Probability Model

- Random experiment
  - *Repeating the experiment produces a different (a priori unknown) outcome each time*
    - Tossing a coin
    - Throwing dice
    - Setting a thermostat
    - Turning on a function generator
    - Powering up a flip-flop circuit
    - Dialing a phone number

Experiment: select ball from urn

- 3 balls, labeled 0, 1, and 2
  - *Blindly select one ball*
  - *Record label*
  - *Replace ball*
  - *Repeat*

a priori unknown!

… 0 2 2 1 0 2 1 1 0 2 0 0 1 2 1 2 0 1 2 0 ? …

a series of outcomes

outcome is element of the sample space $S = \{0,1,2\}$

Relative frequency of outcome

0 2 2 1 0 2 1 1 0 2 0 0 1 2 1 2 0 1 2 0 ? …

$N_k(n)$: total # times outcome *k* has occurred after *n* trials

$$N_0(20) = 7 \qquad N_1(20) = 6 \qquad N_2(20) = 7$$

$f_k(n) = \dfrac{N_k(n)}{n}$: Relative frequency (of occurrence) of outcome *k*

statistical regularity: $\lim_{n\to\infty} f_k(n) = p_k$

constant

probability of the outcome

Changing the experiment (condition)

- 3 balls, labeled 0, 0, and 2
  - *Blindly select one ball*
  - *Record label*
  - *Replace ball*
  - *Repeat*

a priori unknown!

a series of outcomes

… 0 2 2 0 0 2 0 0 0 2 0 0 0 2 0 2 0 0 2 0 ? …

each outcome is an element of the sample space $S = \{0, 2\}$

Relative frequency of outcome

0 2 2 0 0 2 0 0 0 2 0 0 0 2 0 2 0 0 2 0 ? ...

$N_0(20) = 13$ $\quad$ $N_2(20) = 7$

changed how?

$$f_k(n) = \frac{N_k(n)}{n}:$$ Relative frequency (of occurrence) of outcome *k*

statistical regularity: $\lim_{n\to\infty} f_k(n) = p_k$ probability of the outcome

constant

changed how?

Conditions under which a random experiment is performed determine probabilities of the outcomes of the experiment

Axioms

- A random experiment has been defined, and a set *S* of all possible outcomes has been identified
- A class of subsets of *S* – called events – has been specified
- Each event *A* has been assigned a number, *P*[*A*], in such a way that the following axioms are satisfied:

1. $0 \le P[A] \le 1$
2. $P[S] = 1$
3. $P[A \; or \; B] = P[A] + P[B]$

   when *A* and *B* are events that cannot occur simultaneously.

do these look familiar?

Random experiments $\begin{cases} \text{experimental procedure} \\ \text{measurement/observation} \end{cases}$

- E_1: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.
- E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.
- E_3: Toss a coin three times and note the sequence of heads and tails.
- E_4: Toss a coin three times and note the number of heads.
- E_5: Count the number of voice packets containing only silence produced from a group of *N* speakers in a 10ms period.
- E_6: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
- E_7: Pick a number at random between zero and one.
- E_8: Measure the time between two message arrivals at a message center.

Sample Space

- The sample space S of a random experiment is defined as the set of all possible outcomes.
- An outcome or sample point ζ of a random experiment is a result that cannot be decomposed into other results. $\zeta \in S$
- One and only one outcome occurs when a random experiment is performed.
  - *Outcomes are mutually exclusive – they cannot occur simultaneously*

Sample Spaces – example experiments

- E_1: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

$$S_1 = \{1, 2, \ldots, 50\}$$

- E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

$$S_2 = \{(1,b),(2,b),(3,w),(4,w)\}$$

- E_3: Toss a coin three times and note the sequence of heads and tails.

$$S_3 = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

- E_4: Toss a coin three times and note the number of heads.

$$S_4 = \{0, 1, 2, 3\}$$

sample spaces are sets

Sample Spaces

- E_5: Count the number of voice packets containing only silence produced from a group of *N* speakers in a 10ms period.

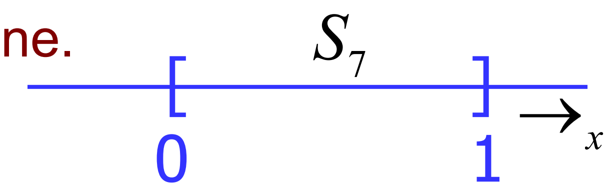
$$S_5 = \{0,1,2,\ldots,N\}$$

- E_6: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

$$S_6 = \{1,2,3,\ldots\}$$

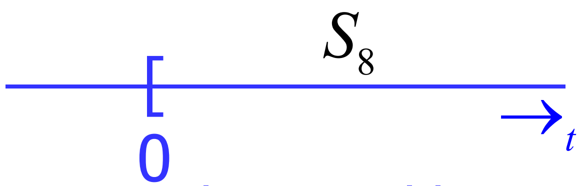
- E_7: Pick a number at random between zero and one.

$$S_7 = \{x : 0 \le x \le 1\} = [0,1]$$



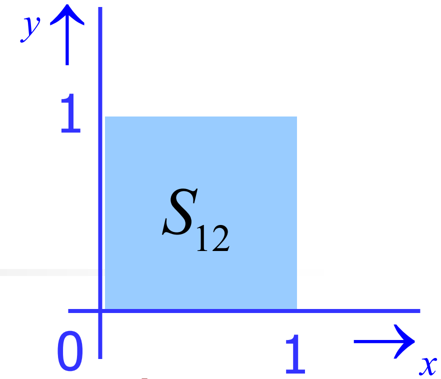
- E_8: Measure the time between two message arrivals at a message center.

$$S_8 = \{t : t \ge 0\} = [0,\infty)$$



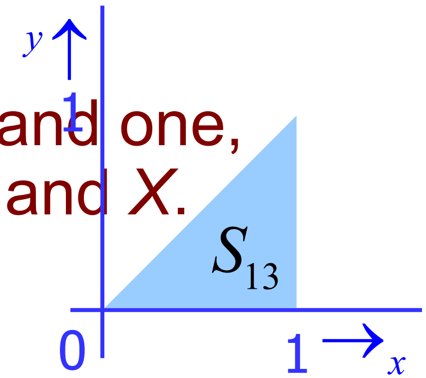
sample spaces visualized as intervals on real line

Sample Spaces



- E_{12}: Pick two numbers at random between zero and one.

$$S_{12} = \{(x, y) : 0 \le x \le 1 \; and \; 0 \le y \le 1\}$$



- E_{13}: Pick a number *X* at random between zero and one, then pick a number *Y* at random between zero and *X*.

$$S_{13} = \{(x, y) : 0 \le y \le x \le 1\}$$

- E_{14}: A system component is installed at time t=0. For t≥0 let $X(t)$=1 as long as the component is functioning, and let $X(t)$=0 after the component fails.

$$S_{14} = \left\{ X(t) : X(t) = \begin{cases} 1 & 0 \le t < t_0 \\ 0 & t \ge t_0 \end{cases}, \; time \; of \; component \; failure \; t_0 \right\}$$

sample spaces visualized as regions of the plane

Sample spaces

- Finite, countably infinite, uncountably infinite
- Discrete sample space
  - *If S is countable*
    - Outcomes can be put in 1-to-1 correspondence with the positive integers
- Continuous sample space
  - *If S is not countable*
- Can be multi-dimensional
  - *Can sometimes be written as Cartesian product of other sets*

Events

- A subset of *S*
  - *Does the outcome satisfy certain conditions?*
    - E_{10}: Determine the value of a voltage waveform at time t_1.

$$S_{10} = \{v : -\infty < v < \infty\} = (-\infty, \infty)$$

  - *Is the voltage negative?*
- Event *A* occurs iff the outcome of the experiment ζ is in the subset

iff = if and only if

$$A = \{\zeta : -\infty < \zeta < 0\}$$

Special events

- Certain event S
  - *Consists of all outcomes, hence occurs always*
- Impossible or null event $\varnothing$
  - *Contains no outcomes, hence never occurs*

Events – example experiments

- E_1: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

  "An even-numbered ball is selected" $A_1 = \{2, 4, \ldots, 48, 50\}$

- E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

  "The ball is white and even-numbered" $A_2 = \{(4, w)\}$

- E_3: Toss a coin three times and note the sequence of heads and tails.

  "The three tosses give the same outcome" $A_3 = \{HHH, TTT\}$

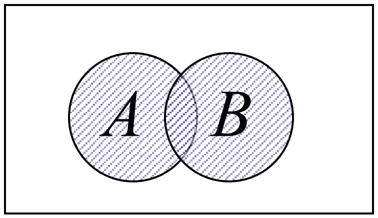
- E_4: Toss a coin three times and note the number of heads.

  "The number of heads equals the number of tails" $A_4 = \varnothing$

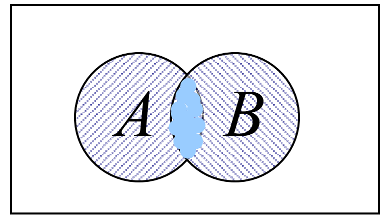
Set (event) operations

- Union $A \bigcup B$
- Intersection $A \bigcap B$
  - *Mutually exclusive* $A \bigcap B = \varnothing$
- Complement $A^c \text{ such that } A^c \bigcup A = S \text{ and } A^c \bigcap A = \varnothing$
- Implies
  - *all outcomes in A are also outcomes in B* $A \subset B$
- Equal
  - *A and B contain the same outcomes* $A = B$

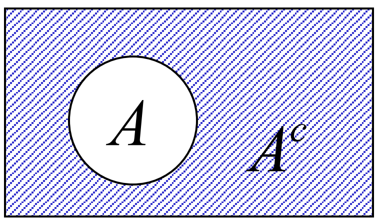
Venn diagrams

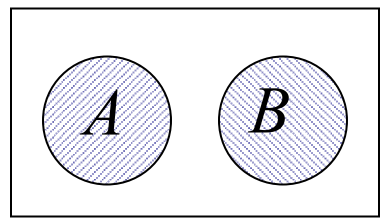


$A \bigcup B$

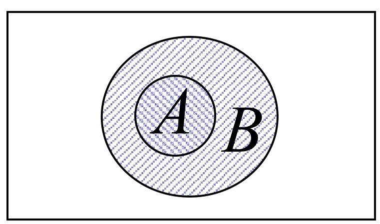


$A \bigcap B$





$A \bigcap B = \varnothing$



$A \subset B$

properties

- Commutative $A \bigcup B = B \bigcup A$ $A \bigcap B = B \bigcap A$
- Associative $A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C$

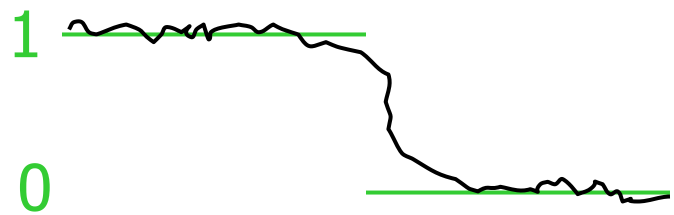
  $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$
- Distributive $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

  $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$
- deMorgan's rules $(A \bigcap B)^c = A^c \bigcup B^c$

  $(A \bigcup B)^c = A^c \bigcap B^c$

Conditional probability

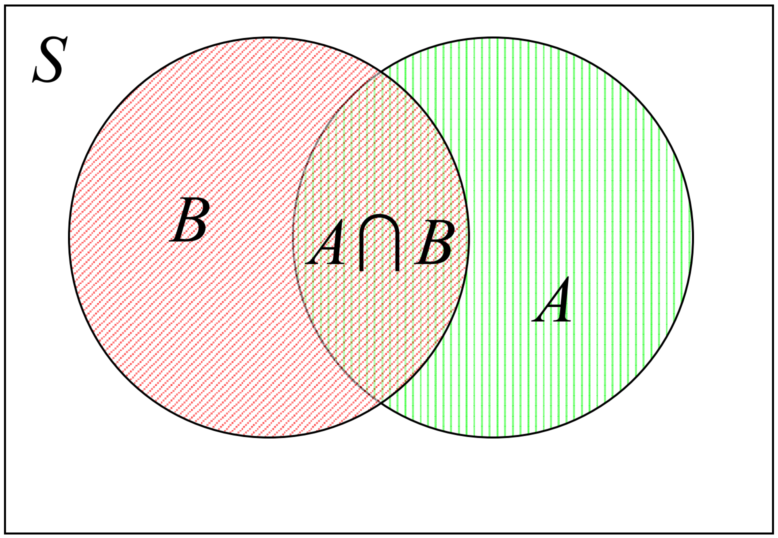
- Are two events *A* and *B* related, in the sense that one tells us something about the other?
  - *We observe/measure something to learn about something else that's not directly measurable*



- Conditional probability of event *A* given that event *B* has occurred

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0$$

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad for\ P[B] > 0$$



event B has occurred $\zeta \in B$

$$S_{.|B} = B$$

$P[A \mid B]$ deals with $\zeta \in A \cap B$

a renormalization of probability for the reduced sample space

Ex Conditional probability

- Select a ball from an urn containing 2 black balls, labeled 1 and 2, and 2 white balls, labeled 3 and 4

$$S = \{(1,b),(2,b),(3,w),(4,w)\}$$

- Assuming equi-probable outcomes, find *P*[*A*|*B*] and *P*[*A*|*C*], where

A={(1,b),(2,b)} "black ball selected"

B={(2,b),(4,w)} "even-numbered ball selected"

C={(3,w),(4,w)} "number of ball is >2"

$$\left.\begin{array}{l} P[A \cap B] = P[(2,b)] \\ P[A \cap C] = P[\varnothing] = 0 \end{array}\right\}$$

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{.25}{.5} = .5 = P[A]$$

$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{0}{.5} = 0 \neq P[A]$$

knowing B doesn't help, knowing C does