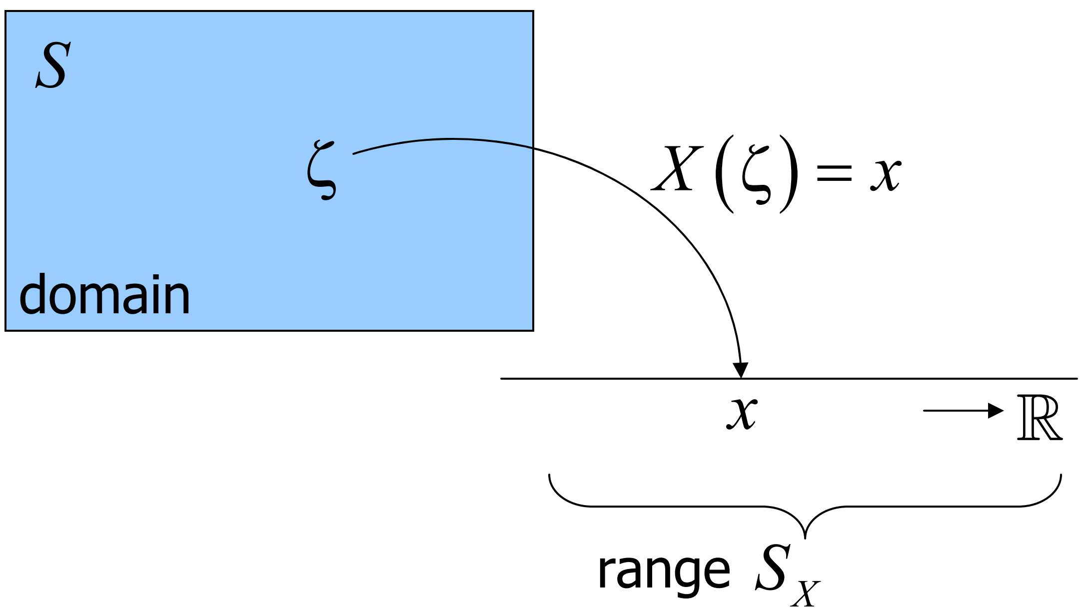
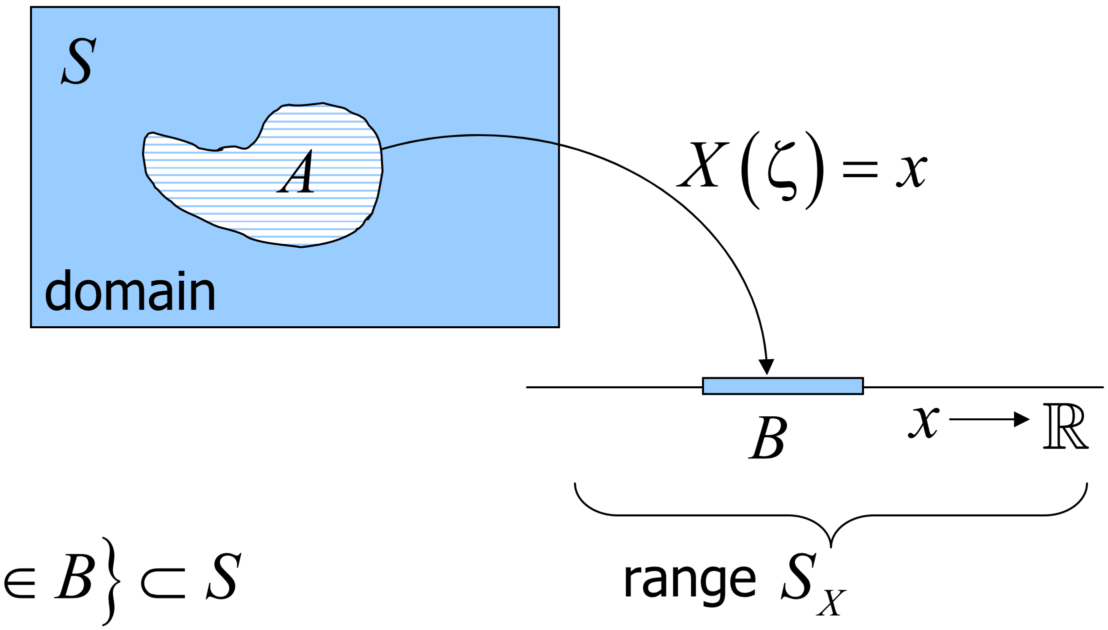
Random variable

- A random variable X is a function that assigns a real number, $X(\zeta)$, to each outcome ζ in the sample space of a random experiment



Finding probabilities involving r.v. X



new sample space

$$B \subset S_X$$

$$A = \{\zeta : X(\zeta) \in B\} \subset S$$

B in S_X occurs when *A* in *S* occurs

$$P[B] = P[A] = P\left[\{\zeta : X(\zeta) \in B\}\right] = P\left[X^{-1}(B)\right]$$

A and *B* are equivalent events

Ex

- Event $\{X=k\}=\{k$ Heads in three coin tosses$\}$ occurs when the outcome of the coin tossing experiment contains three Heads. The probability of the event $\{X=k\}$ is given by the sum of the probabilities of the corresponding outcomes or elementary events.

$$p_0 = P[X=0] = P\{TTT\} = (1-p)^3$$

$$p_1 = P[X=1] = P\{HTT\} + P\{THT\} + P\{TTH\} = 3(1-p)^2\, p$$

$$p_2 = P[X=2] = P\{HHT\} + P\{HTH\} + P\{THH\} = 3(1-p)\, p^2$$

$$p_3 = P[X=3] = P\{HHH\} = p^3$$

- The p_k's can be used to find the probabilities of all events involving X, i.e. we can deal with sample space S_X and p_k's, instead of dealing with sample space S and the probabilities of ζ

Cumulative distribution function (cdf)

- Defined as: $F_X(x) \triangleq P[X \leq x]$ for $-\infty < x < \infty$

$$F_X(x) = P\left[X \in (-\infty, x]\right] = P\left[\zeta : X(\zeta) \leq x\right]$$

function of variable *x*

convenient way of specifying the probability of
all semi-infinite intervals of the real line

events of interest – when dealing with numbers – are intervals of the real line,
and their complements, unions, and intersections
and probabilities of all these can be expressed in terms of the cdf!

CDF properties

i. $0 \le F_X(x) \le 1$

ii. $\lim_{x\to\infty} F_X(x) = 1$

iii. $\lim_{x\to-\infty} F_X(x) = 0$

iv. $F_X(a) \le F_X(b)$ for $a < b$

non-decreasing

v. $F_X(b) = \lim_{h\to 0} F_X(b+h) = F_X(b^+)$ for $h > 0$ Δ

continuous from the right

More CDF properties

$$\{X \le a\} \cup \{a < X \le b\} = \{X \le b\}$$

$$\Downarrow$$

$$F_X(a) + P\left[\{a < X \le b\}\right] = F_X(b)$$

$$\Downarrow$$

vi. $$P\left[\{a < X \le b\}\right] = F_X(b) - F_X(a)$$

$$\Downarrow$$

$$P\left[\{b - \varepsilon < X \le b\}\right] = F_X(b) - F_X(b - \varepsilon)$$

$$\Downarrow \; \varepsilon \to 0$$

vii. $$P\left[X = b\right] = F_X(b) - F_X\left(b^-\right)$$

magnitude of jump in CDF at *b*

if CDF is continuous at *b*, then $\{X = b\}$ is a zero-probability event

Other types of interval

$$\{a \le X \le b\} = \{X = a\} \cup \{a < X \le b\}$$

$$\Downarrow$$

$$P\left[\{a \le X \le b\}\right] = P[X = a] + F_X(b) - F_X(a)$$

$$= F_X(a) - F_X\left(a^-\right) + F_X(b) - F_X(a)$$

$$= F_X(b) - F_X\left(a^-\right)$$

for CDF that is continuous at *a* and *b*

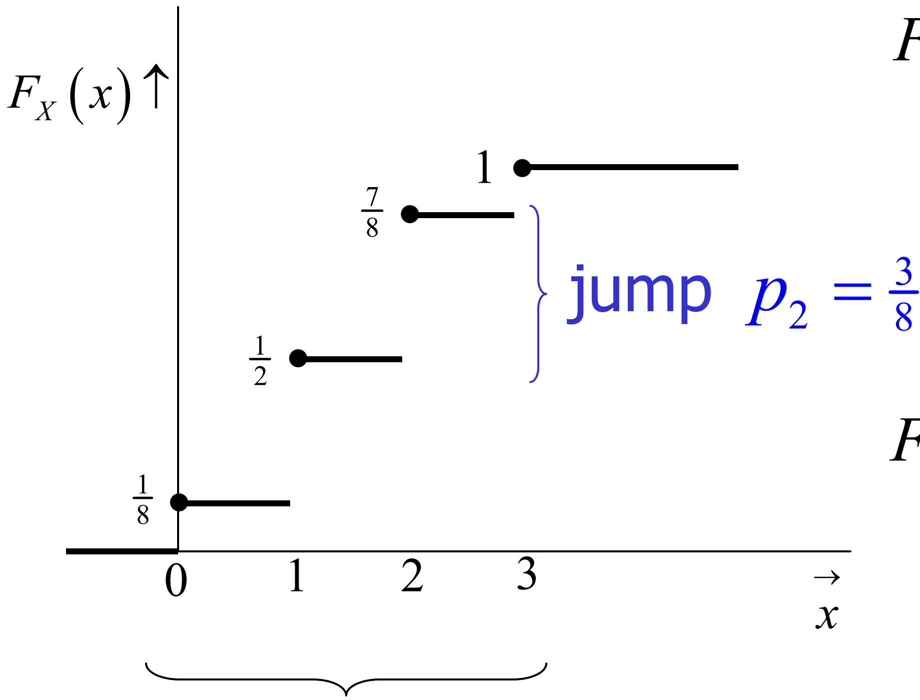
$$P[a < X < b] = P[a \le X < b] = P[a < X \le b] = P[a \le X \le b]$$

$$\{X \le x\} \cup \{X > x\} = \{-\infty < X < \infty\}$$

$$\Downarrow$$

viii. $P[X > x] = 1 - F_X(x)$

Ex CDF # heads in three coin tosses



$$S_X = \{0,1,2,3\}$$

$$p_i = \left\{\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\right\} \Longrightarrow F_X(x) = \sum_{i:x_i \le x} p_i = \sum_{i=0}^{3} p_i u(x - x_i)$$

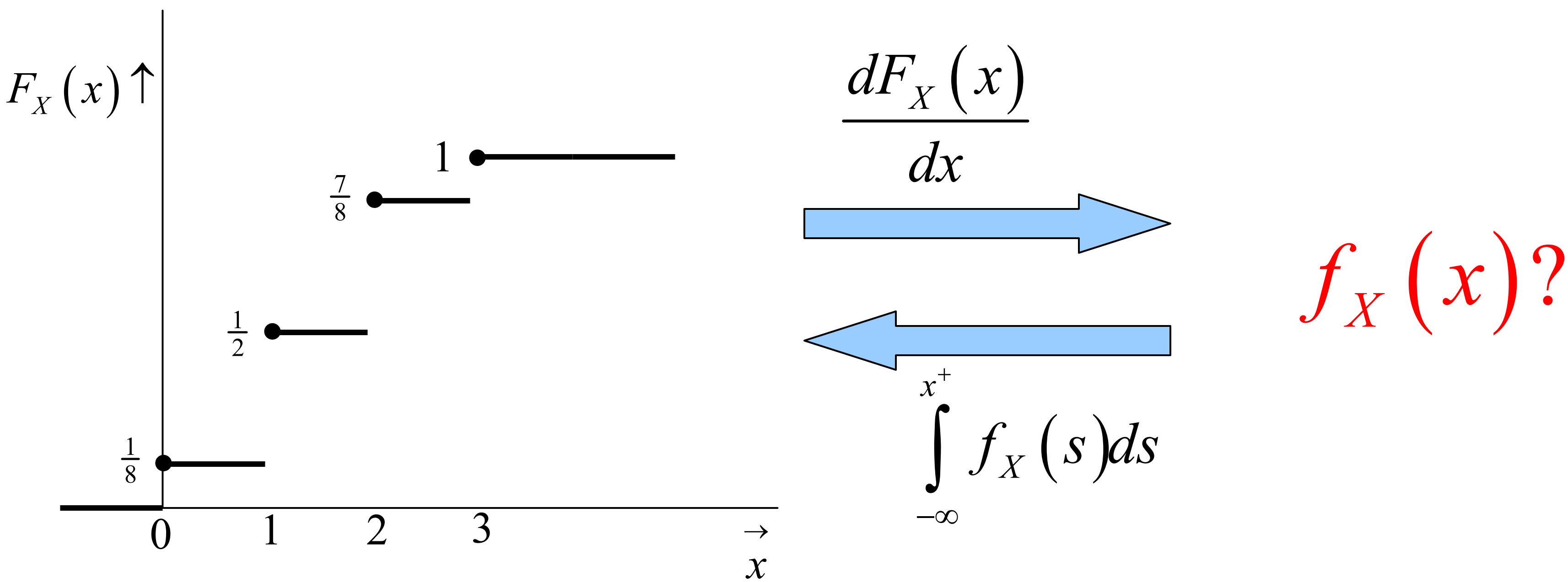
$$F_X(2-\delta) = P[X \le 2-\delta] = P[\{0 \; or \; 1 \; heads\}] = p_0 + p_1$$

Δ

$$F_X(2+\delta) = P[X \le 2+\delta] = P[\{0 \; or \; 1 \; or \; 2 \; heads\}] = p_0 + p_1 + p_2$$

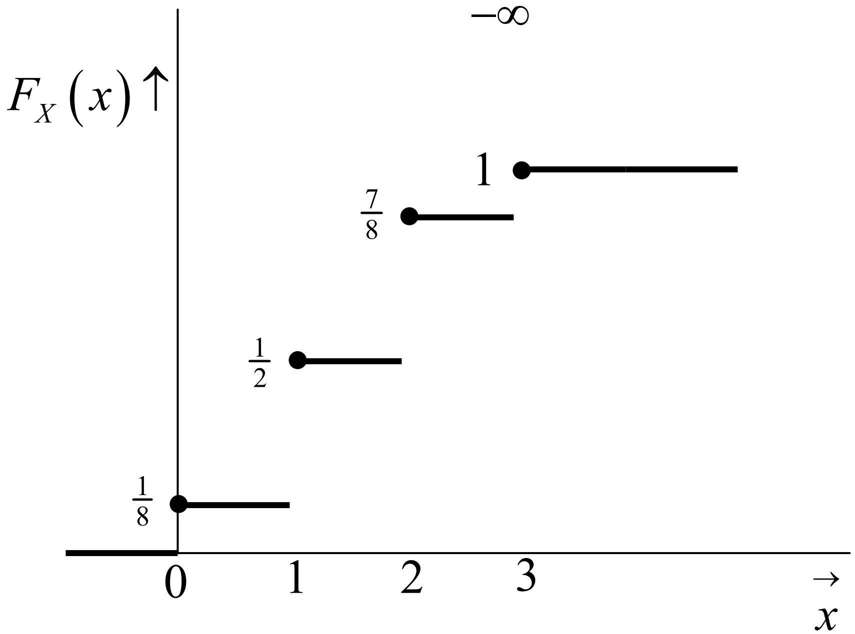
Derivative of CDF?

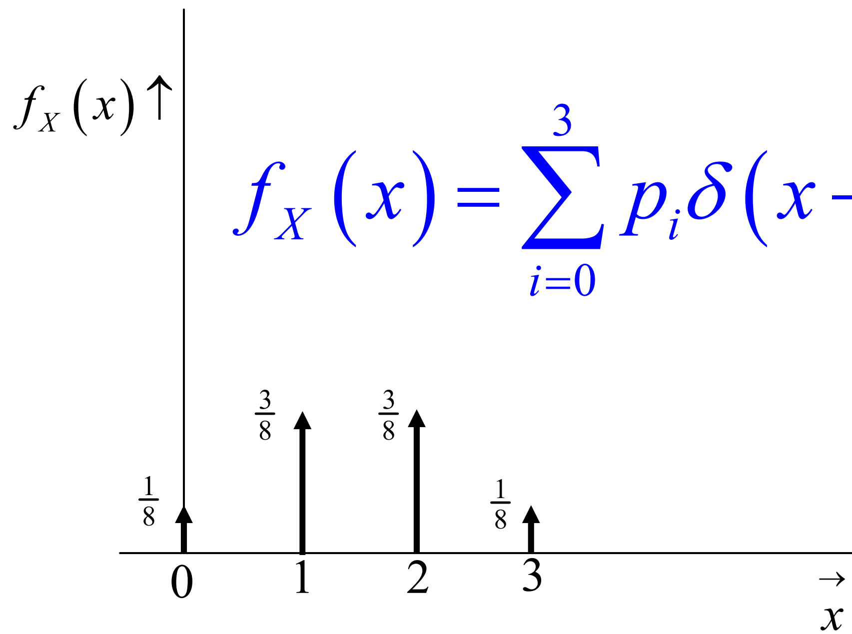
$$\text{Find } f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X(x^+) - F_X(-\infty) = F_X(x)$$



Derivative of CDF?

$$f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X(x^+) - F_X(-\infty) = F_X(x)$$





$$f_X(x) = \sum_{i=0}^{3} p_i \delta(x - x_i)$$

$$\int_{-\infty}^{x^+} \delta(s-a)ds = u(x-a)$$

Ex 3.5 message transmission time

- The transmission time *X* of messages in a communication system obeys the exponential probability law with parameter λ

$$P[X > x] = e^{-\lambda x} \quad x > 0$$

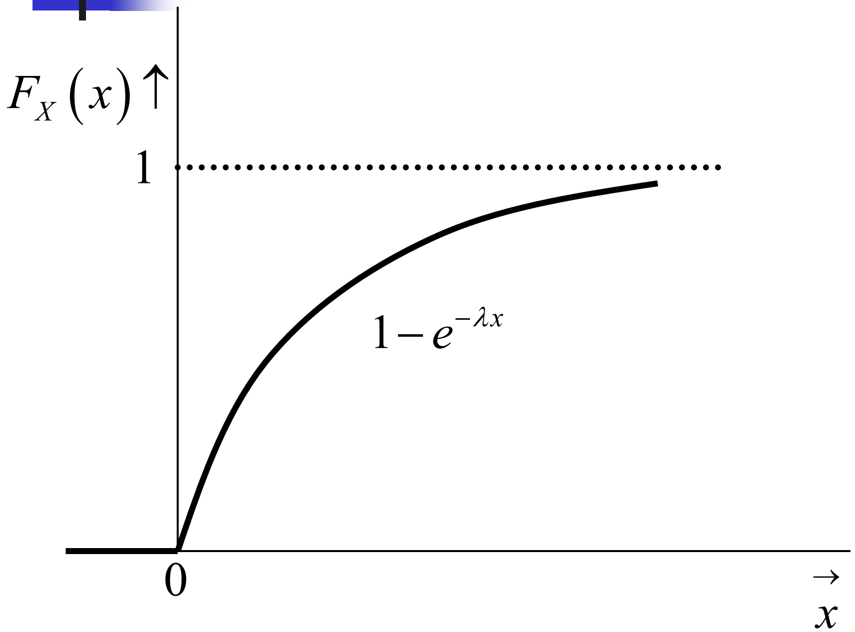
Find cdf $F_X(\mathrm{x})$. Find $P[T < X < 2T]$, where $T = \lambda^{-1}$.

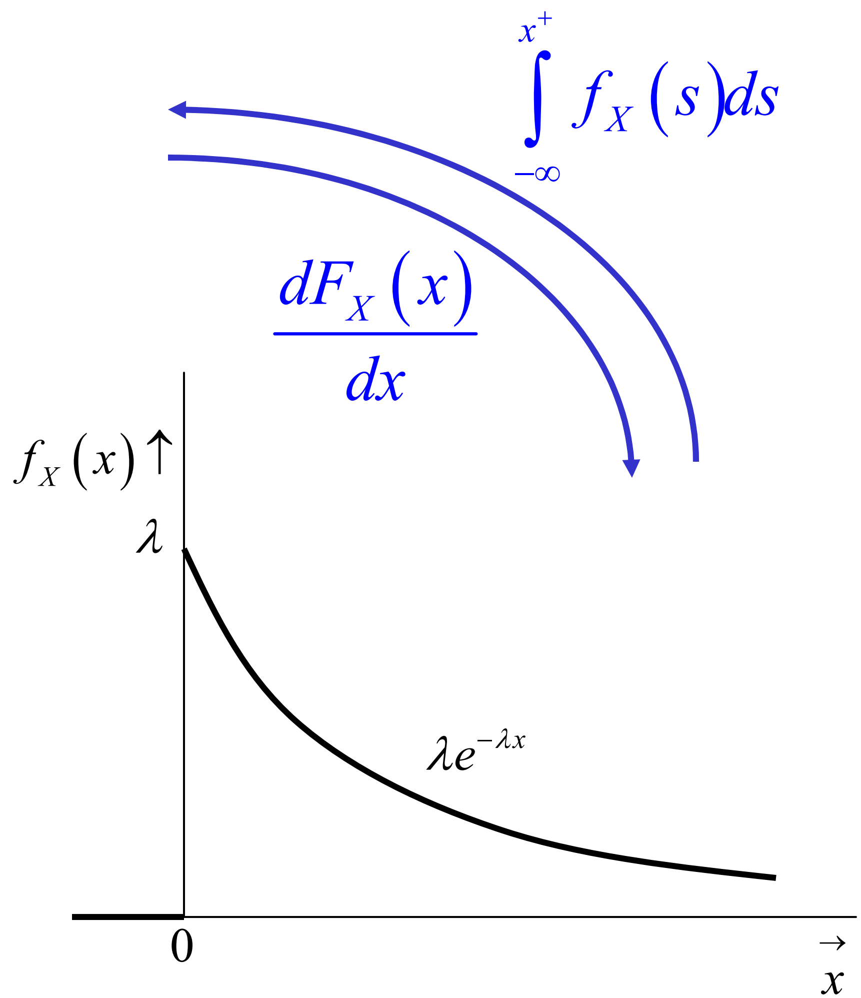
$$F_X(x) = P[X \le x] = 1 - P[X > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$P[T < X \le 2T] = F_X(2T) - F_X(T)$$

$$= 1 - e^{-\lambda 2T} - \left(1 - e^{-\lambda T}\right) = -e^{-2} + e^{-1}$$

Ex 3.5 message transmission time





continuous for all *x*

⇓

derivative exists everywhere except for *x*=0