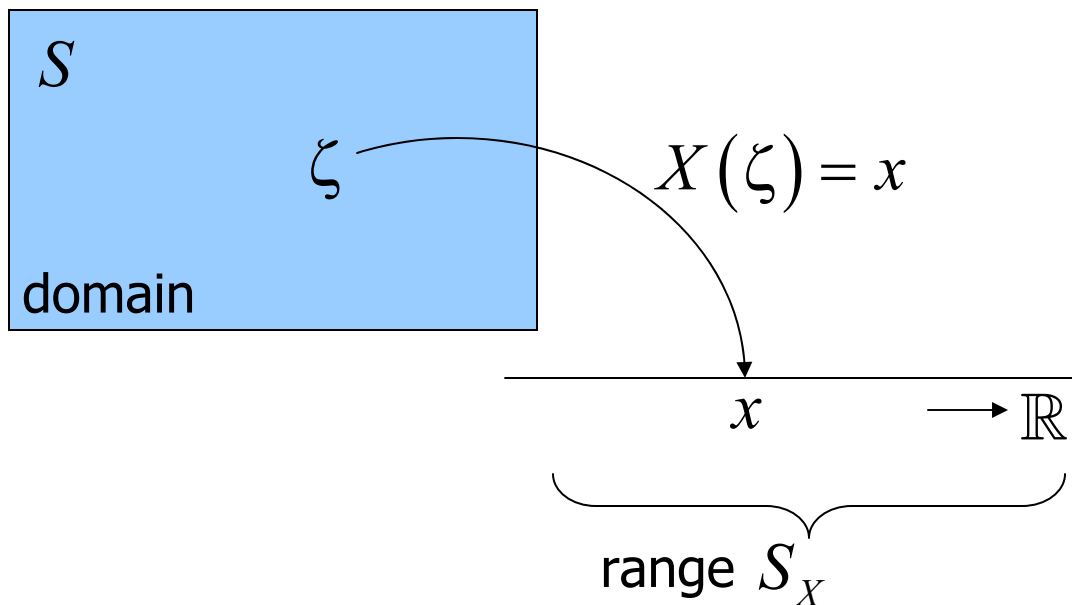
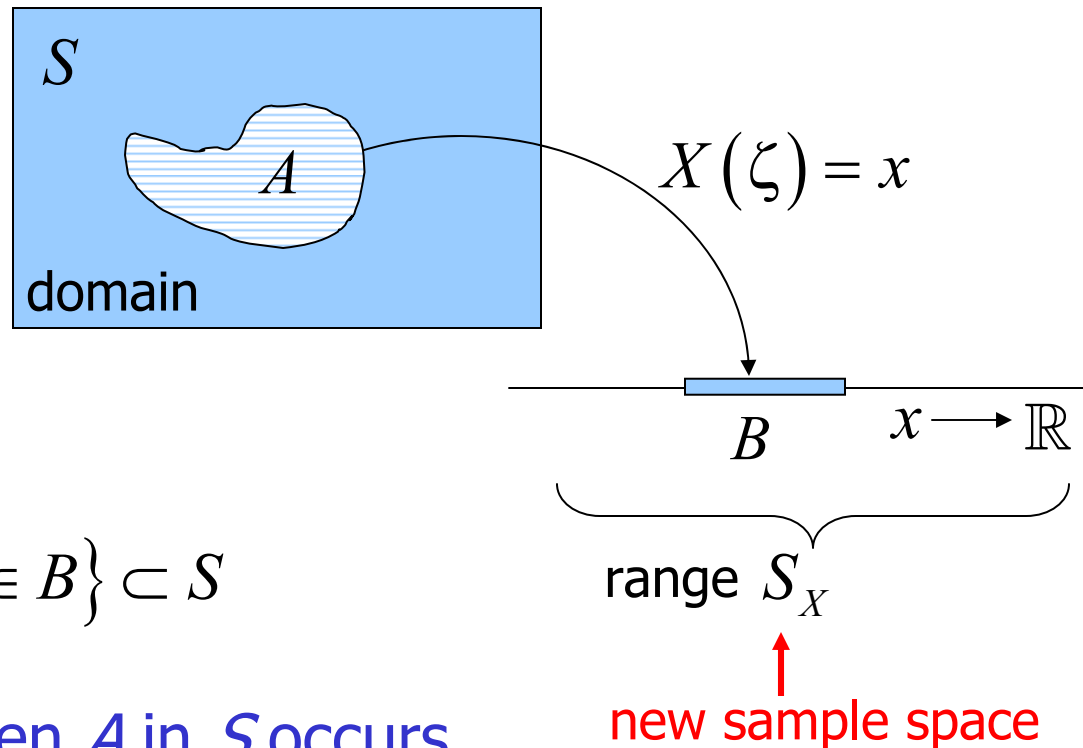


Random variable

- A random variable X is a function that assigns a real number, $X(\zeta)$, to each outcome ζ in the sample space of a random experiment



Finding probabilities involving r.v. X



$$B \subset S_X$$

$$A = \{\zeta : X(\zeta) \in B\} \subset S$$

B in S_X occurs when A in S occurs



$$P[B] = P[A] = P[\{\zeta : X(\zeta) \in B\}] = P[X^{-1}(B)]$$

A and B are equivalent events



Ex

- Event $\{X=k\}=\{k \text{ Heads in three coin tosses}\}$ occurs when the outcome of the coin tossing experiment contains three Heads. The probability of the event $\{X=k\}$ is given by the sum of the probabilities of the corresponding outcomes or elementary events.

$$p_0 = P[X = 0] = P\{TTT\} = (1-p)^3$$

$$p_1 = P[X = 1] = P\{HTT\} + P\{THT\} + P\{TTH\} = 3(1-p)^2 p$$

$$p_2 = P[X = 2] = P\{HHT\} + P\{HTH\} + P\{THH\} = 3(1-p) p^2$$

$$p_3 = P[X = 3] = P\{HHH\} = p^3$$

- The p_k 's can be used to find the probabilities of all events involving X , i.e. we can deal with sample space S_X and p_k 's, instead of dealing with sample space S and the probabilities of ζ



Cumulative distribution function (cdf)

- **Defined as:** $F_X(x) \triangleq P[X \leq x]$ for $-\infty < x < \infty$

$$F_X(x) = P[X \in (-\infty, x]] = P[\zeta : X(\zeta) \leq x]$$

function of variable x

convenient way of specifying the probability of
all semi-infinite intervals of the real line

events of interest – when dealing with numbers – are intervals of the real line,
and their complements, unions, and intersections
and probabilities of all these can be expressed in terms of the cdf!



CDF properties

i. $0 \leq F_X(x) \leq 1$

ii. $\lim_{x \rightarrow \infty} F_X(x) = 1$

iii. $\lim_{x \rightarrow -\infty} F_X(x) = 0$

iv. $F_X(a) \leq F_X(b)$ for $a < b$

non-decreasing

v. $F_X(b) = \lim_{h \rightarrow 0} F_X(b+h) = F_X(b^+)$ for $h > 0$

continuous from the right





More CDF properties

$$\{X \leq a\} \cup \{a < X \leq b\} = \{X \leq b\}$$



$$F_X(a) + P[\{a < X \leq b\}] = F_X(b)$$



vi. $P[\{a < X \leq b\}] = F_X(b) - F_X(a)$



$$P[\{b - \varepsilon < X \leq b\}] = F_X(b) - F_X(b - \varepsilon)$$



vii. $P[X = b] = F_X(b) - F_X(b^-)$

magnitude of jump in CDF at b

if CDF is continuous at b , then $\{X=b\}$ is a zero-probability event



Other types of interval

$$\{a \leq X \leq b\} = \{X = a\} \cup \{a < X \leq b\}$$

$$\begin{aligned} P[\{a \leq X \leq b\}] &\stackrel{\downarrow}{=} P[X = a] + F_X(b) - F_X(a) \\ &= F_X(a) - F_X(a^-) + F_X(b) - F_X(a) \\ &= F_X(b) - F_X(a^-) \end{aligned}$$

for CDF that is continuous at a and b

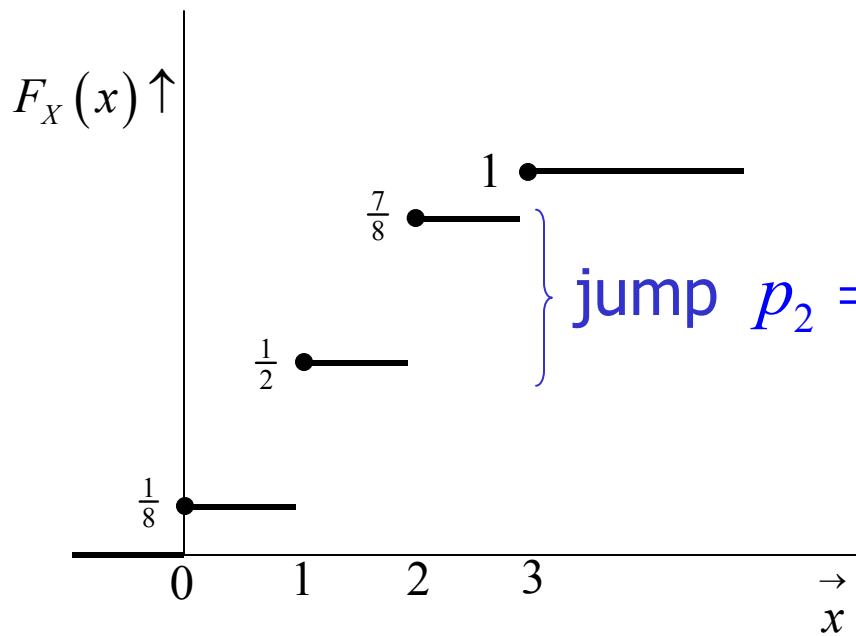
$$P[a < X < b] = P[a \leq X < b] = P[a < X \leq b] = P[a \leq X \leq b]$$

$$\{X \leq x\} \cup \{X > x\} = \{-\infty < X < \infty\}$$

$$\text{viii. } P[X > x] \stackrel{\downarrow}{=} 1 - F_X(x)$$

Ex

CDF # heads in three coin tosses



$$\begin{aligned}
 F_X(2 - \delta) &= P[X \leq 2 - \delta] \\
 &= P[\{0 \text{ or } 1 \text{ heads}\}] \\
 &= p_0 + p_1
 \end{aligned}$$

△

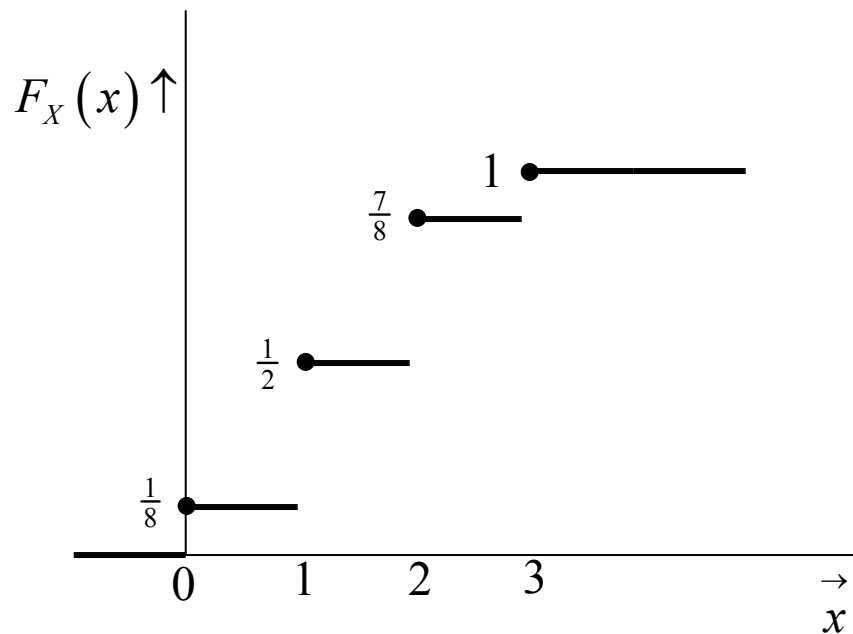
$$\begin{aligned}
 F_X(2 + \delta) &= P[X \leq 2 + \delta] \\
 &= P[\{0 \text{ or } 1 \text{ or } 2 \text{ heads}\}] \\
 &= p_0 + p_1 + p_2
 \end{aligned}$$

$$S_X = \{0, 1, 2, 3\}$$

$$p_i = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \right\} \implies F_X(x) = \sum_{i: x_i \leq x} p_i = \sum_{i=0}^3 p_i u(x - x_i)$$

Derivative of CDF?

Find $f_X(x) \ni \int_{-\infty}^{x^+} f_X(s) ds = F_X(x^+) - F_X(-\infty) = F_X(x)$



$$\frac{dF_X(x)}{dx}$$

→

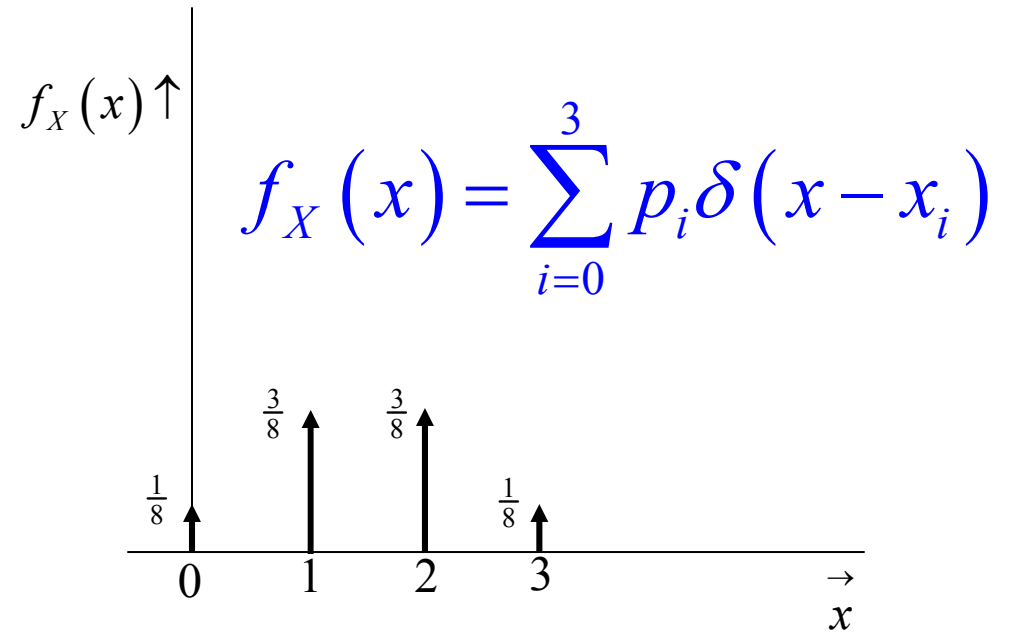
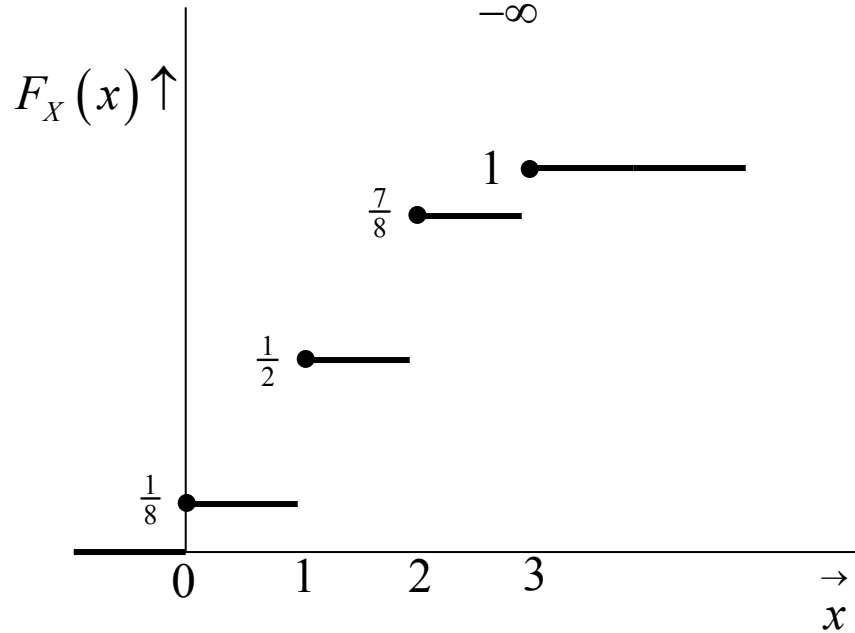
$$\int_{-\infty}^{x^+} f_X(s) ds$$

←

$f_X(x)?$

Derivative of CDF?

$$f_X(x) \ni \int_{-\infty}^{x^+} f_X(s) ds = F_X(x^+) - F_X(-\infty) = F_X(x)$$



$$\int_{-\infty}^{x^+} \delta(s - a) ds = u(x - a)$$



Ex 3.5 message transmission time

- The transmission time X of messages in a communication system obeys the exponential probability law with parameter λ

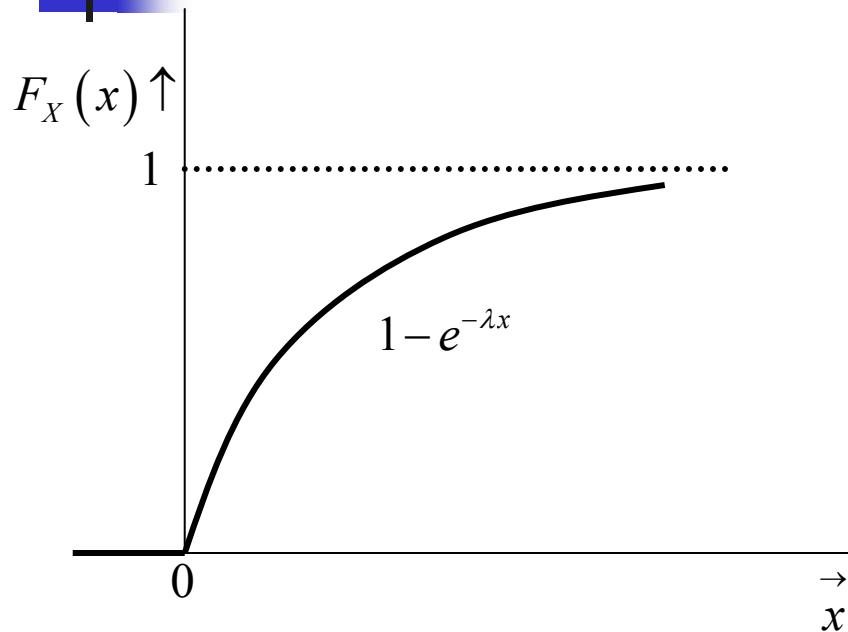
$$P[X > x] = e^{-\lambda x} \quad x > 0$$

Find cdf $F_X(x)$. Find $P[T < X < 2T]$, where $T = \lambda^{-1}$.

$$F_X(x) = P[X \leq x] = 1 - P[X > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\begin{aligned} P[T < X \leq 2T] &= F_X(2T) - F_X(T) \\ &= 1 - e^{-\lambda 2T} - (1 - e^{-\lambda T}) = -e^{-2} + e^{-1} \end{aligned}$$

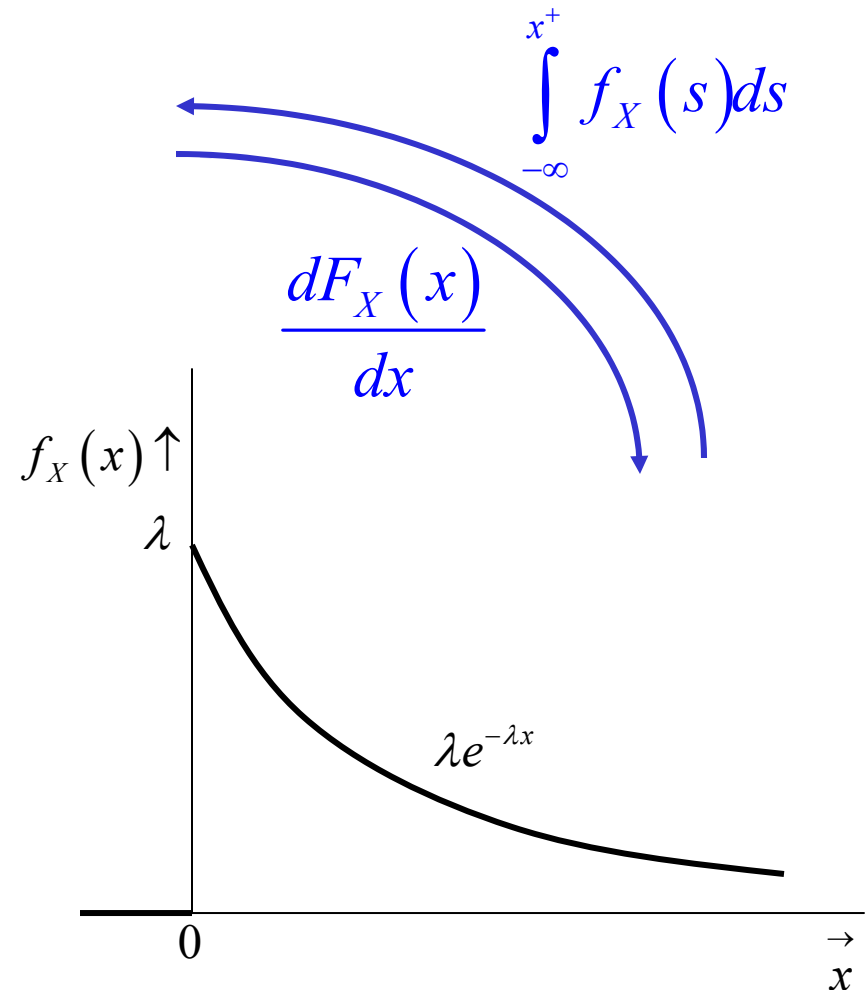
Ex 3.5 message transmission time



continuous for all x



derivative exists everywhere
except for $x=0$



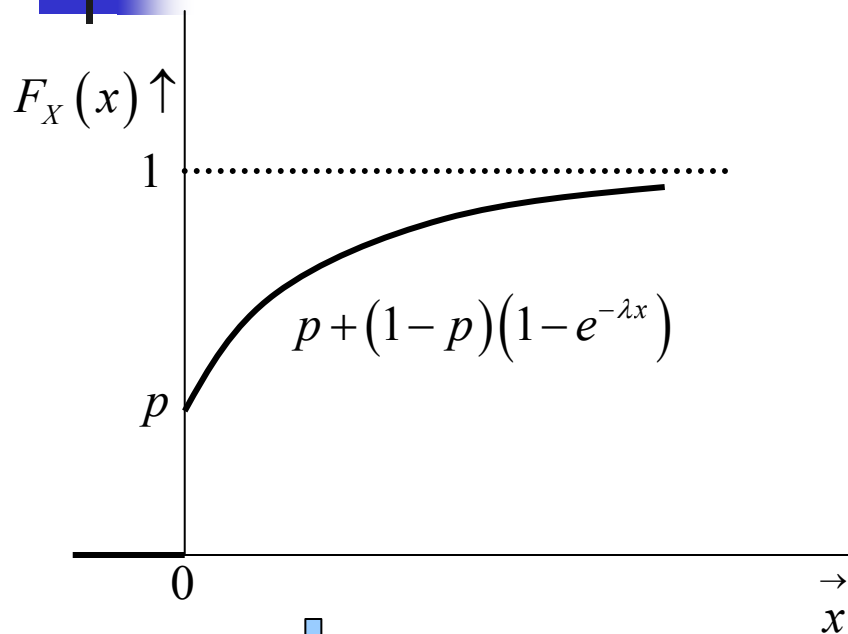


Ex 3.6 waiting time in queue

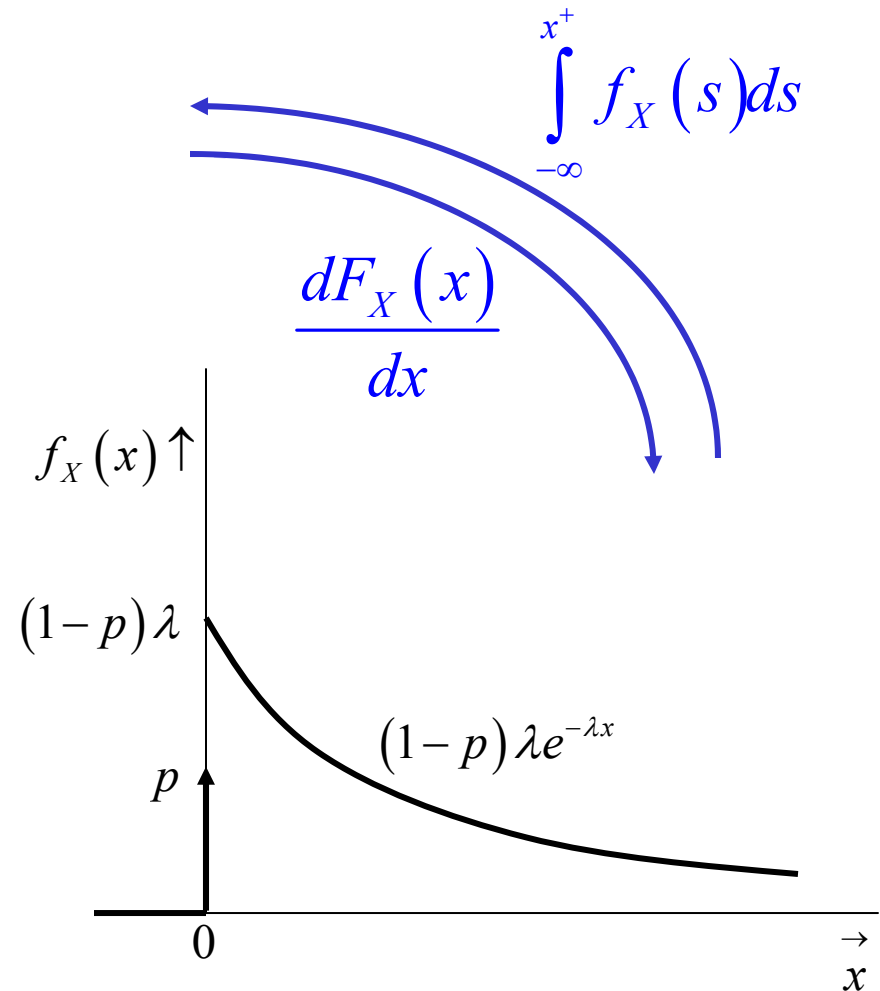
- Customer waiting time X is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy; the probability for finding the system idle is p .

$$\begin{aligned}F_X(x) &= P[X \leq x] \\ &= P[X \leq x \mid \textit{idle}]p + P[X \leq x \mid \textit{busy}](1-p) \\ &= u(x)p + (1 - e^{-\lambda x})(1-p)u(x)\end{aligned}$$

Ex 3.6 waiting time in queue



sum of step function
and continuous function





Types of random variables

- **Discrete**

- *CDF is right-continuous, staircase function of x , with jumps at a countable set of points x_k*
- *Probability mass function $p_X(x_k)$*
- *CDF can be written as weighted sum of unit step functions*

- **Continuous**

- *CDF is continuous everywhere*
- *CDF can be written as the integral of a non-negative function (its derivative)*
- *$P[X=x]=0$ for all x*



Types of random variables

- **Mixed**

- *CDF that has jumps on a countable set of points x_0, x_1, x_2, \dots and also increases continuously over at least one interval of values of x .*
- *$F_X(x) = pF_D(x) + (1-p)F_C(x)$ with $0 < p < 1$*
- *Can be viewed as being produced by a two-step process*
 - *An unfair coin is tossed: if Heads, a discrete r.v. is generated according to $F_D(x)$; if Tails, a continuous r.v. is generated according to $F_C(x)$*



More PDF properties

$$\text{ii. } P[a \leq X \leq b] = \int_{a^-}^{b^+} f_X(x) dx$$

$$\text{iii. } F_X(x) = \int_{-\infty}^{x^+} f_X(t) dt$$

the PDF completely specifies the behavior of continuous r.v.'s

$$\text{iv. } \int_{-\infty}^{\infty} f_X(t) dt = F_X(\infty) = 1 \quad \text{unit probability mass}$$

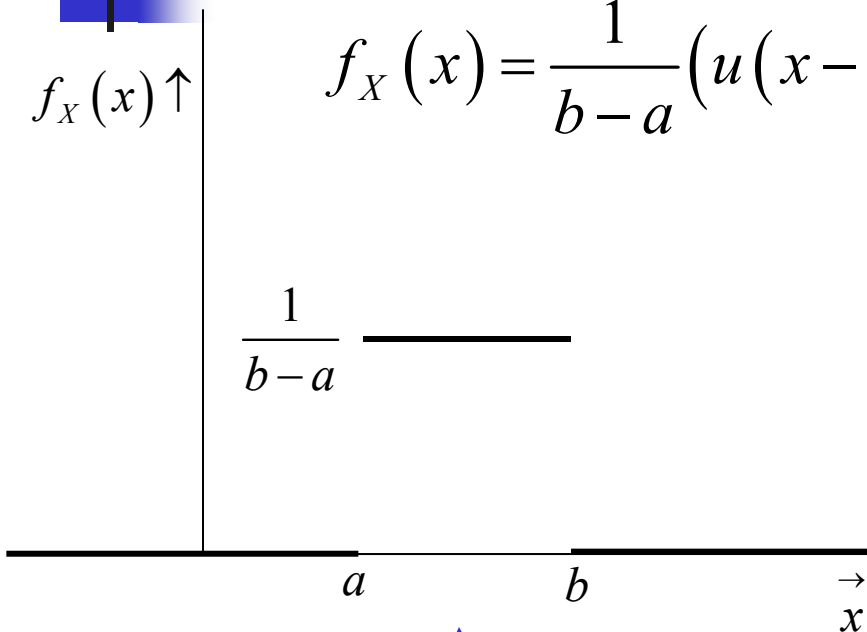
PDF is like "mass" density

a valid PDF can be formed from any nonnegative,
piecewise continuous, integrable function

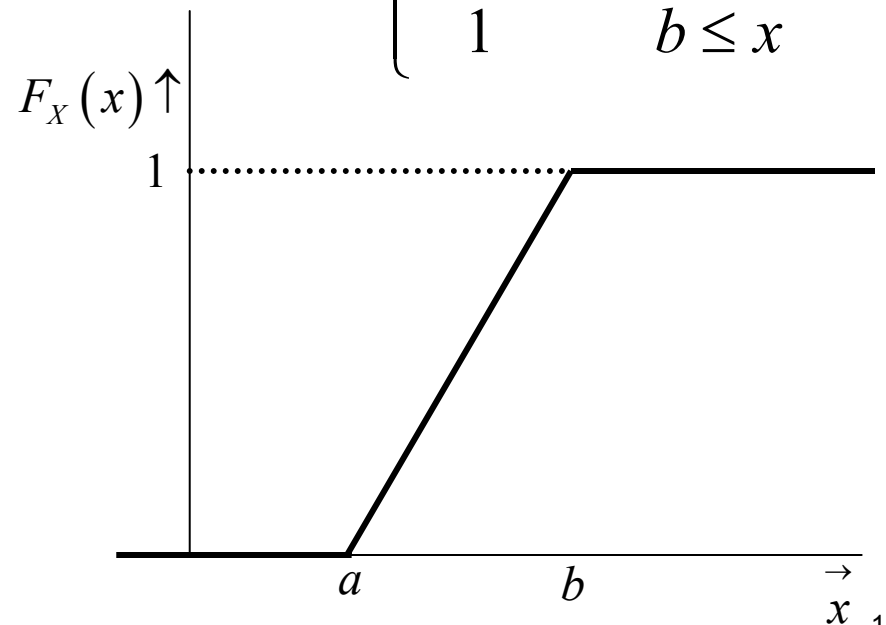
PDF – like CDF – is defined over $(-\infty, \infty)$

Ex 3.7 Uniform random variable

$$f_X(x) = \frac{1}{b-a} (u(x-a) - u(x-b))$$



$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$$



$$\int_{-\infty}^{x^+} f_X(s) ds$$

$\frac{dF_X(x)}{dx}$

Ex 3.8 normalization

not magnitude

- PDF of **sample values of speech** waveforms is found to decay exponentially - at a rate α - for positive and negative values

$$f_X(x) = ce^{-\alpha|x|} \quad -\infty < x < \infty$$

normalization:

$$\int_{-\infty}^{\infty} ce^{-\alpha|x|} dx = 2 \int_0^{\infty} ce^{-\alpha|x|} dx = 2c \frac{e^{-\alpha\infty} - e^{-\alpha 0}}{-\alpha} = 1$$

\downarrow
 $c = \frac{\alpha}{2}$

$$P[|X| < \nu] = \frac{\alpha}{2} \int_{-\nu}^{\nu} e^{-\alpha|x|} dx = \alpha \int_0^{\nu} e^{-\alpha x} dx = \alpha \frac{e^{-\alpha\nu} - e^{-\alpha 0}}{-\alpha} = 1 - e^{-\alpha\nu}$$

Bernoulli r.v.

- A is an event related to the outcomes of a random experiment

indicator function for A : $I_A(\zeta) \triangleq \begin{cases} 0 & \zeta \notin A \\ 1 & \zeta \in A \end{cases}$

↓

$$S_X = \{0, 1\}$$

assigns # to outcome

↓
 $I_A(\zeta)$ is a r.v.

pmf: $p_I(0) = 1 - p$; $p_I(1) = p = P[A]$

$I_A(\zeta) = 1 \sim$ "success" \rightarrow Bernoulli r.v.

"tossing of a biased coin": Bernoulli r.v. is a model for this fundamental mechanism for generating randomness"



Binomial r.v.

- Random experiment repeated n independent times. Let X be the number of times event A occurs in these n trials.

$$X = I_1 + I_2 + \cdots + I_n$$

sum of Bernoulli r.v.'s
(indicator functions for A in trial j)

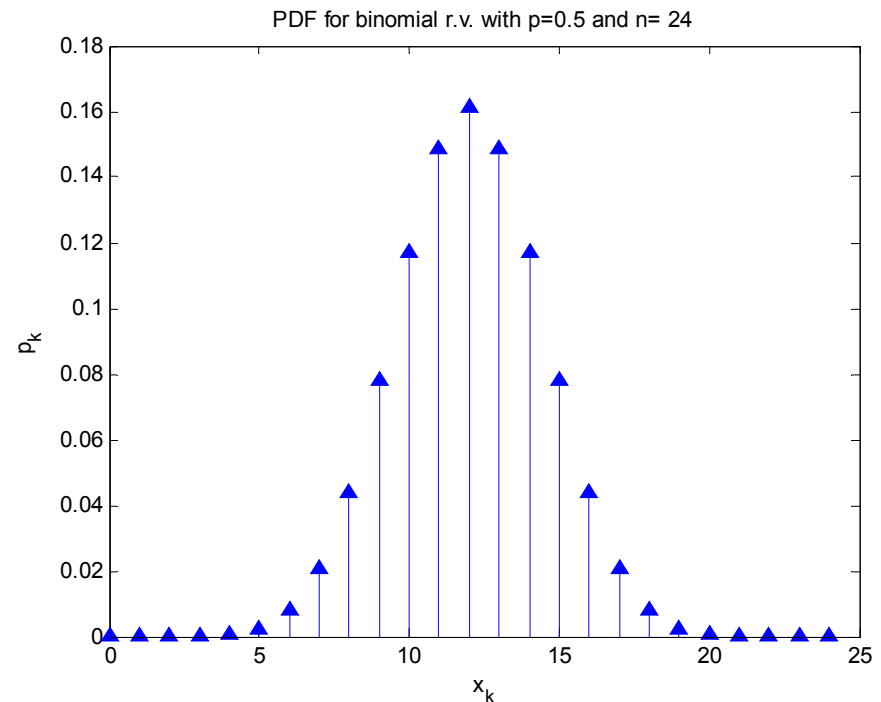
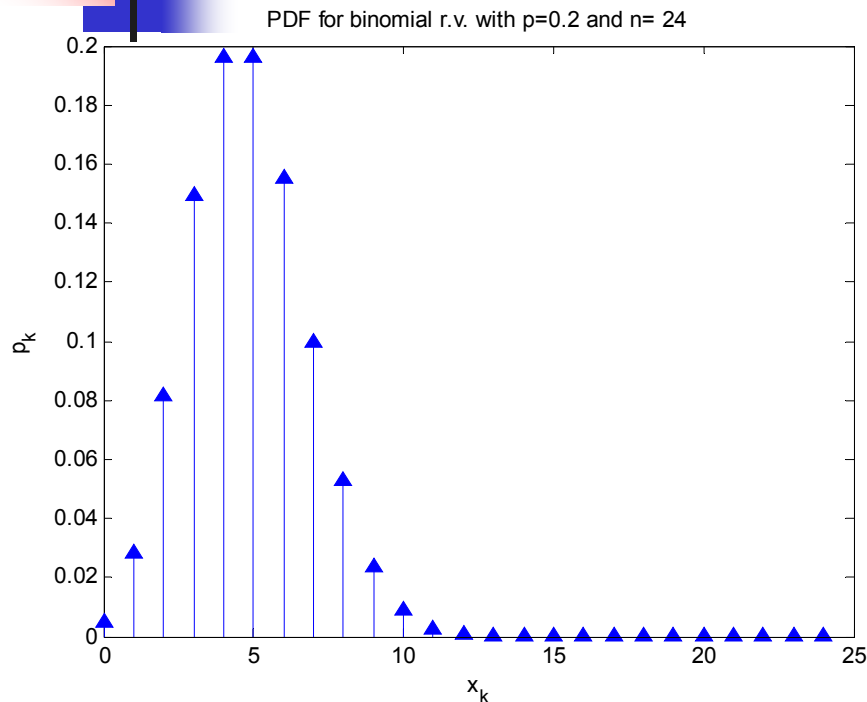
$$S_X = \{0, 1, \dots, n\}$$

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n$$

Binomial PDF

$$k_{\max} = \arg \max_k P[X = k] = \lfloor (n+1)p \rfloor$$

if $\lfloor (n+1)p \rfloor == \lceil (n+1)p \rceil$ then also max at $k_{\max-1}$



Arises in applications where there are two types of object (heads/tails, good/defective, correct/in-error, active/silent), and we're interested in the number of type 1 objects in a randomly selected batch of size n , and the type of each object is independent of the types of the other objects in the batch



Geometric r.v.

- # of independent Bernoulli trials until first occurrence of “success”

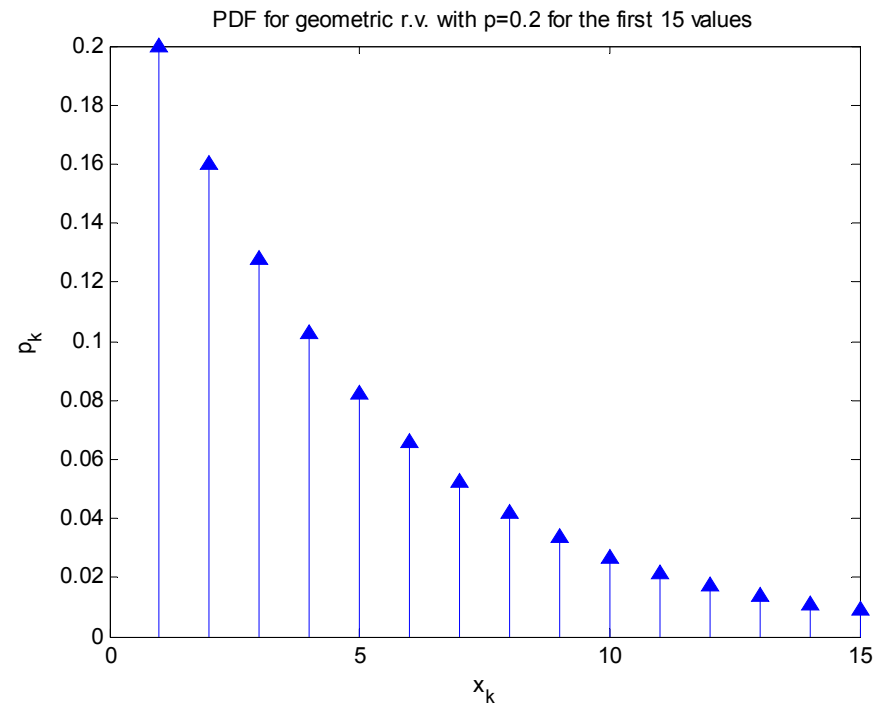
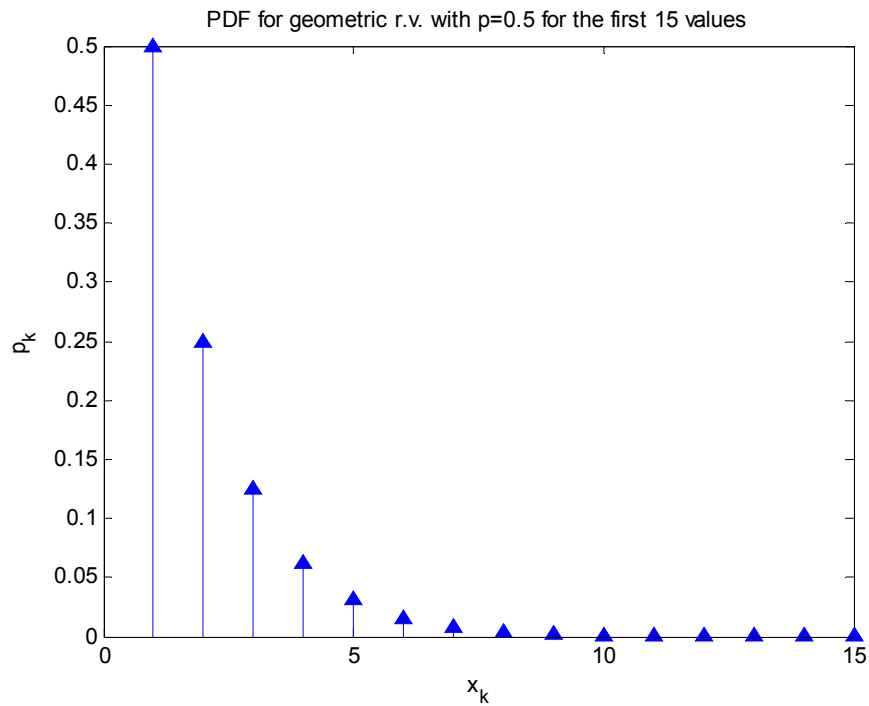
$$S_X = \{1, 2, \dots\}$$

$p = P[A] = P[\text{"success"}]$ in each Bernoulli trial

$$P[M = k] = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, \dots$$

↙ geometric decay

Geometric PDF



decay like 0.5^k and 0.8^k respectively



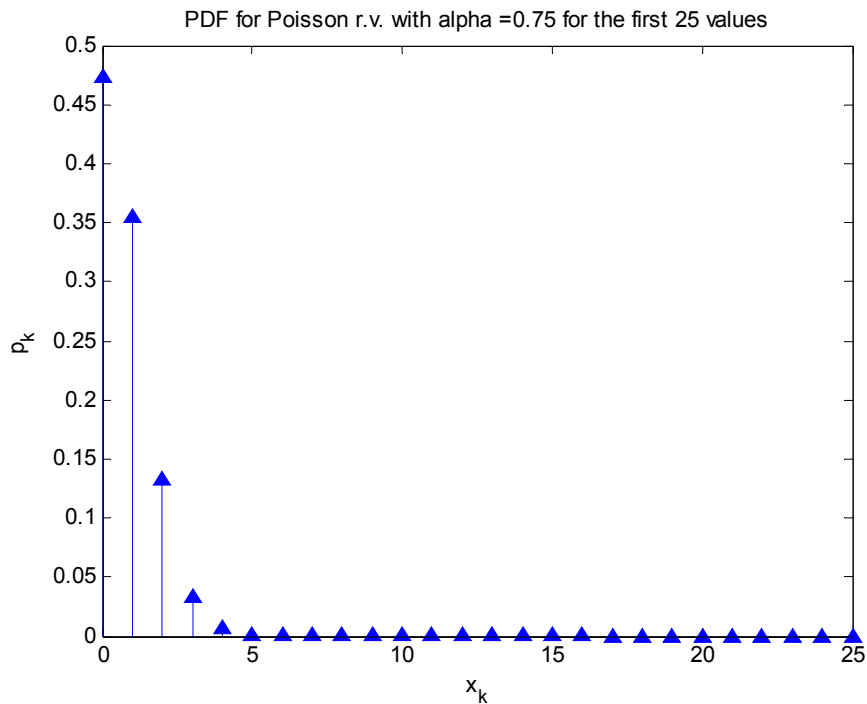
Poisson r.v.

- Interested in counting the # occurrences of an event in a certain time period or in a certain region in space
 - *Events occur completely “at random”*
 - Emissions from radioactive substances
 - Counts of demands for telephone connections
 - Counts of defects in a semiconductor chip

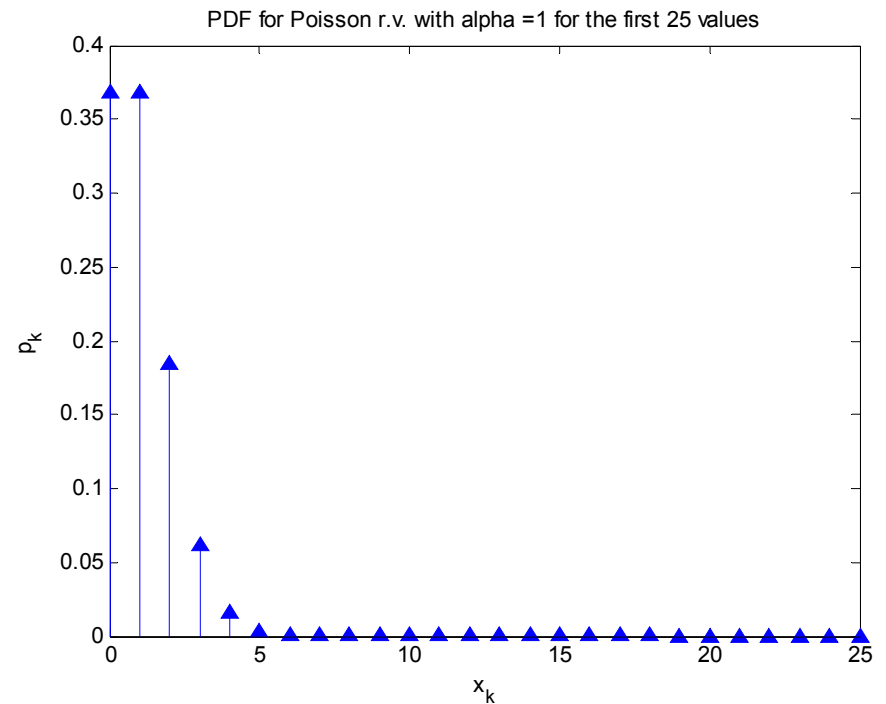
$$P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha} \text{ for } k = 0, 1, 2, \dots$$

α is the average # of event occurrences in a specified interval or region in space

Poisson PDF



$\max P[N = k]$ is at 0 for $\alpha < 1$

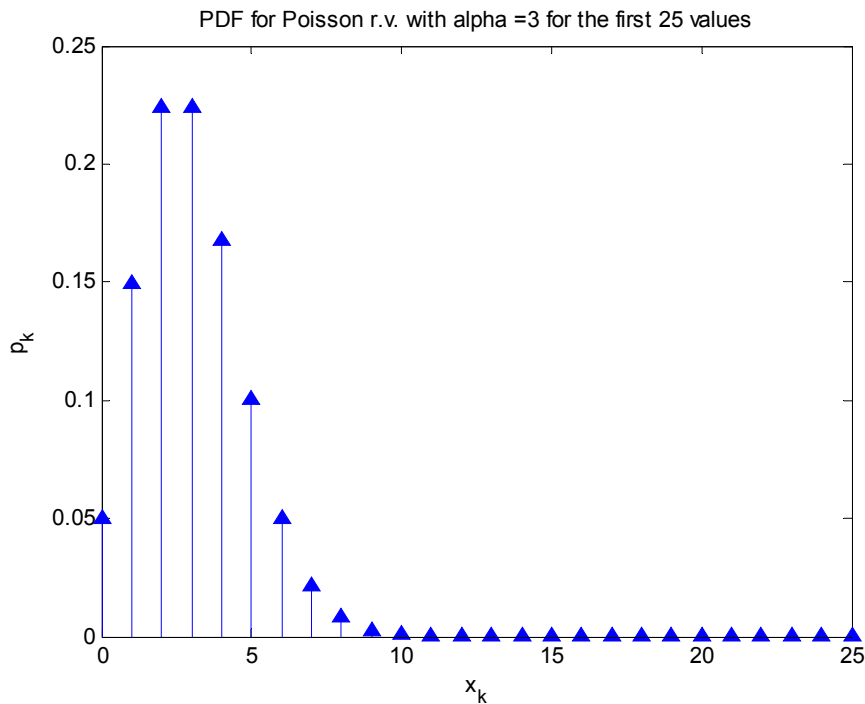


$\max P[N = k]$ is at $k = \alpha$ and
at $k = \alpha - 1$ for integer $\alpha \geq 1$

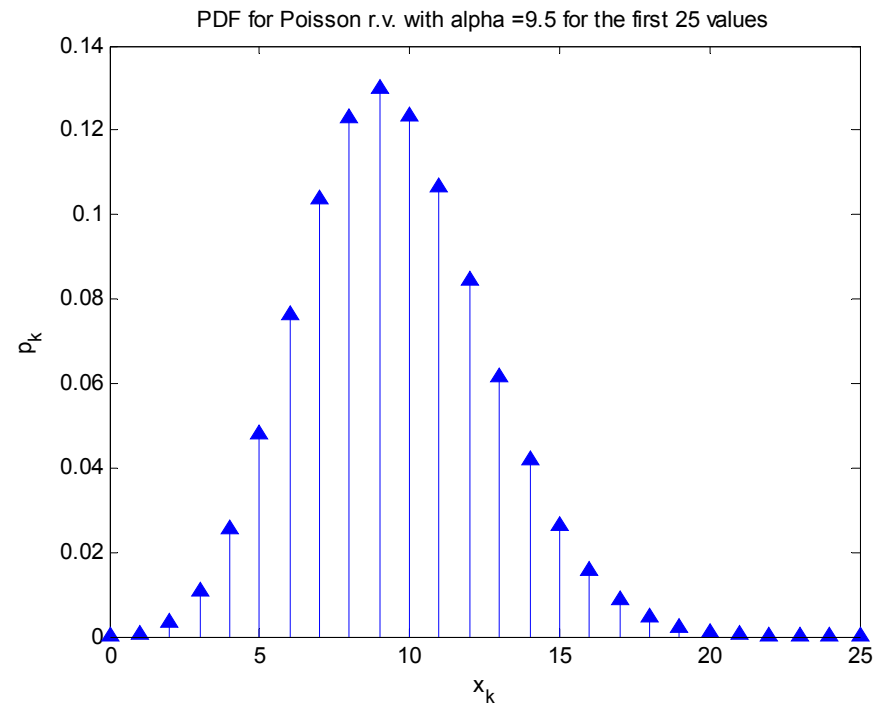
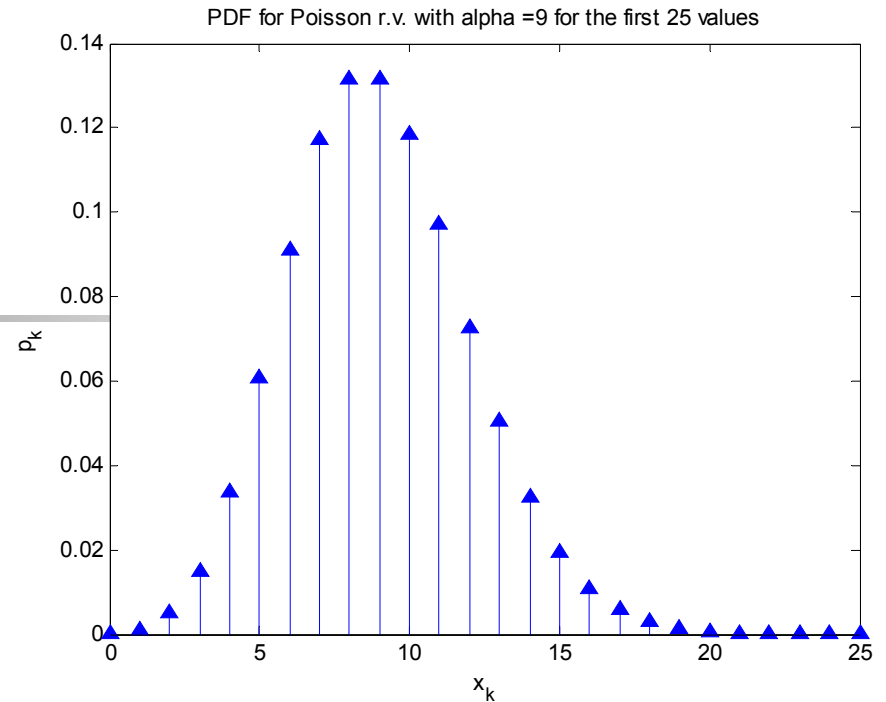


Poisson PDF

$\max P[N = k]$ is at $k = \alpha$ and
at $k = \alpha - 1$ for integer $\alpha \geq 1$



$\max P[N = k]$ is at $\lfloor \alpha \rfloor$ for $\alpha > 1$





Exponential r.v.

- Arises in modeling of the time between occurrence of events, and in modeling lifetime of devices and systems; λ is the rate at which events occur

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$$F_X(x) = (1 - e^{-\lambda x}) u(x)$$

shown earlier

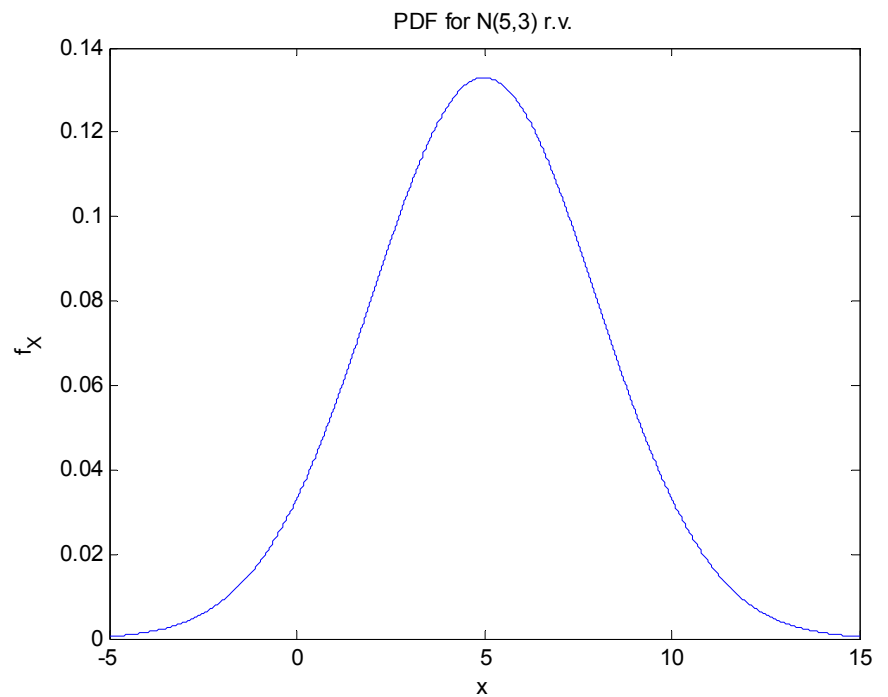
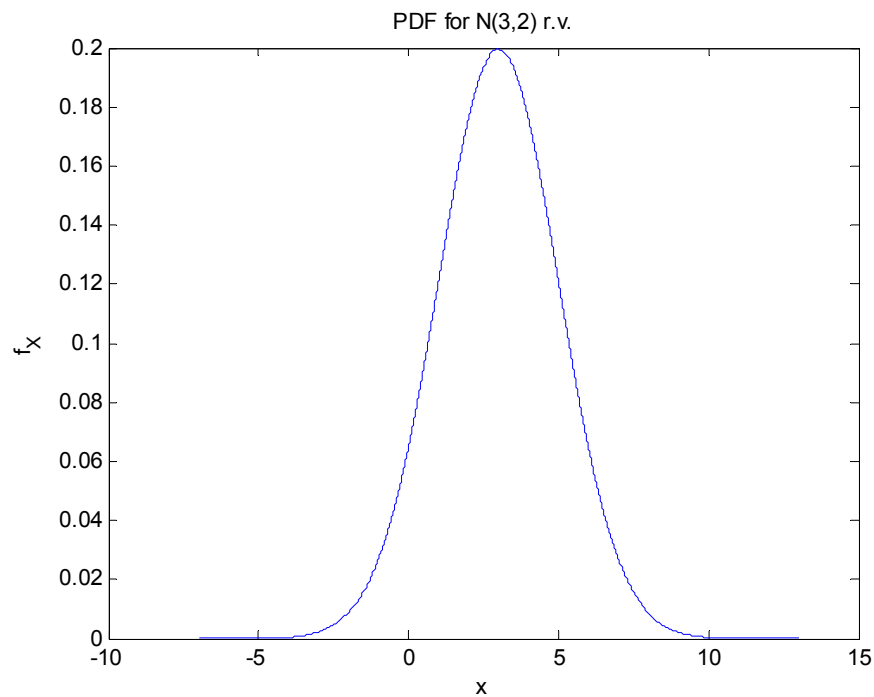


Gaussian (normal) r.v.

- In many situations in man-made and natural phenomena one deals with a r.v. X that consists of a large sum of “small” r.v.’s
 - *Exact PDF becomes complex and unwieldy*
- Under fairly general conditions, as the number of components becomes large (CLT), the CDF approaches that of the normal r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty$$
$$= N(m, \sigma)$$

Gaussian (normal) r.v. - PDF



the "bell-shaped" curve

Gaussian (normal) r.v. - CDF

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(s-m)^2}{2\sigma^2}} ds$$

↓ COV: $t = \frac{s-m}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(x-m)}{\sigma}} e^{-t^2/2} dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right) \text{ where } \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$



CDF for $N(0,1)$ r.v.

“standard normal”

Q-function

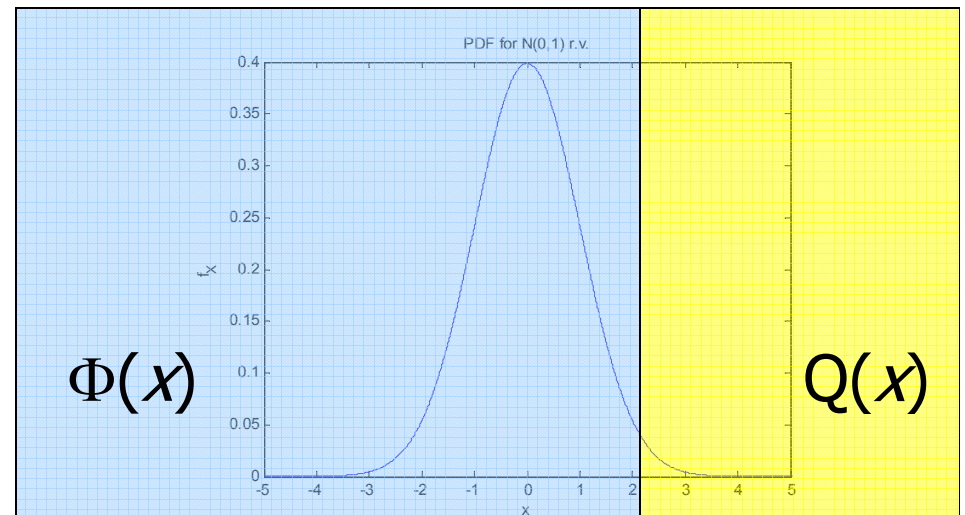
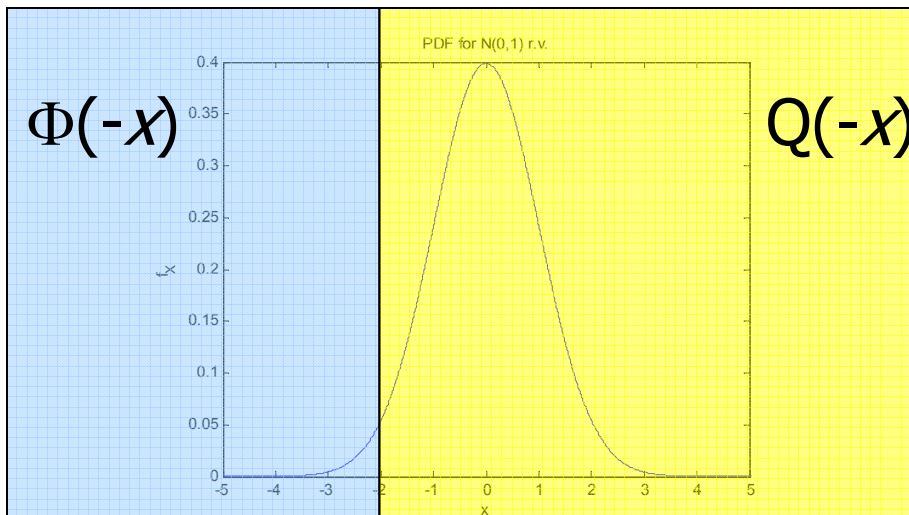
used by EE as error probability

$$Q(x) \triangleq 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$$

probability of the "tail"

$$Q(0) = 0.5$$

$$Q(-x) = 1 - \Phi(-x) = 1 - Q(x)$$





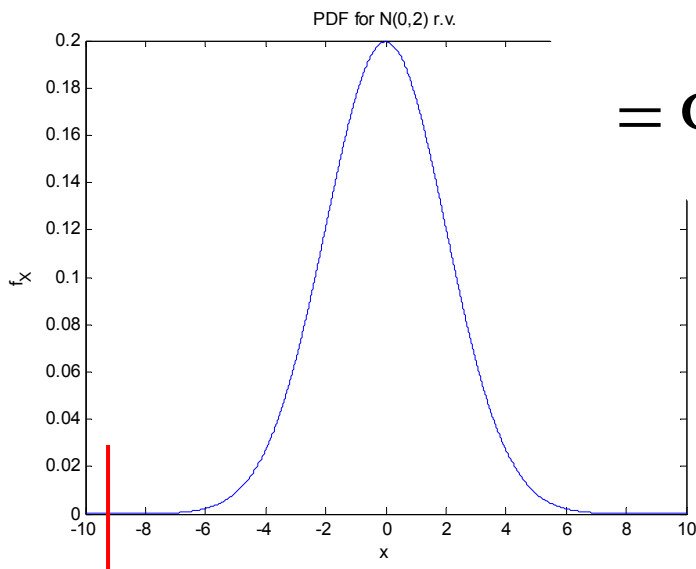
Gaussian (normal) r.v.

- Plays an important role in communication systems, where transmission of signals is subject to noise
 - *Noise resulting from the thermal motion of electrons, can – from physical principles – be shown to have a Gaussian PDF*

Ex

- A communication system accepts a positive voltage V as input and outputs a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and N is $\sim N(0, 2)$. Find $V \ni P[Y < 0] = 10^{-6}$

$$P[Y < 0] = P[\alpha V + N < 0] = P[N < -\alpha V]$$



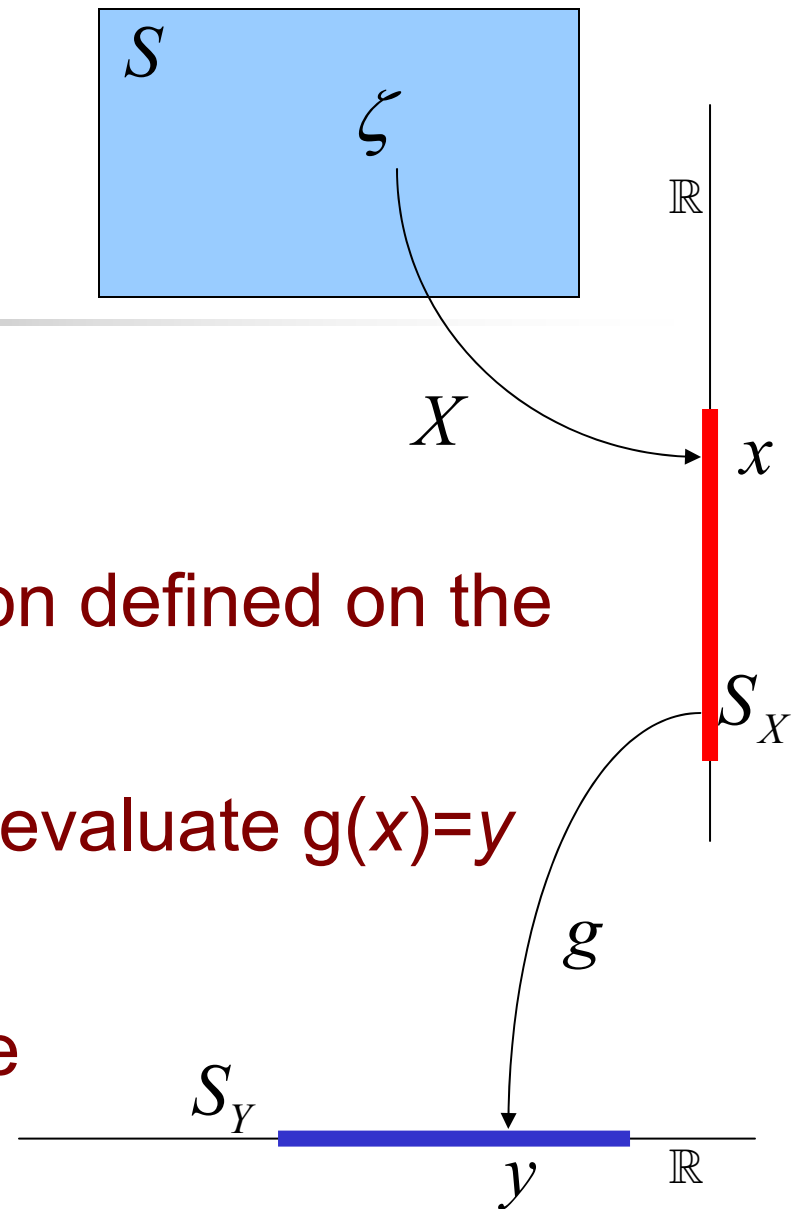
$$= \Phi\left(\frac{-\alpha V}{\sigma}\right) = Q\left(\frac{\alpha V}{\sigma}\right) = 10^{-6}$$

Table

$$\frac{\alpha V}{\sigma} = 4.7535$$

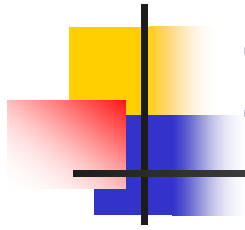
$$V = 4.7535 \frac{\sigma}{\alpha} = 950.6$$

Functions of a r.v.



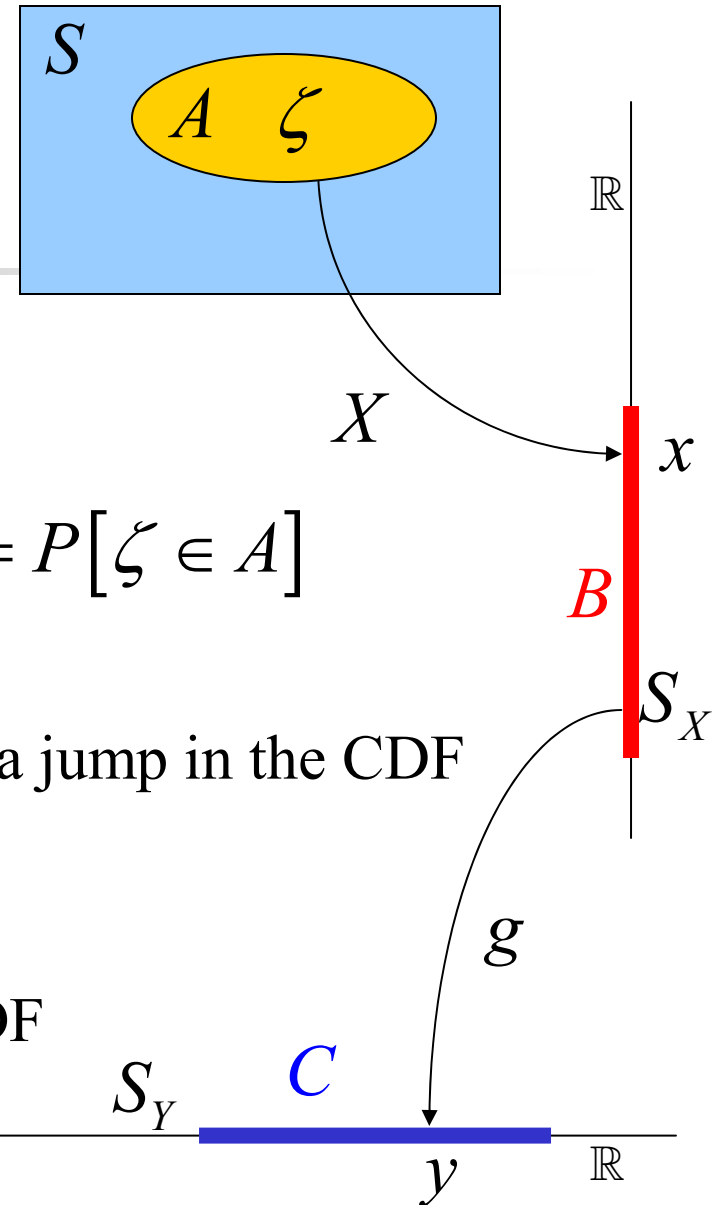
- X is a random variable
- $g(x)$ is a real-valued function defined on the real line
- $Y=g(X)$, i.e. for every $X=x$, evaluate $g(x)=y$ and assign it to Y
- Y is also a random variable
- Find CDF and PDF of Y

ultimately probabilities are induced by the underlying experiment



Induced probability

equivalent events



$$P[Y \in C] = P[g(X) \in C] = P[X \in B] = P[\zeta \in A]$$

useful events:

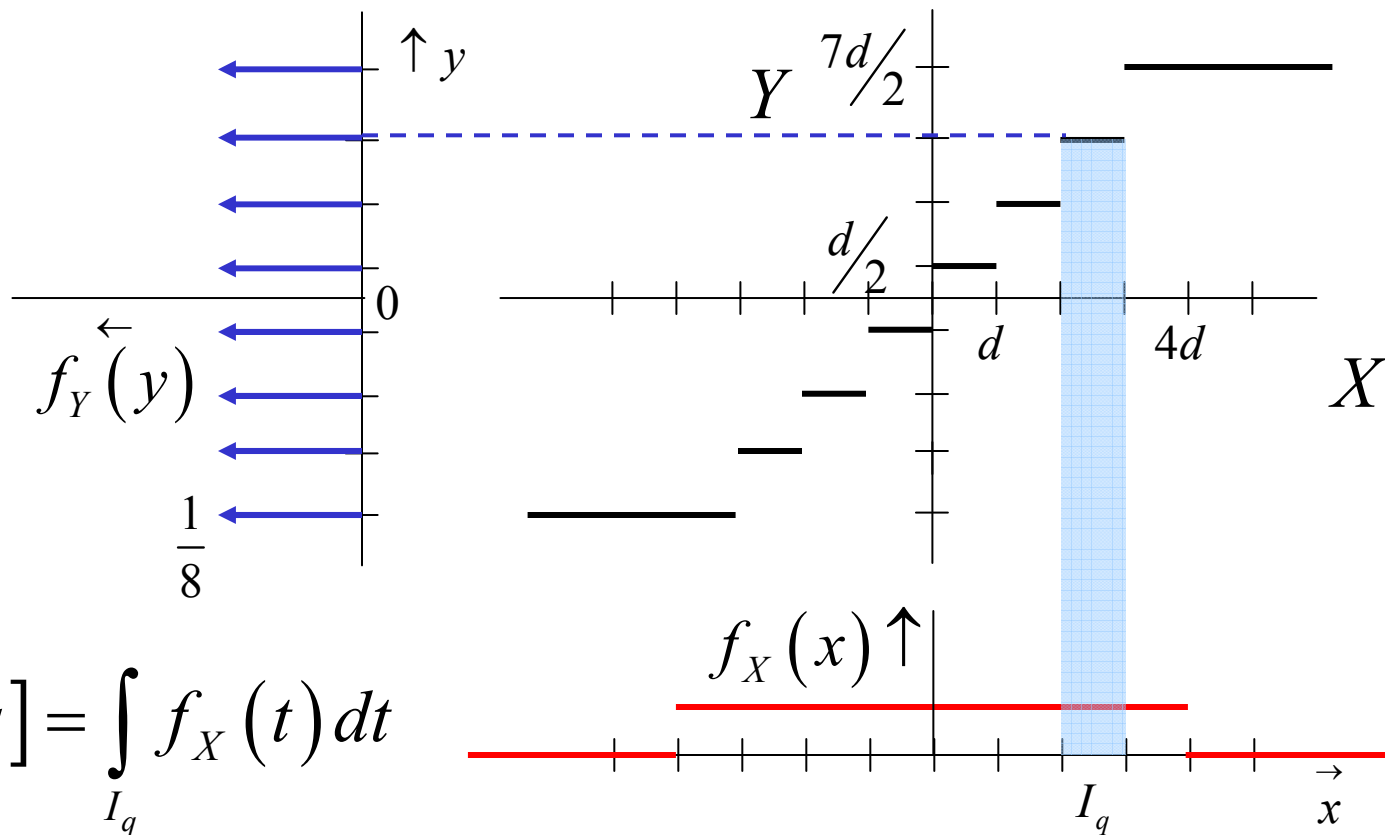
$\{g(X) = y_k\}$ is used to find the magnitude of a jump in the CDF

$\{g(X) \leq y\}$ is used to directly find the CDF

$\{y < g(X) \leq y + h\}$ is useful in finding the PDF

Ex 3.22 8-level uniform quantizer

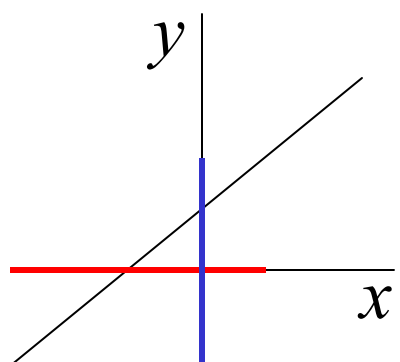
- Let X be a sample voltage of a speech waveform; assume X is uniform over $[-4d, 4d]$



$$P[Y = q] = \int_{I_q} f_X(t) dt$$

Ex 3.23 a linear function $Y = aX + b \quad a \neq 0$

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = P[aX \leq y - b]$$



$$= \begin{cases} P\left[X \leq \frac{y-b}{a}\right] & a > 0 \\ P\left[X \geq \frac{y-b}{a}\right] & a < 0 \end{cases} = \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

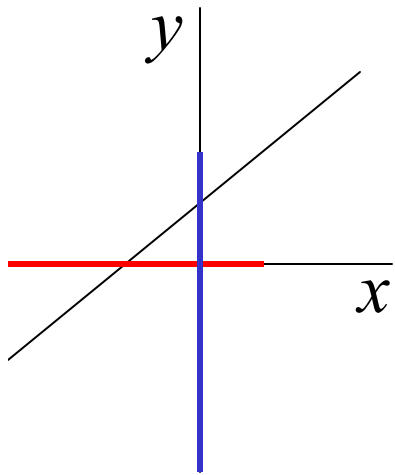
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$



Ex 3.24 linear function of Gaussian r.v.

$$Y = aX + b \quad a \neq 0$$



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

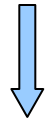
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{|a\sigma|\sqrt{2\pi}} e^{-\frac{(y-b-am)^2}{2(a\sigma)^2}}$$

linear function of a Gaussian r.v. is also a Gaussian r.v.

Ex 3.25 square law device

$$Y = X^2$$

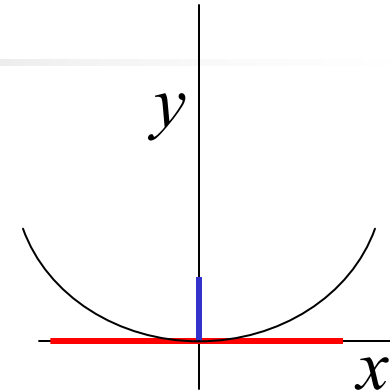


$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y]$$

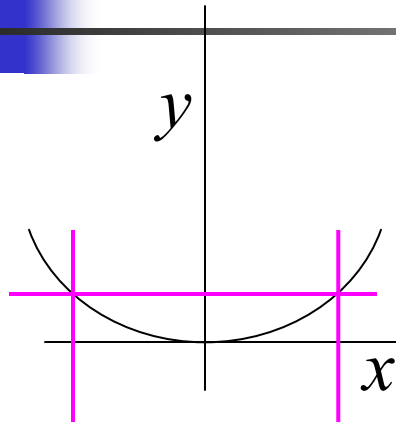
$$= P[-\sqrt{y} \leq X \leq \sqrt{y}] = [F_X(\sqrt{y}) - F_X(-\sqrt{y})]u(y)$$

$$f_Y(y) = \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})]u(y)$$

$$= \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \right] u(y)$$



from Ex 3.26



$$y_0 = g(x) \leftarrow x_0, x_1$$

produces 2 terms in PDF

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

Redo Ex 3.27

for $y < 0$: $y = x^2$ has no solutions $\Rightarrow f_Y(y) = 0$

for $y \geq 0$: $y = x^2$ has two solutions: $x_0 = \sqrt{y}$; $x_1 = -\sqrt{y}$

$$\begin{aligned} f_Y(y) &= \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \left[\frac{f_X(x)}{|2x|} \right]_{x=x_0} + \left[\frac{f_X(x)}{|2x|} \right]_{x=x_1} \\ &= \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \right] u(y) \end{aligned}$$



Nonlinear function $Y=g(X)$

$$P[C_y] = P[B_y]$$

equivalent events \downarrow induce equal probabilities

$$f_Y(y)|dy| = f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \sum_k \left[f_X(x) \left| \frac{dx}{dy} \right| \right]_{x=x_k}$$

function of y

Ex 3.28

$$X \sim U(0, 2\pi]$$

$$Y = \cos(X)$$

for $y < -1$ or $y > 1$: no sol^s

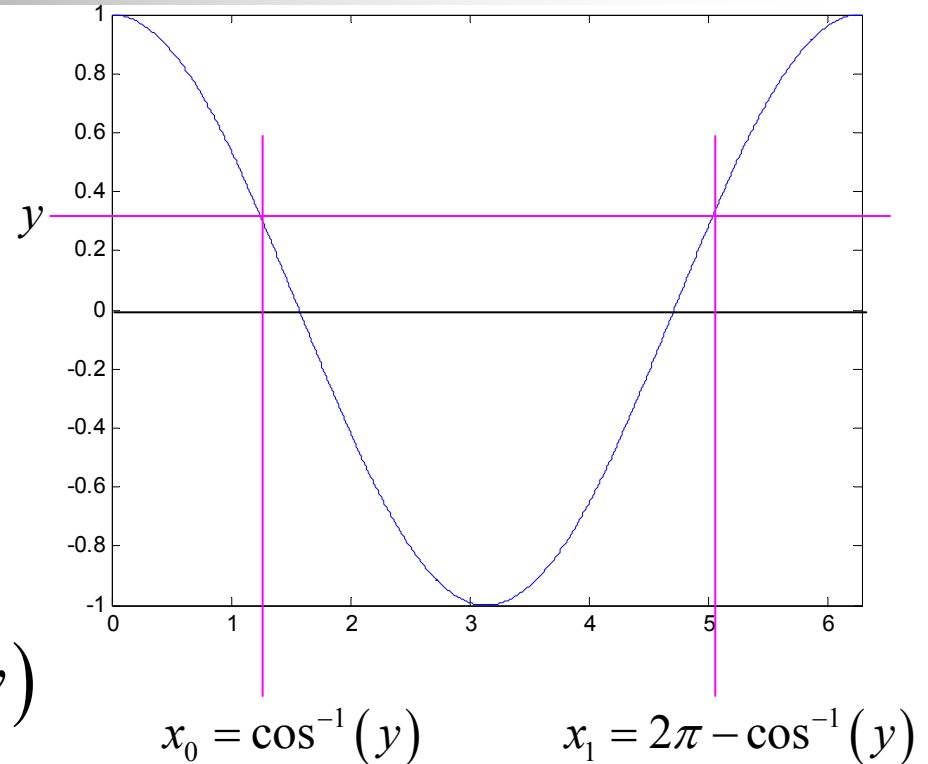
$$f_Y(y) = 0$$

for $-1 \leq y \leq 1$:

$$x_0 = \cos^{-1}(y); x_1 = 2\pi - \cos^{-1}(y)$$

$$\left. \frac{dy}{dx} \right|_{x=x_0} = -\sin(x_0) = -\sin\{\cos^{-1}(y)\} = -\sqrt{1-y^2}$$

$$Y = \cos(X)$$





$$Y = \cos(X)$$

$$X \sim U(0, 2\pi] \rightarrow f_X(x) = \frac{1}{2\pi} [u(x) - u(x - 2\pi)]$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right]_{x=x_k} = \frac{1}{2\pi \left| -\sqrt{1-y^2} \right|} + \frac{1}{2\pi \left| \sqrt{1-y^2} \right|} = \frac{1}{\pi \sqrt{1-y^2}} \quad -1 \leq y \leq 1$$

$$F_Y(y) = \frac{1}{2} + \frac{\sin^{-1}(y)}{\pi} \text{ for } -1 \leq y \leq 1$$

Y has the arcsine distribution

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$