ECE5984

Orthogonal Frequency Division Multiplexing and Related Technologies

Fall 2007

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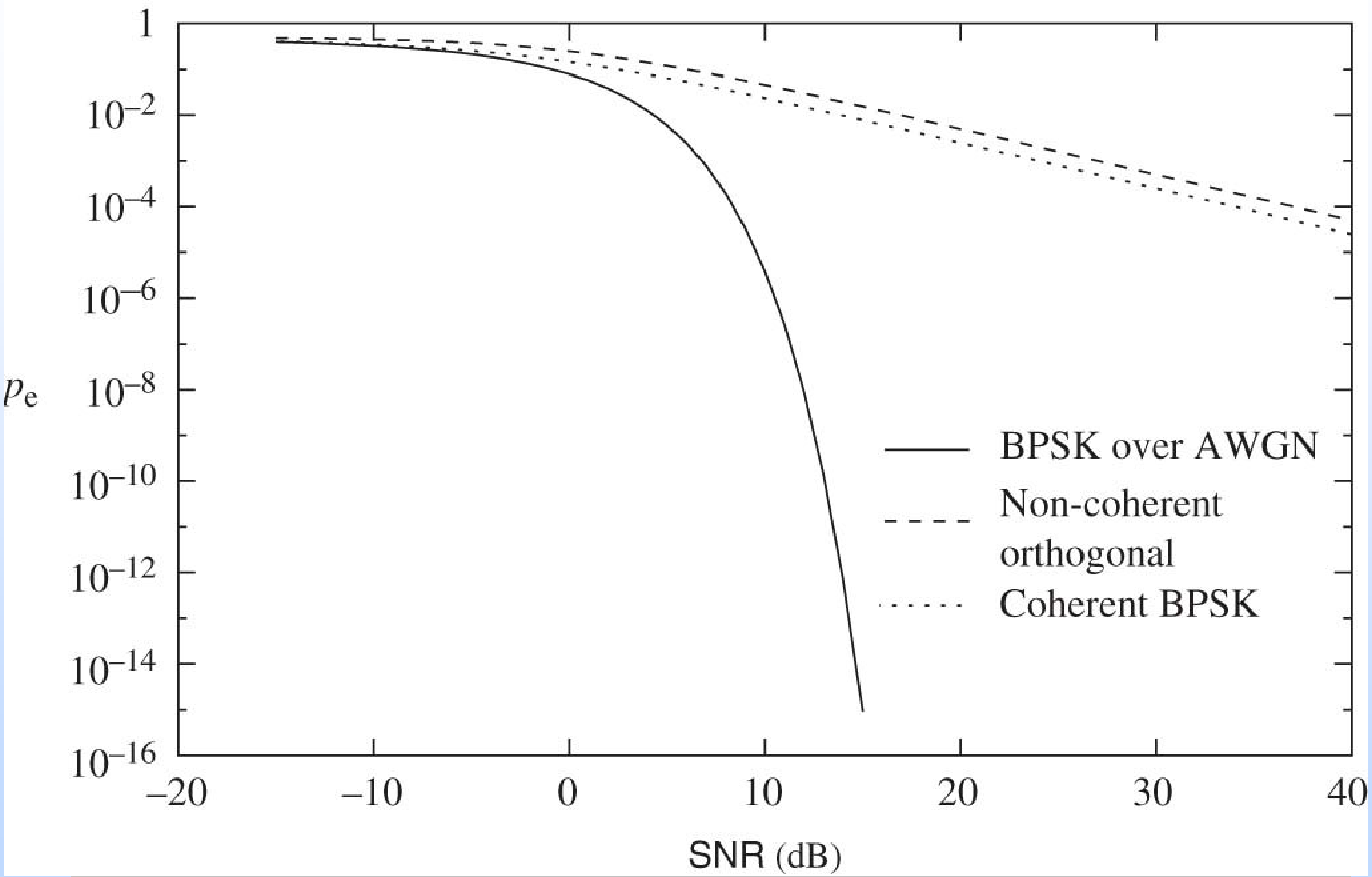
Performance of OFDM in Gaussian and Fading channels

Matlab Assignment #1
Thursday 4 October 2007

- **Develop an OFDM system with the following components**
  - S/P
  - Mapping model (modulation techniques)
  - Coding model (conv, turbo)
  - IFFT
  - CP
  - Channel (Gaussian, SFFF channel)
  - Mapping decoding
  - Decoding model
  - FFT
  - CP removal
  - Channel Estimation (later)
- **Input :pulse shaping, Number of subcarriers, symbol rate, BW, CP ratio**
- **Output: Signal in time, spectrum, BER, ICI (later), ISI (later)**

Single-tap, Flat Fading (Rayleigh) vs AWGN



<u>Why</u> do we have this huge degradation in performance/reliability?

Rayleigh Flat Fading Channel

$$y = hx + w$$

$$h \sim \mathcal{CN}(0, 1)$$

BPSK: $x = \pm a$. Coherent detection.

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$$

Looks like AWGN, but…

***p_e* needs to be "unconditioned"**

Averaged over h,

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1+\mathsf{SNR}}}\right) \approx \frac{1}{4\mathsf{SNR}}$$

To get a much poorer scaling

at high SNR.

This is a very discouraging result. To get an error probability $p_e = 10^{-3}$ one would require $\mathsf{SNR} \approx 500$ (27 dB). Stupendous amounts of power would be required for more reliable communication.

Typical Error Event

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$$

When $|h|^2 >> \frac{1}{\mathsf{SNR}}$, **the error probability is very small.**

When $|h|^2 < \frac{1}{\mathsf{SNR}}$, **the error probability is large:**

$$p_e \approx P\left(|h|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{\mathsf{SNR}}$$

$$|h|^2 \sim \exp(1).$$

$$\mathbb{P}\left\{|h|^2 \mathsf{SNR} < 1\right\} = \int_0^{1/\mathsf{SNR}} e^{-x} dx$$

Typical error event is due to: channel (h) being in deep fade!
… rather than (additive) noise being large.

BER vs. SNR in a flat fading channel

In a flat fading channel (or narrowband system), the CIR (channel impulse response) reduces to a single impulse scaled by a time-varying complex coefficient.

The received (equivalent lowpass) signal is of the form

$$r(t) = a(t) e^{j\phi(t)} s(t) + n(t)$$

We assume that the phase changes "slowly" and can be perfectly tracked

=> important for coherent detection

BER vs. SNR (cont.)

We assume:

- the time-variant complex channel coefficient changes slowly (=> constant during a symbol interval)
- the channel coefficient magnitude (= attenuation factor) a is a Rayleigh distributed random variable
- coherent detection of a binary PSK signal (assuming ideal phase synchronization)

Let us define instantaneous SNR and average SNR:

$$\gamma = a^2 E_b / N_0 \qquad\qquad \gamma_0 = E\left\{a^2\right\} \cdot E_b / N_0$$

BER vs. SNR (cont.)

Since

$$p(a) = \frac{2a}{E\{a^2\}} e^{-a^2 / E\{a^2\}} \qquad a \geq 0,$$

using

$$p(\gamma) = \frac{p(a)}{|d\gamma / da|}$$

we get

$$p(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \qquad \gamma \geq 0.$$

Rayleigh distribution

Exponential distribution

BER vs. SNR (cont.)

The average bit error probability is

$$P_e = \int_0^{\infty} P_e(\gamma)\, p(\gamma)\, d\gamma$$

Important formula for obtaining statistical average

where the bit error probability for a certain value of a is

$$P_e(\gamma) = Q\left(\sqrt{2a^2 E_b / N_0}\right) = Q\left(\sqrt{2\gamma}\right).$$

2-PSK

We thus get

$$P_e = \int_0^{\infty} Q\left(\sqrt{2\gamma}\right) \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} d\gamma = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right).$$

BER vs. SNR (cont.)

Approximation for large values of average SNR is obtained in the following way. First, we write

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) = \frac{1}{2}\left(1 - \sqrt{1 + \frac{-1}{1+\gamma_0}}\right)$$

Then, we use

$$\sqrt{1+x} = 1 + x/2 + \ldots$$

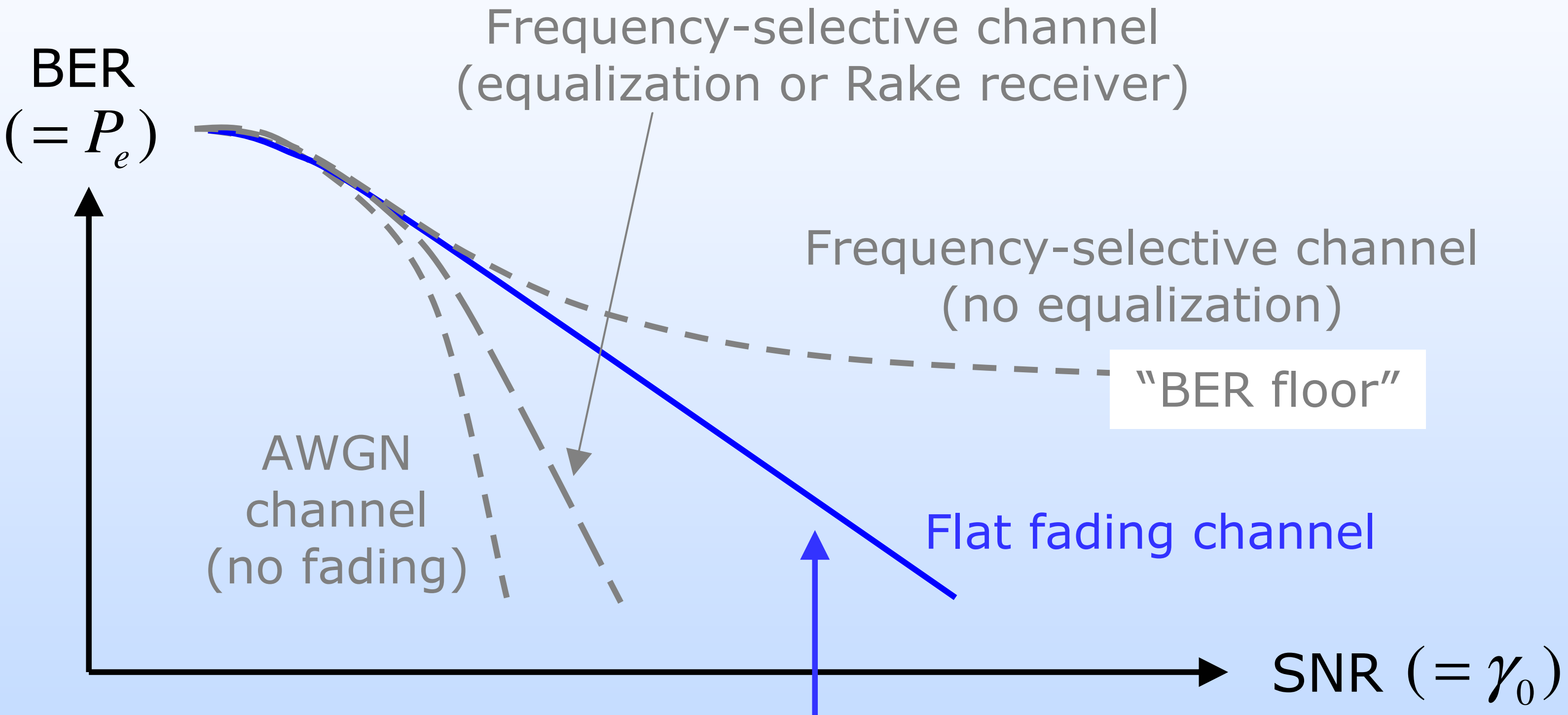
which leads to

$$P_e \approx 1/4\gamma_0$$ for large γ_0 .

BER vs. SNR, summary

Modulation	$P_e(\gamma)$	P_e	P_e (for large γ_0)
2-PSK	$Q\left(\sqrt{2\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{1+\gamma_0}}\right)$	$1/4\gamma_0$
DPSK	$e^{-\gamma}/2$	$1/(2\gamma_0+2)$	$1/2\gamma_0$
2-FSK (coh.)	$Q\left(\sqrt{\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)$	$1/2\gamma_0$
2-FSK (non-c.)	$e^{-\gamma/2}/2$	$1/(\gamma_0+2)$	$1/\gamma_0$

BER vs. SNR (cont.)



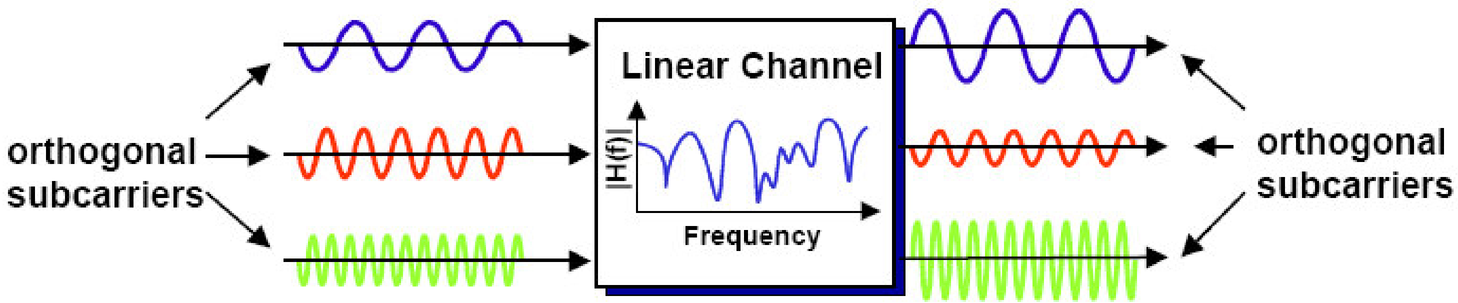
$P_e \approx 1/4\gamma_0$ means a straight line in log/log scale

OFDM Transmission Technique - Channel

Influence of the linear time-(in)variant radio channel

Transmitted signal:

$$s_{i,n} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_{i,k} e^{j2\pi\frac{nk}{N}}$$



Received signal:

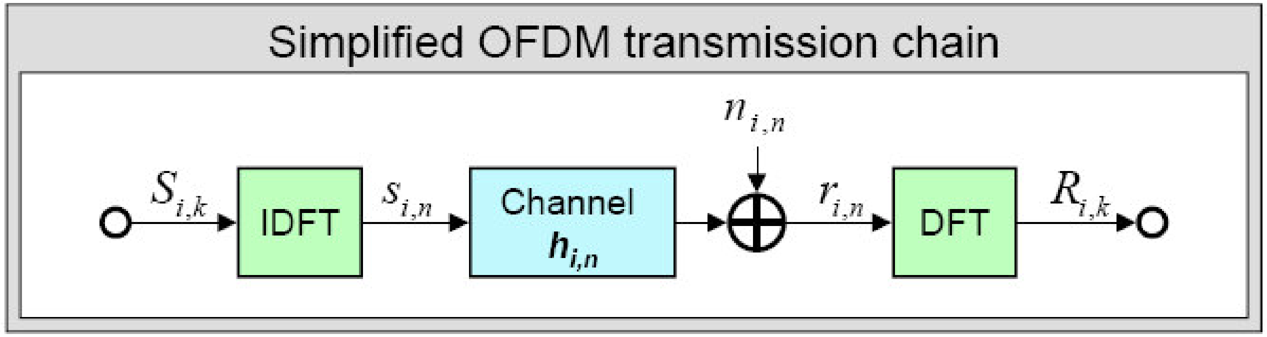
$$r_{i,n} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_{i,k} H_{i,k} e^{j2\pi\frac{nk}{N}}$$

- linear operations
- Eigenfunctions of the channel
- Subcarrier-wise Channel Transfer Factors

TUHH

Transmission over mutipath channel

OFDM symbols can be treated seperately, since an **ISI-free transmission** can be assumed (use of **guard interval**)

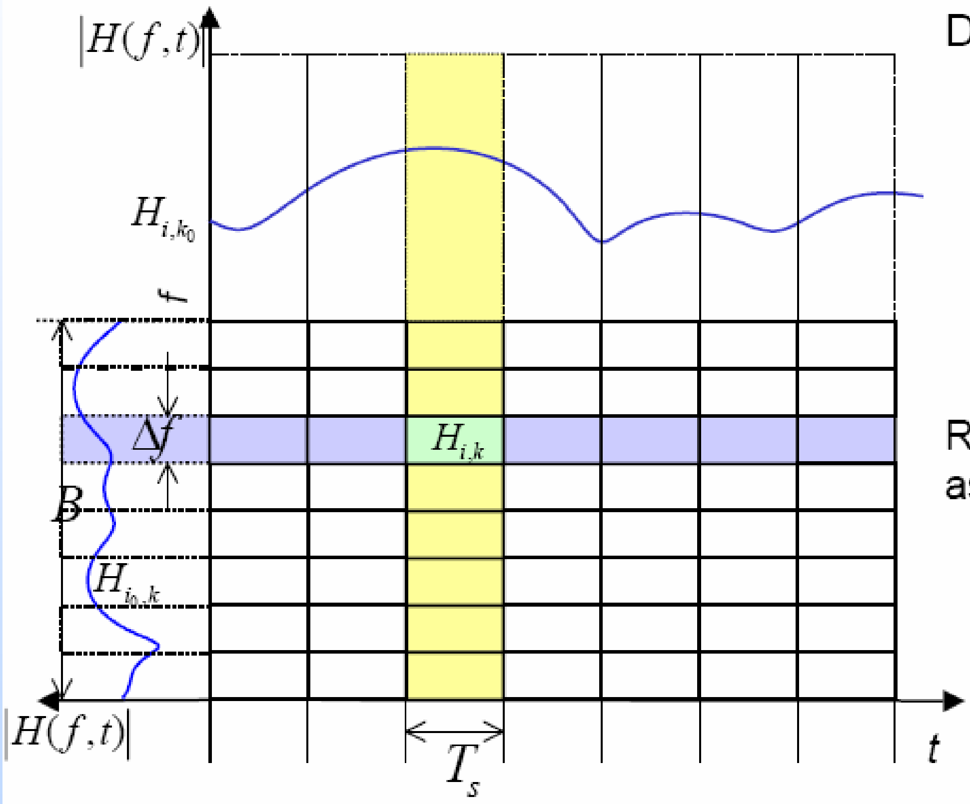


$$r_{i,n} = s_{i,n} * h_{i,n} + n_{i,n}$$

$$\Leftrightarrow \quad r_{i,n} = \left[\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_{i,k}\cdot e^{j2\pi\frac{nk}{N}}\right] * \left[\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} H_{i,k}\cdot e^{j2\pi\frac{nk}{N}}\right] + \left[\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} N_{i,k}\cdot e^{j2\pi\frac{nk}{N}}\right]$$

$$R_{i,k} = S_{i,k} \cdot H_{i,k} + N_{i,k}$$

Radio channel parameters for OFDM



Design:

$$\Delta f \cdot << B_c = \frac{1}{\tau_{\max}}$$

$$T_S << T_c = \frac{1}{f_{D,\max}}$$

Radio channel behaves as:

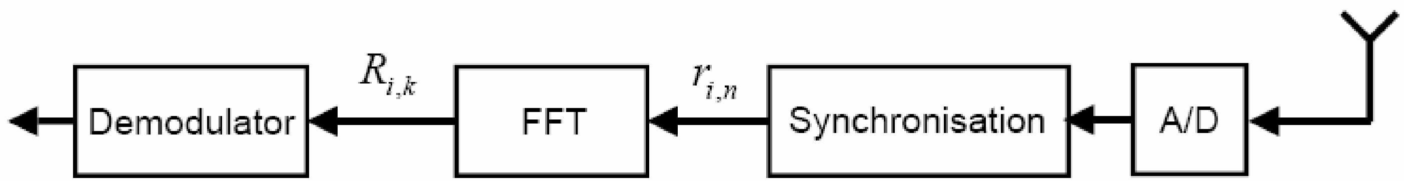
$$H_{i,k} = H(k\Delta f, iT_s)$$

time in-variant inside each OFDM symbol duration T_s

frequency non-selective inside each subcarrier bandwidth Δf

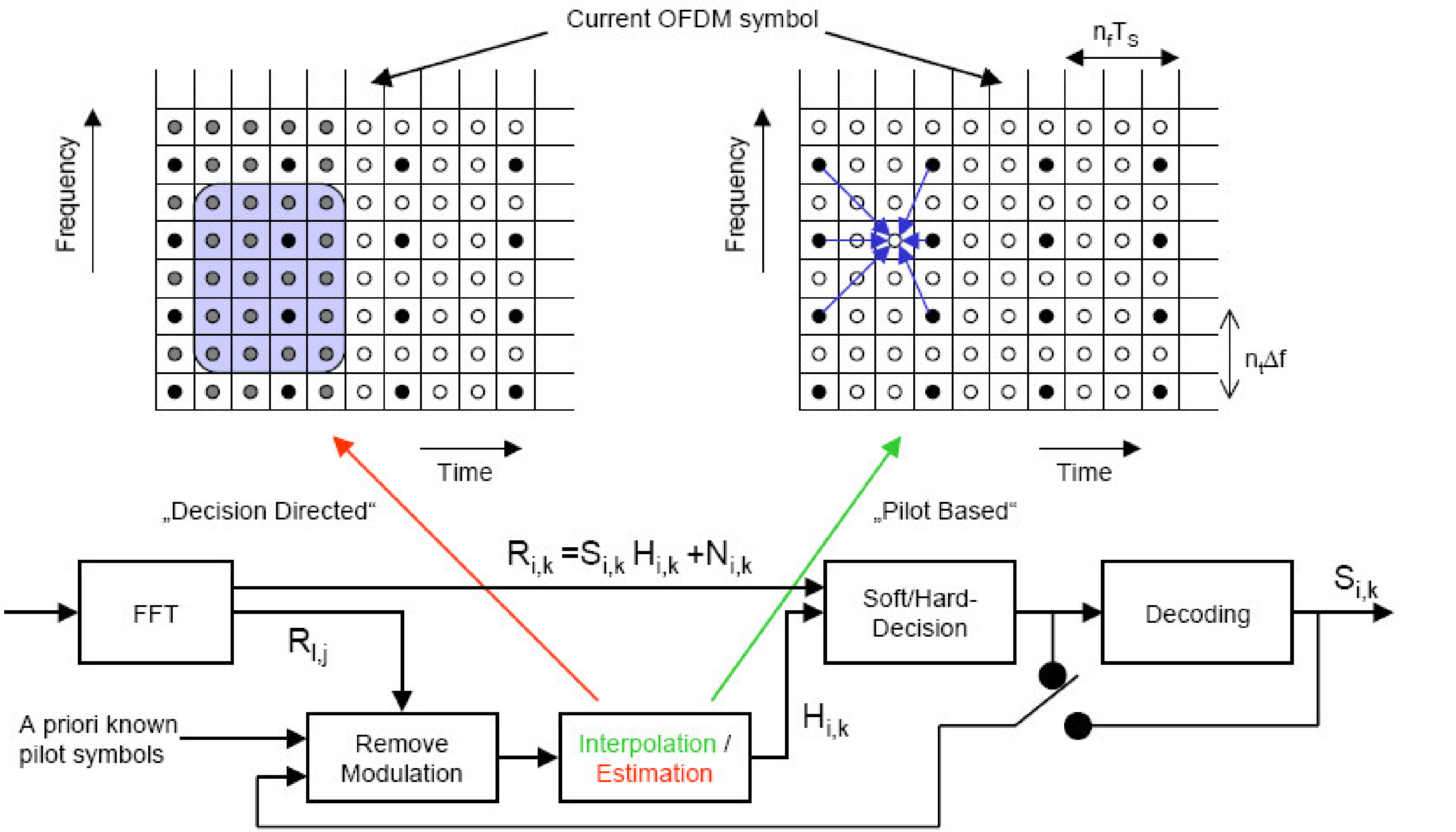
OFDM Transmission Technique - Receiver

- Received time-continuous signal of the i^{th} OFDM block
  - **Time-Domain:** $r_i(t) = s_i(t) * h_i(t) + n_i(t)$
  - **Frequency-Domain:** $R_i(f) = S_i(f) \cdot H_i(f) + N_i(f)$



- Received time-discrete signal of the i^{th} OFDM block
  - **Time-Domain:** $r_{i,n} = s_{i,n} * h_{i,n} + n_{i,n}$
  - **Frequency-Domain:** $R_{i,k} = S_{i,k} \cdot H_{i,k} + N_{i,k}$

Pilot-Based / Blind Channel Estimation



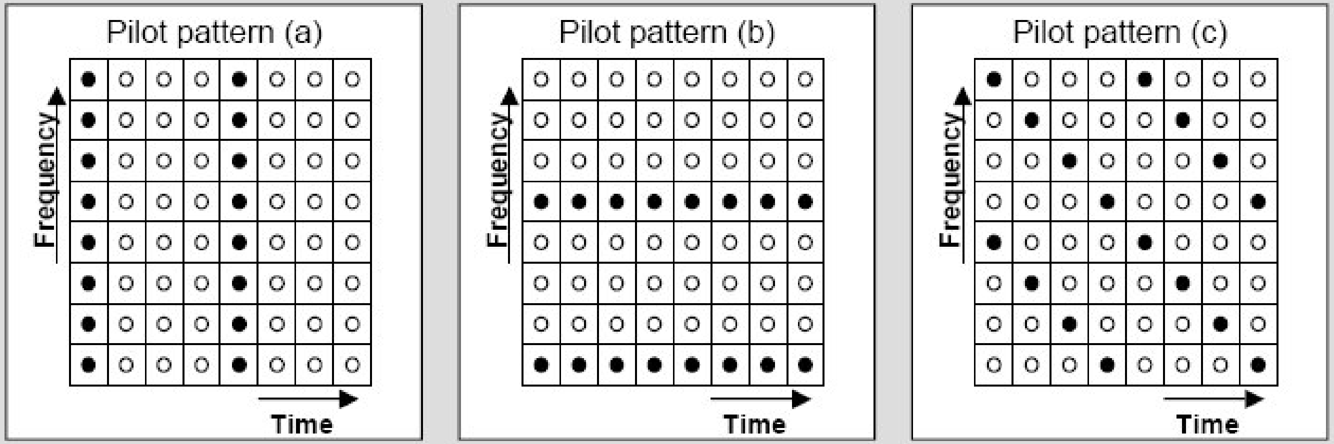
Pilot Symbols

- Complex channel factors $H_{i,k}$ required for coherent demodulation

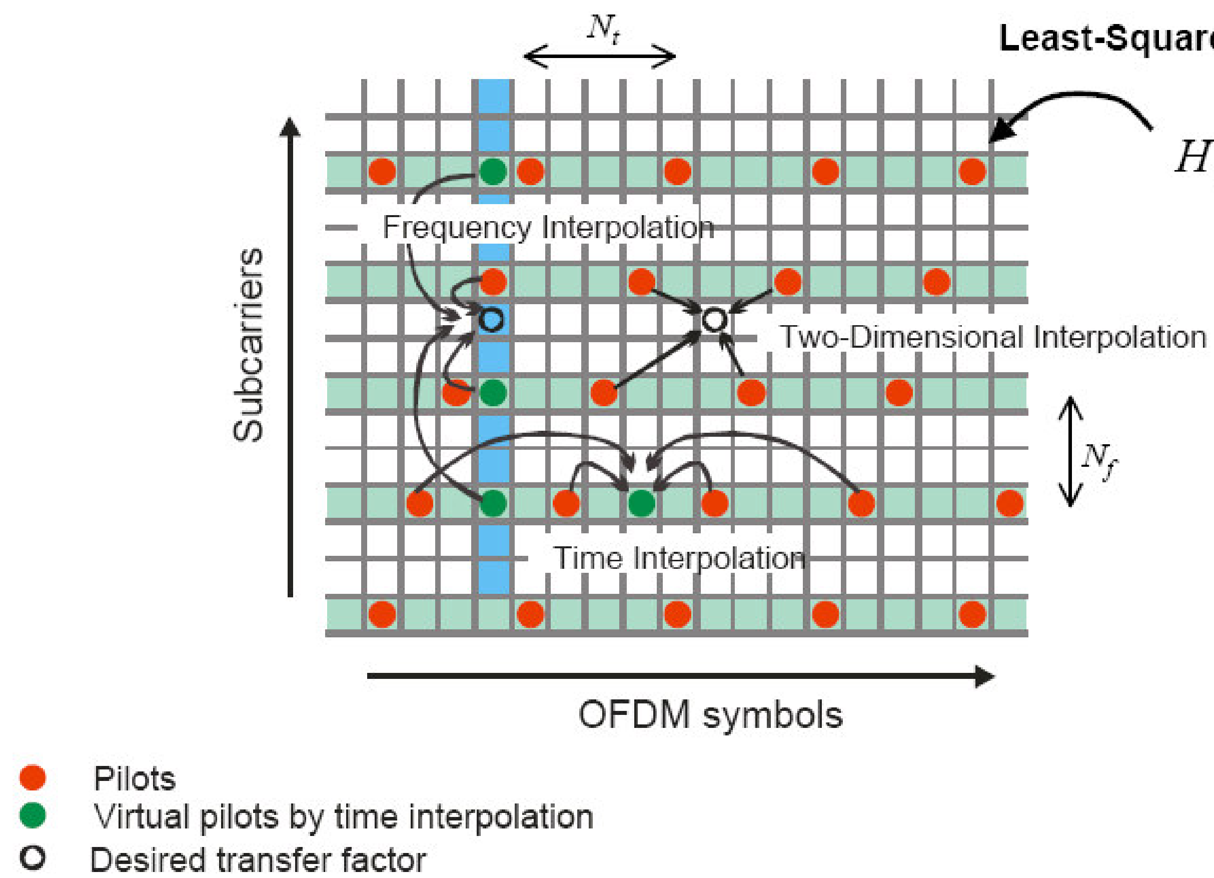
$$\hat{H}_{i,k} = \frac{R_{i,k}}{S_{i,k}} = H_{i,k} + \frac{N_{i,k}}{S_{i,k}} \approx H_{i,k}$$

⇒ Known symbols $\boldsymbol{S_{i,k}}$ (***Pilots***) can be used to estimate the channel

Three sample pilot patterns for channel estimation in OFDM



Pilot-Based Channel Estimation



Least-Squares Estimation:

$$H_{i,k} = \frac{R_{i,k}}{S_{i,k}}$$

Interpolation Methods

- Linear interpolation
- Second order interpolation
- Low pass interpolation
- Spline cubic interpolation
- Wiener-Filter interpolation
- Time domain interpolation

Better performance through diversity

Diversity ⇔ the receiver is provided with multiple copies of the transmitted signal. The multiple signal copies should experience *uncorrelated fading* in the channel.

In this case the probability that *all* signal copies fade simultaneously is reduced dramatically with respect to the probability that a *single* copy experiences a fade.

As a rough rule:

$$P_e \text{ is proportional to } \frac{1}{\gamma_0^{\,L}}$$

BER

Average SNR

Diversity of *L*:th order

OFDM Performance over AWGN channel

- **The performance of OFDM in AWGN is identical to that of single carrier modulation.**

$$p_{e,BPSK}(\gamma) = Q(\sqrt{2\gamma}),$$
$$p_{e,QPSK}(\gamma) = Q(\sqrt{\gamma}),$$
$$p_{e,16QAM}(\gamma) = 0.75 \cdot Q\left(\sqrt{\frac{\gamma}{5}}\right) 0.25 \cdot Q\left(\sqrt{\frac{\gamma}{5}}\right),$$

- **However:**
  - Problem of PAPR in OFDM: will discuss later in the course
  - Problem of Guard interval

$$P_{b,AWGN}^{B,coherent} = P_{b,AWGN}^{Q,coherent} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\gamma_b'}\right)$$

$$\gamma_b' = \frac{t_s}{T_s}\gamma_b = (1-\alpha_G)\gamma_b$$

OFDM Performance over Flat fading channel

- **OFDM performance over Raleigh flat fading channel is similar to single carrier modulation**

$$P_{b,fading}^{B,coherent} = P_{b,fading}^{Q,coherent} = \int_0^\infty \frac{1}{2}\operatorname{erfc}\left(\sqrt{\gamma_b'}\right) p(\gamma_b')\, d\gamma_b'$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma_b'}}{1+\overline{\gamma_b'}}}\right)$$

OFDM Performance over Frequency Selective Fading channel

- **OFDM performance over frequency selective fading channel is similar to single carrier modulation with same modulation in flat fading channels**

$$h(\tau; t) = \sum_{l=1}^{L_1+L_2} \alpha_l(t)\,\delta(\tau - \tau_l)$$

$$0 < \tau_l \le \Delta_G,\ (l = 1, \ldots, L_1)$$

$$\Delta_G < \tau_l \le T_s,\ (l = L_1 + 1, \ldots, L_1 + L_2)$$

$$r_{ni} = \left\{ \sum_{l=1}^{L_1} \frac{c_{ni}}{t_s} e^{-j2\pi f_n \tau_l} \int_{iT_s}^{iT_s+t_s} \alpha_l(t)\, dt + \sum_{l=L_1+1}^{L_1+L_2} \frac{c_{ni}}{t_s} e^{-j2\pi f_n \tau_l} \int_{iT_s-\Delta_G+\tau_l}^{iT_s+t_s} \alpha_l(t)\, dt \right\}$$

$$+\left\{\sum_{l=1}^{L_1}\sum_{\substack{k=1\\k\neq n}}^{N_{SC}}\frac{c_{ki}}{t_s}e^{-j2\pi f_k\tau_l}\int_{iT_s}^{iT_s+t_s}\alpha_l(t)\,e^{-j2\pi(f_k-f_n)(t-iT_s)}\,dt\right.$$

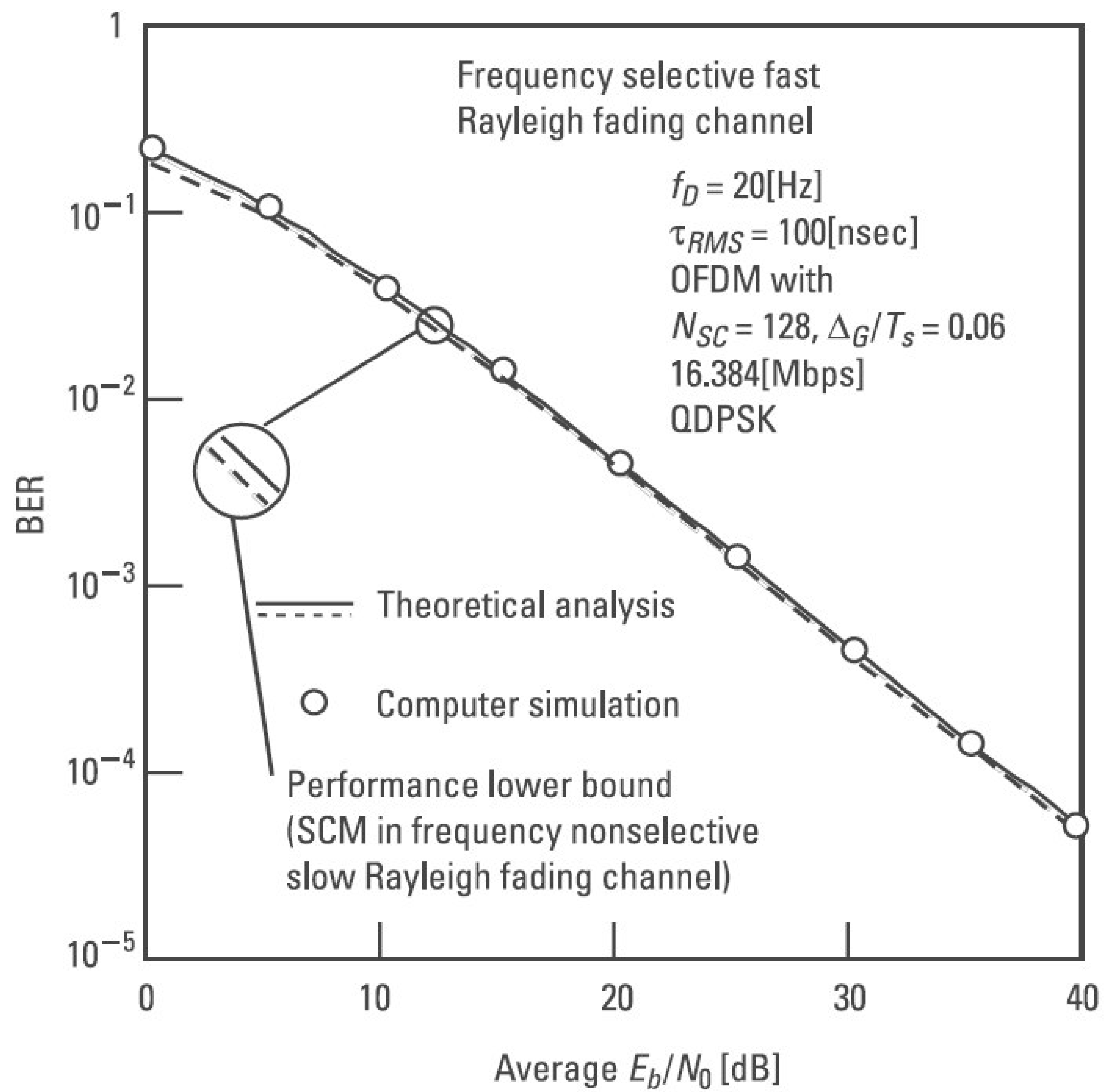
$$+\sum_{l=L_2+1}^{L_1+L_2}\sum_{\substack{k=1\\k\neq n}}^{N_{SC}}\frac{c_{ki}}{t_s}e^{-jf_k\tau_l}\int_{iT_s-\Delta_G+\tau_l}^{iT_s+t_s}\alpha_l(t)\,e^{-j2\pi(f_k-f_n)(t-iT_s)}\,dt$$

$$\left.+\sum_{l=L_1+1}^{L_1+L_2}\sum_{\substack{k=1\\k\neq n}}^{N_{SC}}\frac{c_{k(i-1)}}{t_s}e^{-j2\pi f_k(\tau_l-T_s)}\int_{iT_s}^{iT_s-\Delta_G+\tau_l}\alpha_l(t)\,e^{-j2\pi(f_k-f_n)(t-iT_s)}\,dt\right\}$$

$$+\sum_{l=L_1+1}^{L_1+L_2}\frac{c_{n(i-1)}}{t_s}e^{-j2\pi f_n(\tau_l-T_s)}\int_{iT_s}^{iT_s-\Delta_G+\tau_l}\alpha_l(t)\,dt+n_{ni}$$

In (4.33), the first term means the desired signal component, the second and third terms mean the intersubcarrier interference and ISI, respectively, and the fourth term means the Gaussian noise component with average of zero and power of σ_n^2. σ_l^2 and σ_n^2 satisfies the following equation:

$$\overline{\gamma_b} = \frac{\sum_{l=1}^{L_1+L_2} \sigma_l^2}{k_M \sigma_n^2} \tag{4.34}$$



ure 4.9 BER of a QDPSK-based OFDM system in a frequency selective fast Rayleigh fading channel.

Thank you