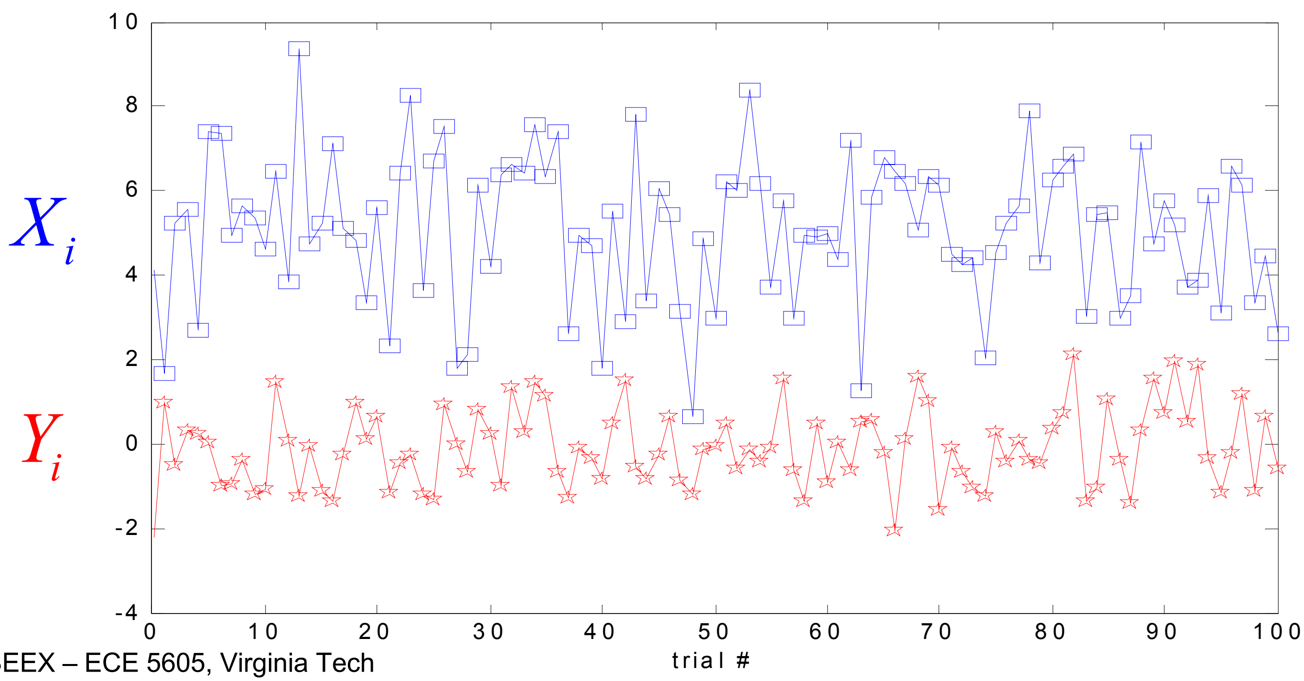
Expected value of r.v.'s

- CDF or PDF are complete (probabilistic) descriptions of the behavior of a random variable. Sometimes we are interested in less information; in a partial characterization.



different center
different spread

Expected value or MEAN

$$E[X] \triangleq \int_{-\infty}^{\infty} t f_X(t)dt \qquad E[X] \triangleq \sum_k x_k p_X(x_k)$$

defined if integral or sum converges absolutely

$$E[|X|] = \int_{-\infty}^{\infty} |t| f_X(t)dt < \infty \qquad E[|X|] = \sum_k |x_k| p_X(x_k) < \infty$$

there are random variables for which the above do not converge
we then say "the mean does not exist"

$E[X]$ represents the "center of mass"

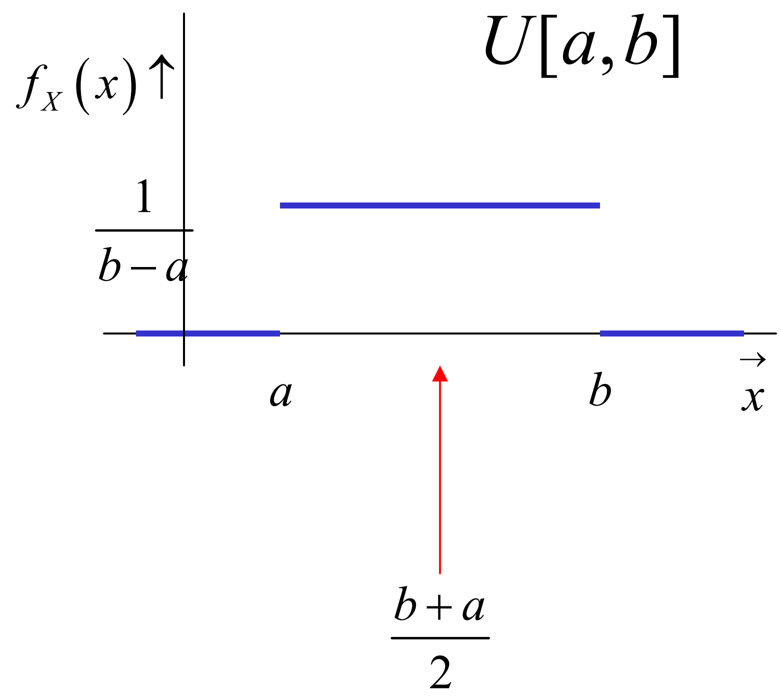
arithmetic average of large # independent observations of a r.v. will tend to the mean; it's like the "average of *X*"

Ex 3.29 mean of U[a,b]

$$E[X] \triangleq \int_{-\infty}^{\infty} t f_X(t) dt$$

$$= \int_a^b t \frac{1}{b-a} dt$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$



midpoint of interval [a,b]

Ex 3.31 mean of exponential r.v.

- Time *X* between customer arrivals at a service station has an exponential PDF with parameter λ. Find the mean inter-arrival time.

$$\int u\,dv = uv - \int v\,du$$

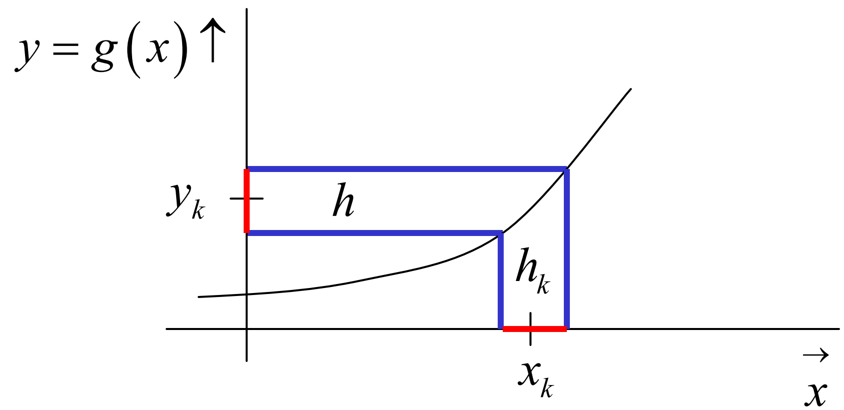
$$E[X] = \int_0^\infty \underbrace{t}_{u}\,\overbrace{\lambda e^{-\lambda t}\,dt}^{dv} = t\,e^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\,dt$$

$$= \lim_{t\to\infty} t e^{-\lambda t} - 0 + \frac{e^{-\lambda t}}{-\lambda}\Bigg|_0^\infty = \frac{1}{\lambda}$$

As λ is customer arrival rate in customers per second, the mean inter-arrival time of λ^{-1} seconds/customer makes sense!

Expected value of a function of a r.v.

$$E[Y] \triangleq \int_{-\infty}^{\infty} t f_Y(t) dt$$



alternative in terms of *X*

$$E[Y] \simeq \sum_k y_k f_Y(y_k) h$$

equivalent events

$$= \sum_k g(x_k) f_X(x_k) h_k$$

limit h→0

$$= \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$\Theta \sim U(0, 2\pi]$$

$$Y = a\cos(\omega t + \Theta)$$

Ex 3.33

- Sampling a sinusoid with random phase. Find the expected value of *Y* and of Y^2, the power of *Y*.

$$E[Y] = E[a\cos(\omega t + \Theta)]$$
$$= \int_0^{2\pi} a\cos(\omega t + \theta)\frac{1}{2\pi}d\theta$$
$$= \frac{-a}{2\pi}\sin(\omega t + \theta)\Big|_0^{2\pi}$$
$$= \frac{-a}{2\pi}[\sin(\omega t + 2\pi) - \sin(\omega t)] = 0$$

$$E[Y^2] = E[a^2\cos^2(\omega t + \Theta)]$$
$$= \frac{a^2}{2}E[1 + \cos(2\omega t + 2\Theta)]$$
$$= \frac{a^2}{2}\left[1 + \int_0^{2\pi}\cos(2\omega t + 2\theta)\frac{1}{2\pi}d\theta\right]$$
$$= \frac{a^2}{2}$$

agreement with time-averages: "DC" value of 0, power $a^2/2$

Variance of *X*

- Provides information about a random variable – in addition to its mean value – regarding its deviations from the mean

$$D \triangleq X - E[X]$$ deviation from the mean

measures of "width/spread"

$$VAR[X] \triangleq E\left[\left(X - E[X]\right)^2\right]$$ variance of r.v.

$$STD[X] \triangleq \sqrt{VAR[X]}$$ standard deviation of r.v.

$$VAR[X] = E\left[X^2 - 2E[X]X + E^2[X]\right]$$
$$= E\left[X^2\right] - 2E[X]E[X] + E^2[X]$$
$$= E\left[X^2\right] - E^2[X]$$

Ex 3.36 variance of X~U[a,b]

$$E[X]=\frac{b+a}{2}$$

$$VAR[X]=\frac{1}{b-a}\int_a^b\left(x-\frac{b+a}{2}\right)^2dx$$

$$=\frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2}y^2dy=\frac{y^3}{3(b-a)}\Bigg|_{-(b-a)/2}^{(b-a)/2}=\frac{(b-a)^2}{12}$$

$$U[-2,4]\rightarrow E[X]=\frac{4+(-2)}{2}=1;VAR[X]=\frac{(4-(-2))^2}{12}=3$$

3.38 variance of Gaussian r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$\Big\downarrow \int_{-\infty}^{\infty}$$

$$\sigma\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-(x-m)^2 / 2\sigma^2} dx$$

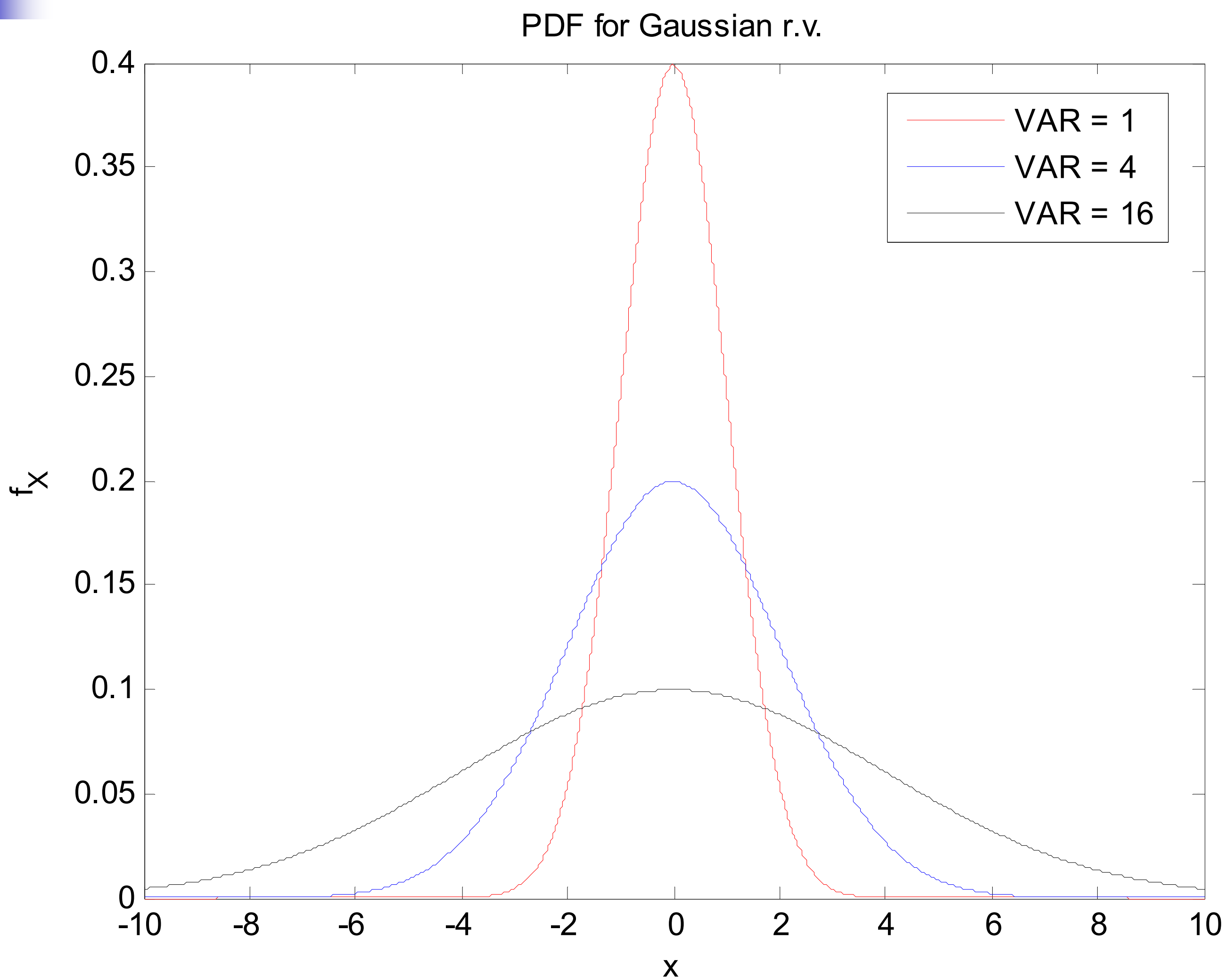
$$\Big\downarrow \frac{d}{d\sigma}$$

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} \frac{(x-m)^2}{\sigma^3} e^{-(x-m)^2 / 2\sigma^2} dx$$

$$\Big\downarrow$$

$$VAR[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^2 e^{-(x-m)^2 / 2\sigma^2} dx = \sigma^2$$

Effect of VAR on the PDF of a Gaussian r.v.



Mean & variance

- Are the two most important parameters for (partially) characterizing the PDF of a r.v.
- Others are sometimes used, e.g.

$$skewness \triangleq \frac{E\left[\left(X - E[X]\right)^3\right]}{\sigma^3}$$

  which measures the degree of asymmetry about the mean
  - *Skewness is zero for a symmetric PDF*
- Each involves n^{th} moment of r.v. X: $E\left[X^n\right] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$
- $VAR[X] = E\left[X^2\right] - E^2[X]$
- Under certain conditions, a PDF is completely specified if all moments are known (more on that later)

$$VAR[c] = 0 \quad VAR[X + c] = VAR[X] \quad VAR[cX] = c^2 VAR[X]$$