$

$$= (1-\varepsilon)p + \varepsilon(1-p)$$

$$P[T_0 \mid R_1] = \frac{P[R_1 \mid T_0]P[T_0]}{P[R_1]} = \frac{\varepsilon(1-p)}{(1-\varepsilon)p+\varepsilon(1-p)} \overset{p=0.5}{=} \frac{\varepsilon/2}{1/2} = \varepsilon$$

# Independence of events

- Knowledge of the occurrence of event *B* does not alter the probability of some other event *A*
  - *A does not depend on B*

$$P[A] = P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

events *A* and *B* are independent if

$$P[A \cap B] = P[A]P[B]$$

problematic if $P[B] = 0$

$\Downarrow\Uparrow$ $P[B] \neq 0$

$$P[A \mid B] = P[A]$$

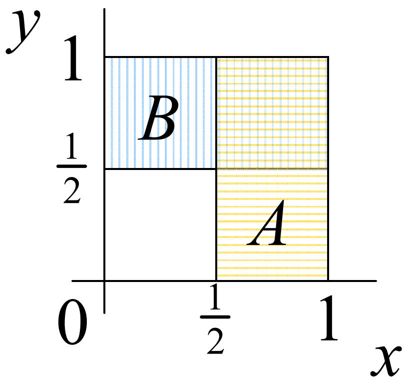
$\Downarrow\Uparrow$ $P[A] \neq 0$

$$P[B \mid A] = P[B]$$

# Ex 2-D continuous

two numbers $x$ and $y$ are selected at random between 0 and 1

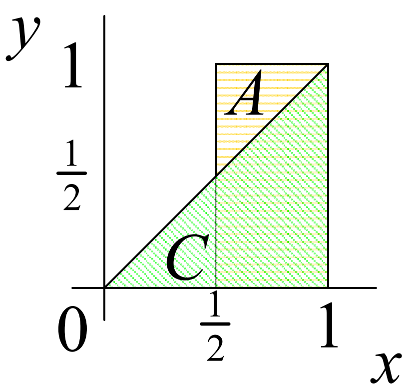
events $A=\{x>0.5\}$, $B=\{y>0.5\}$, $C=\{x>y\}$




$$P[A\,|\,B]=\frac{P[A\cap B]}{P[B]}=\frac{1/4}{1/2}=\frac{1}{2}=P[A]$$

*A* and *B* are independent

ratio of proportions has remained the same




$$P[A\,|\,C]=\frac{P[A\cap C]}{P[C]}=\frac{3/8}{1/2}=\frac{3}{4}\neq\frac{1}{2}=P[A]$$

ratio of proportions has increased

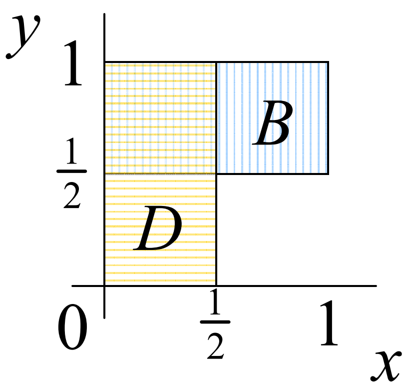
i.e. we gained "knowledge" from measuring *C*

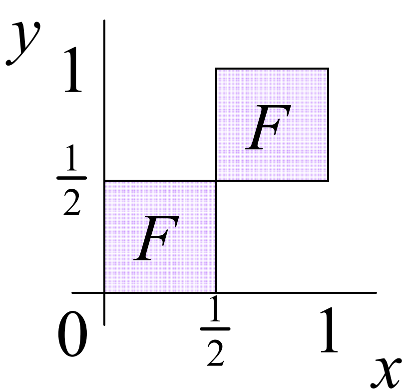
# Ex pairwise independence is not enough

two numbers $x$ and $y$ are selected at random between 0 and 1

events: $B = \{y > 0.5\}$, $D = \{x < 0.5\}$

$$F = \{x < 0.5; y < 0.5\} \cup \{x > 0.5; y > 0.5\}$$




$$P[B \cap D] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[D]$$

$$P[B \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[F]$$

$$P[D \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[D]P[F]$$

pairwise independent

$$P[B \cap D \cap F] = P[\varnothing] = 0$$

$$P[B]P[D]P[F] = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$$

violates $2^{nd}$ condition