

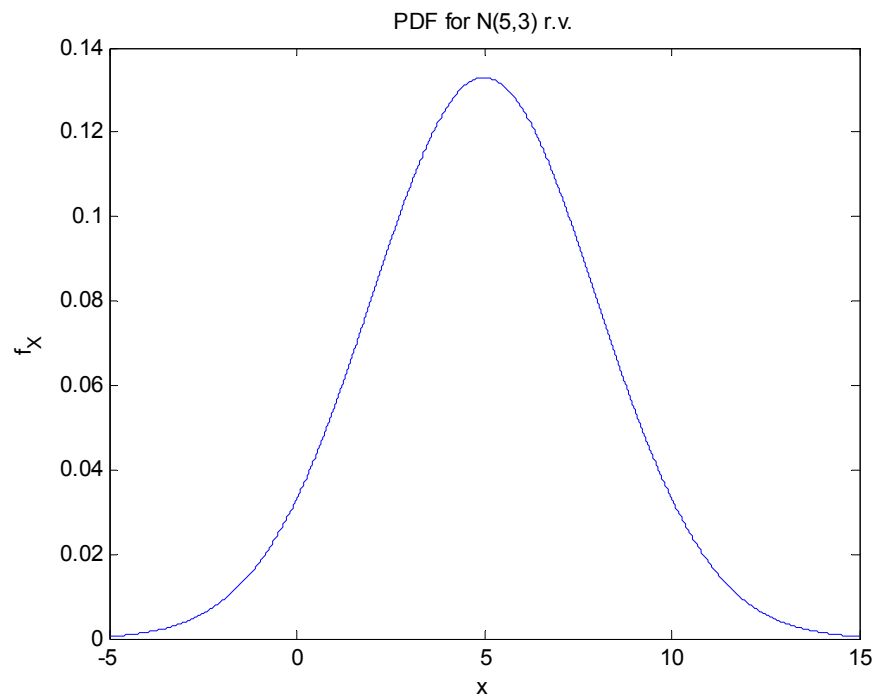
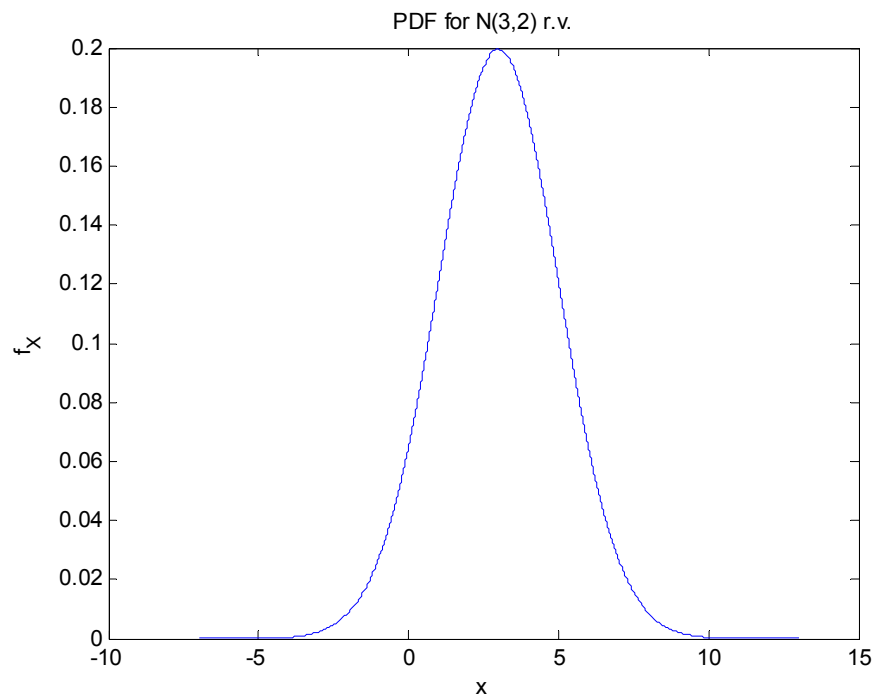


Gaussian (normal) r.v.

- In many situations in man-made and natural phenomena one deals with a r.v. X that consists of a large sum of “small” r.v.’s
 - *Exact PDF becomes complex and unwieldy*
- Under fairly general conditions, as the number of components becomes large (CLT), the CDF approaches that of the normal r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty$$
$$= N(m, \sigma)$$

Gaussian (normal) r.v. - PDF



the "bell-shaped" curve

Gaussian (normal) r.v. - CDF

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(s-m)^2}{2\sigma^2}} ds$$

↓ COV: $t = \frac{s-m}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(x-m)}{\sigma}} e^{-t^2/2} dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right) \text{ where } \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

↑
CDF for $\mathcal{N}(0,1)$ r.v.

“standard normal”

Q-function

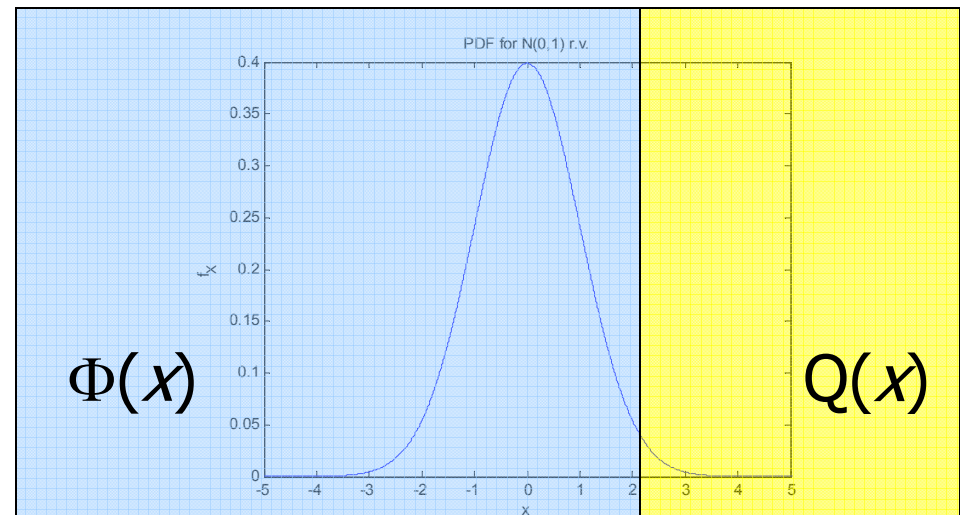
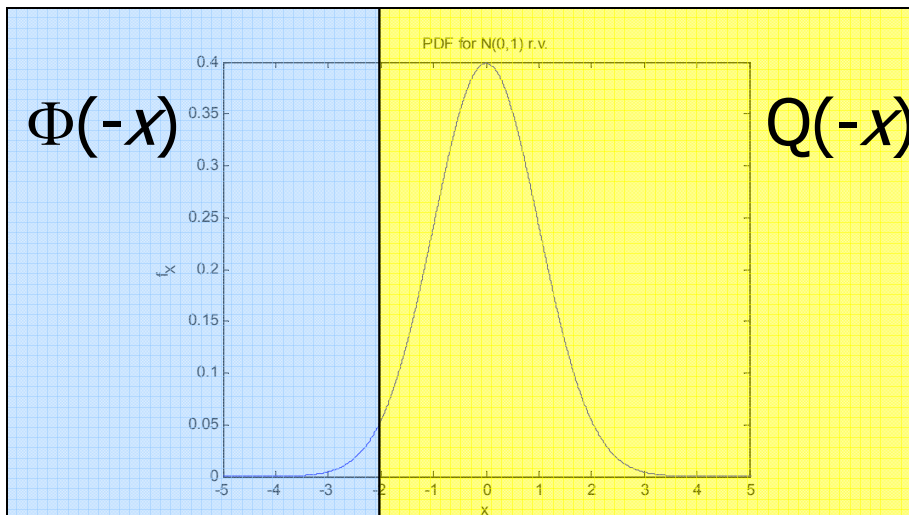
used by EE as error probability

$$Q(x) \triangleq 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$$

probability of the "tail"

$$Q(0) = 0.5$$

$$Q(-x) = 1 - \Phi(-x) = 1 - Q(x)$$





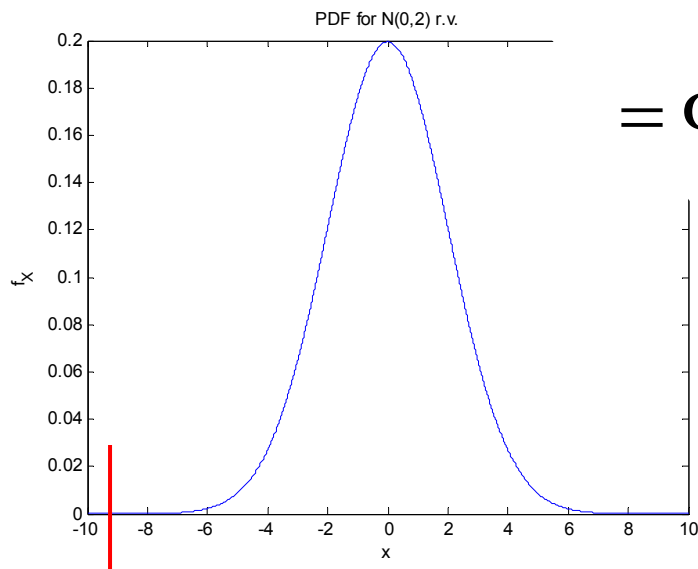
Gaussian (normal) r.v.

- Plays an important role in communication systems, where transmission of signals is subject to noise
 - *Noise resulting from the thermal motion of electrons, can – from physical principles – be shown to have a Gaussian PDF*

Ex

- A communication system accepts a positive voltage V as input and outputs a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and N is $\sim N(0, 2)$. Find $V \ni P[Y < 0] = 10^{-6}$

$$P[Y < 0] = P[\alpha V + N < 0] = P[N < -\alpha V]$$



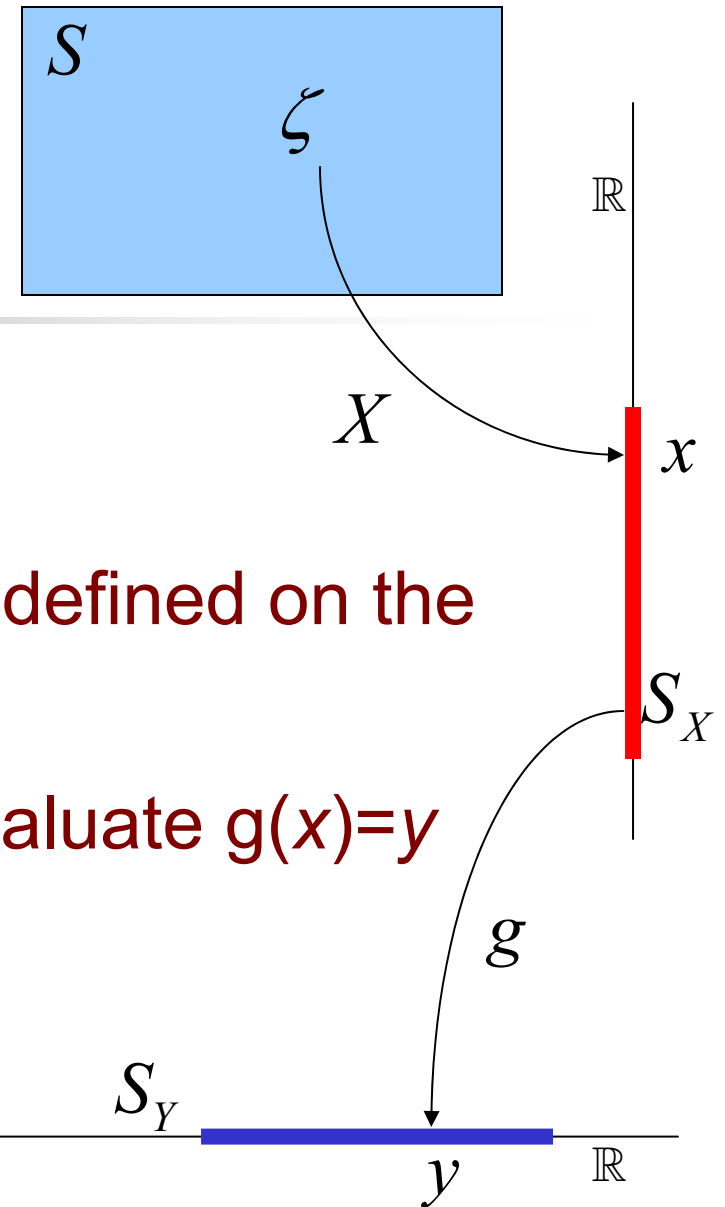
$$= \Phi\left(\frac{-\alpha V}{\sigma}\right) = Q\left(\frac{\alpha V}{\sigma}\right) = 10^{-6}$$

Table

$$\frac{\alpha V}{\sigma} = 4.7535$$

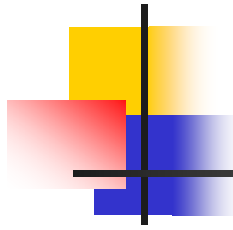
$$V = 4.7535 \frac{\sigma}{\alpha} = 950.6$$

Functions of a r.v.



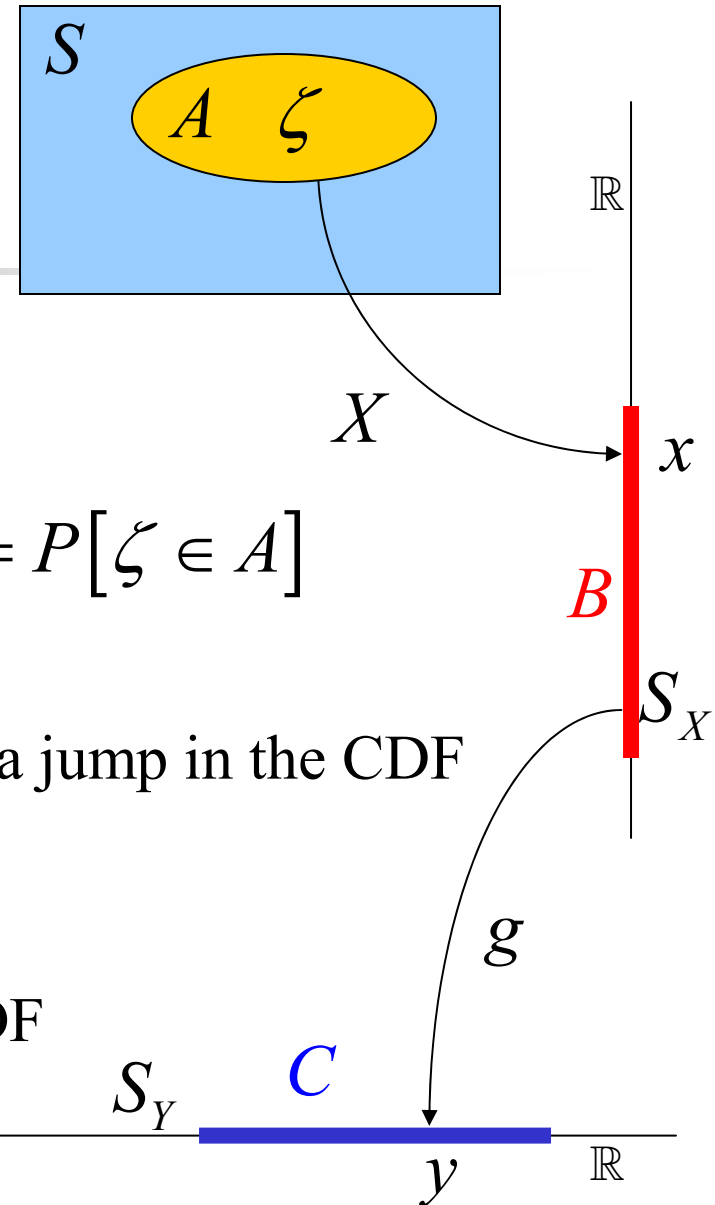
- X is a random variable
- $g(x)$ is a real-valued function defined on the real line
- $Y=g(X)$, i.e. for every $X=x$, evaluate $g(x)=y$ and assign it to Y
- Y is also a random variable
- Find CDF and PDF of Y

ultimately probabilities are induced by the underlying experiment



Induced probability

equivalent events



$$P[Y \in C] = P[g(X) \in C] = P[X \in B] = P[\zeta \in A]$$

useful events:

$\{g(X) = y_k\}$ is used to find the magnitude of a jump in the CDF

$\{g(X) \leq y\}$ is used to directly find the CDF

$\{y < g(X) \leq y + h\}$ is useful in finding the PDF

Ex 3.23 a linear function $Y = aX + b \quad a \neq 0$

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = P[aX \leq y - b]$$

$$= \begin{cases} P\left[X \leq \frac{y-b}{a}\right] & a > 0 \\ P\left[X \geq \frac{y-b}{a}\right] & a < 0 \end{cases} = \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

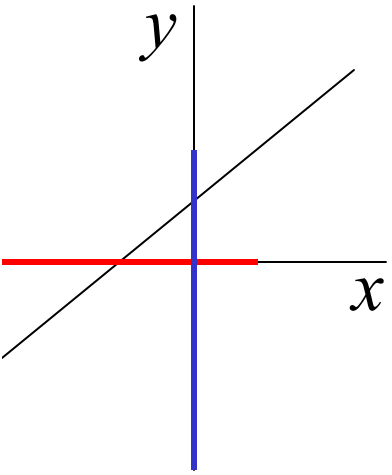
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$



Ex 3.24 linear function of Gaussian r.v.

$$Y = aX + b \quad a \neq 0$$


$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

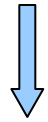
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{|a\sigma|\sqrt{2\pi}} e^{-\frac{(y-b-am)^2}{2(a\sigma)^2}}$$

linear function of a Gaussian r.v. is also a Gaussian r.v.

Ex 3.25 square law device

$$Y = X^2$$

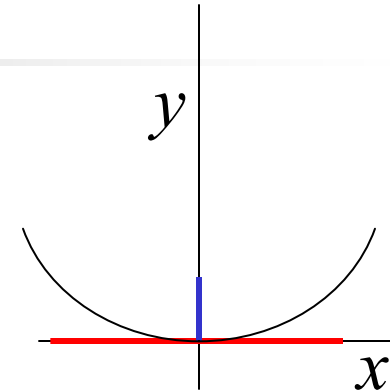


$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y]$$

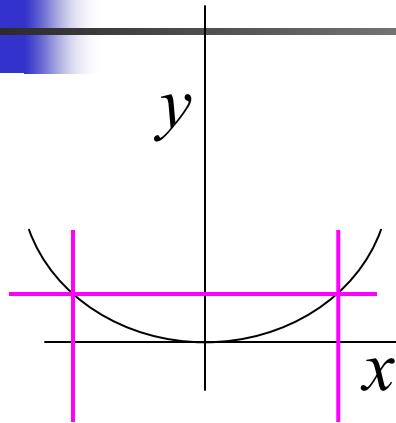
$$= P[-\sqrt{y} \leq X \leq \sqrt{y}] = [F_X(\sqrt{y}) - F_X(-\sqrt{y})]u(y)$$

$$f_Y(y) = \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})]u(y)$$

$$= \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \right] u(y)$$



from Ex 3.26



$$y_0 = g(x) \leftarrow x_0, x_1$$

produces 2 terms in PDF

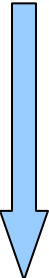
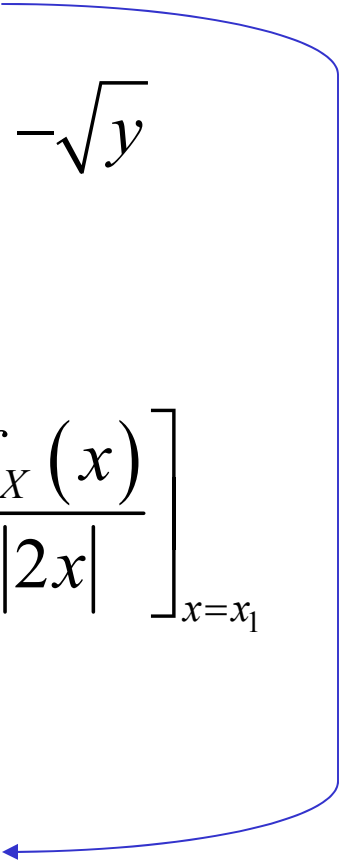
$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$



Redo Ex 3.27

for $y < 0$: $y = x^2$ has no solutions $\Rightarrow f_Y(y) = 0$

for $y \geq 0$: $y = x^2$ has two solutions: $x_0 = \sqrt{y}$; $x_1 = -\sqrt{y}$


$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \left[\frac{f_X(x)}{|2x|} \right]_{x=x_0} + \left[\frac{f_X(x)}{|2x|} \right]_{x=x_1}$$
$$= \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \right] u(y)$$




Nonlinear function $Y=g(X)$

$$P[C_y] = P[B_y]$$

equivalent events \downarrow induce equal probabilities

$$f_Y(y)|dy| = f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \sum_k \left[f_X(x) \left| \frac{dx}{dy} \right| \right]_{x=x_k}$$

function of y

Ex 3.28

$$X \sim U(0, 2\pi]$$

$$Y = \cos(X)$$

for $y < -1$ or $y > 1$: no sol^s

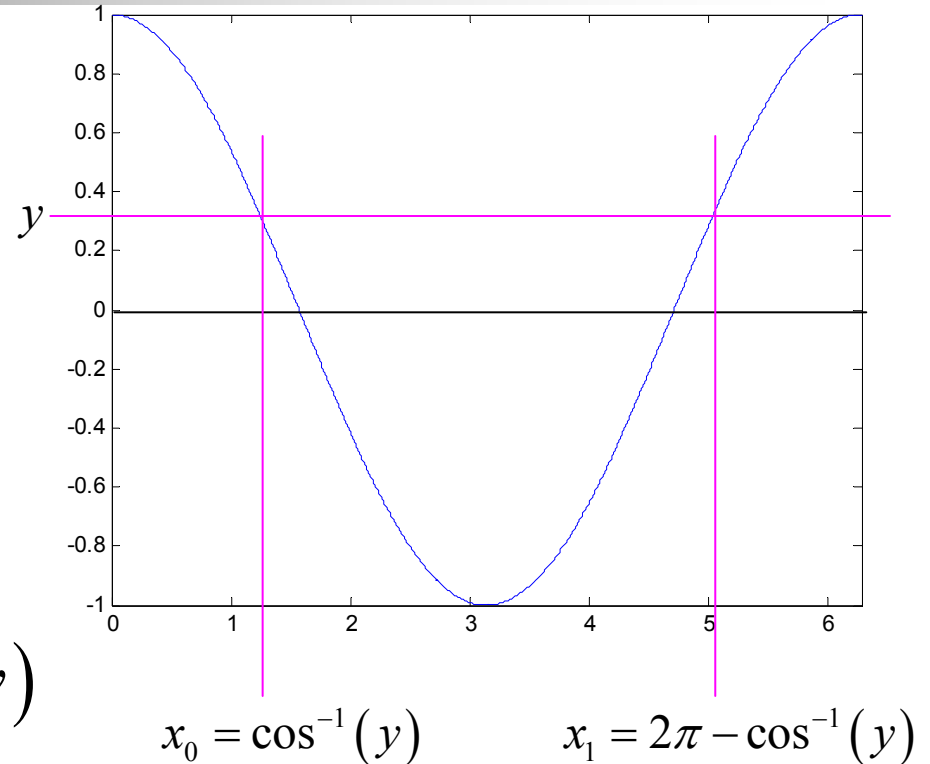
$$f_Y(y) = 0$$

for $-1 \leq y \leq 1$:

$$x_0 = \cos^{-1}(y); x_1 = 2\pi - \cos^{-1}(y)$$

$$\left. \frac{dy}{dx} \right|_{x=x_0} = -\sin(x_0) = -\sin\{\cos^{-1}(y)\} = -\sqrt{1-y^2}$$

$$Y = \cos(X)$$





$$Y = \cos(X)$$

$$X \sim U(0, 2\pi] \rightarrow f_X(x) = \frac{1}{2\pi} [u(x) - u(x - 2\pi)]$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \frac{1}{2\pi \left| -\sqrt{1-y^2} \right|} + \frac{1}{2\pi \left| \sqrt{1-y^2} \right|} = \frac{1}{\pi \sqrt{1-y^2}} \quad -1 \leq y \leq 1$$

$$F_Y(y) = \int_{-\infty}^y \frac{1}{\pi \sqrt{1-t^2}} dt = \frac{1}{2} + \frac{\sin^{-1}(y)}{\pi} \quad \text{for } -1 \leq y \leq 1$$

Y has the arcsine distribution

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$