Random or stochastic processes

Definition

- Random experiment specified by $\zeta \in S$, by the events defined on S, and by the probabilities on these events. To each $\zeta \in S$, a function of time is assigned according to some rule.

$$X(t,\zeta) \quad t \in I$$

$$X(t,\zeta)\big|_{\zeta} \quad \text{is a function of time } t$$

realization, sample path, sample function

$$X(t,\zeta)\big|_{t=t_k} \quad \text{is a function of } \zeta, \text{ i.e. an r.v.}$$

$$\{X(t,\zeta), t \in I\}$$

an indexed family of r.v.'s
or random process
or stochastic process

if I is countable: discrete time SP
if I is continuous (non-negative) real line: continuous SP

Ex 6.1 discrete-time SP

$$\zeta \in S = (0,1]$$

$$\forall \zeta \in S \rightarrow \zeta = \sum_{n=1}^{\infty} b_n 2^{-i}$$ binary expansion

discrete-time SP: $X(n,\zeta) = b_n \; ; n = 1, 2, \ldots$

Ex 6.2 continuous-time SP

$$\zeta \in S = [-1,1]$$

$$X(t,\zeta) = \zeta \cos(2\pi t) \; ; -\infty < t < \infty$$

sinusoids with amplitude ζ

$$\zeta \in S = [-\pi, \pi]$$

time-shifted versions of cos

$$Y(t,\zeta) = \cos(2\pi t + \zeta) \; ; -\infty < t < \infty$$

Specifying a SP

sample the SP $X(t,\zeta)\big|_{t=t_i} \to \text{r.v.'s } X_1, X_2, \cdots, X_k$

joint behavior of the SP at these *k* time instants is specified by the joint CDF of the vector r.v. $\mathbf{X} = (X_1, X_2, \cdots, X_k)$

we can now compute probabilities as before

a SP is specified by the collection of k^{th}-order joint CDFs:

$$F_{X_1,\cdots,X_k}(x_1,\cdots,x_k) = P\left[X_1 \le x_1, X_2 \le x_2, \cdots, X_k \le x_k\right]$$

for any k and any choice of sampling instants $t_1,\cdots t_k$

any   any

$$p_{X_1,\cdots,X_k}(x_1,\cdots,x_k) = P\left[X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k\right]$$

for discrete-time SP

$$f_{X_1,\cdots,X_k}(x_1,\cdots,x_k)$$

for continuous-time SP

Mean, autocorrelation, autocovariance

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1)X(t_2)}(x,y)\,dxdy$$

$$C_X(t_1,t_2) = E\left[\{X(t_1)-m_X(t_1)\}\{X(t_2)-m_X(t_2)\}\right]$$

$$= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

$$VAR[X(t)] = E\left[\{X(t)-m_X(t)\}^2\right] = C_X(t,t)$$

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)C_X(t_2,t_2)}} \Rightarrow |\rho_X(t_1,t_2)| \le 1$$

~predicting one r.v. as a linear function from another

Ex 6.6 random amplitude sinusoid

$$X(t) = A\cos 2\pi t$$

$$m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t$$

mean varies with *t*

process is always =0 for *t*-values for which cos2π*t*=0

$$R_X(t_1,t_2) = E\left[A\cos(2\pi t_1)A\cos(2\pi t_2)\right]$$
$$= E\left[A^2\right]\cos(2\pi t_1)\cos(2\pi t_2)$$

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$
$$= \left\{E\left[A^2\right] - E[A]E[A]\right\}\cos(2\pi t_1)\cos(2\pi t_2)$$
$$= VAR(A)\cos(2\pi t_1)\cos(2\pi t_2)$$

Ex 6.7 random phase sinusoid

$$X(t) = \cos(\omega t + \Theta)\,;\,\Theta \sim U(-\pi, \pi)$$

$$m_X(t) = E\left[\cos(\omega t + \Theta)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t + \theta)\,d\theta = 0$$

mean is constant

$$C_X(t_1, t_2) = R_X(t_1, t_2) = E\left[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left\{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta)\right\}d\theta$$

$$= \frac{1}{2}\cos(\omega(t_1 - t_2))$$

covariance depends on absolute time difference only

Gaussian SP

A random process $X(t)$ is a Gaussian SP

if for any k and any choice of sampling instants $t_1 < t_2 < \ldots < t_k$

the r.v.'s $X_1 = X(t_1), X_2 = X(t_2), \ldots, X_k = X(t_k)$

are jointly Gaussian r.v.'s

discrete & continuous

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k)$$

$$= \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T K^{-1}(\mathbf{x}-\mathbf{m})\right\}}{(2\pi)^{k/2} |K|^{\frac{1}{2}}}$$

k jointly Gaussian r.v.'s

r.v.'s $X_1, X_2, \cdots, X_k$ are said to be jointly Gaussian if their joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k)$$

$$= \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T K^{-1}(\mathbf{x}-\mathbf{m})\right\}}{(2\pi)^{k/2}|K|^{\frac{1}{2}}}$$

$$\mathbf{m} = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_k) \end{bmatrix} \quad K = \left\{C_X(t_i, t_j)\right\}_{i,j}$$

Multiple SPs

- Joint behavior of two or more SPs is specified by the collection of joint distributions for all possible choices of time samples of the SPs
- e.g. for $X(t)$ and $Y(t)$ specify all possible joint density functions of $X(t_1),\ldots, X(t_k)$ and $Y(t_1'),\ldots,Y(t_j')$ for all k,j and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

Multiple SPs

- The SPs $X(t)$ and $Y(t)$ are said to be independent if the vector r.v.'s $(X(t_1),\ldots, X(t_k))$ and $(Y(t_1'),\ldots,Y(t_j'))$ are independent for all k,j and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

cross-correlation $R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$

SPs $X(t)$ and $Y(t)$ are said to be orthogonal if $R_{XY}(t_1,t_2) = 0 \; \forall t_1,t_2$

cross-covariance

$$C_{XY}(t_1,t_2) = E\left[\left\{X(t_1) - m_X(t_1)\right\}\left\{Y(t_2) - m_Y(t_2)\right\}\right]$$

$$= R_{XY}(t_1,t_2) - m_X(t_1)m_Y(t_2)$$

SPs $X(t)$ and $Y(t)$ are said to be uncorrelated if $C_{XY}(t_1,t_2) = 0 \; \forall t_1,t_2$

Ex 6.10 signal + noise

$$Y(t) = X(t) + N(t)$$

cross-correlation

$$R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right]$$

$$= E\left[X(t_1)\left\{X(t_2) + N(t_2)\right\}\right]$$

$$= E\left[X(t_1)X(t_2)\right] + E\left[X(t_1)N(t_2)\right]$$

independence

$$= R_X(t_1, t_2) + E\left[X(t_1)\right]E\left[N(t_2)\right]$$

$$= R_X(t_1, t_2) + m_X(t_1)m_N(t_2)$$

Stationary random processes

- Essentially…the nature of the randomness does not change with time
- Assuming the process started at t=-∞…
  A discrete-time or continuous-time SP $X(t)$ is stationary if the joint distribution of any set of samples does not depend on the placement of the time origin

$$F_{X(t_1),\cdots,X(t_k)}\left(x_1,\cdots,x_k\right)=F_{X(t_1+\tau),\cdots,X(t_k+\tau)}\left(x_1,\cdots,x_k\right);\forall\tau,\forall k,\forall t_1,\cdots,t_k$$

if SP starts at definite time, eg. t=0 or n=0, joint CDF does not change under time shifts to the right

Jointly stationary SPs

- Two processes *X*(*t*) and *Y*(*t*) are said to be jointly stationary if

$$F_{X(t_1),\cdots,X(t_k),Y(t_1^{'}),\cdots,Y(t_j^{'})}\left(x_1,\cdots,x_k,y_1,\cdots,y_j\right)=$$

$$=F_{X(t_1+\tau),\cdots,X(t_k+\tau),Y(t_1^{'}+\tau),\cdots,Y(t_j^{'}+\tau)}\left(x_1,\cdots,x_k,y_1,\cdots,y_j\right)$$

$$\forall\tau,\forall k,j,\forall t_1,\cdots,t_k,\forall t_1^{'},\cdots,t_k^{'}$$

First-order CDF of stationary SP

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) \,;\, \forall \tau, \forall t$$

$$= F_X(x)$$

$$\Downarrow$$

$$m_X(t) = E\left[X(t)\right] = m \,;\, \forall t$$

$$VAR\left(X(t)\right) = E\left[\left(X(t) - m\right)^2\right] = \sigma^2 \,;\, \forall t$$

mean and variance are constant

Second-order CDF of stationary SP

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2)$$

$$= F_{X(0),X(t_2-t_1)}(x_1,x_2);\ \forall t_1,t_2$$

$$\Downarrow$$

$$R_X(t_1,t_2) = R_X(t_2-t_1);\ \forall t_1,t_2$$

$$C_X(t_1,t_2) = C_X(t_2-t_1);\ \forall t_1,t_2$$

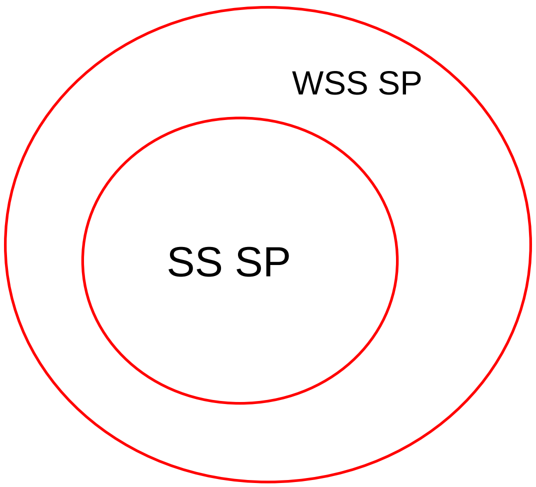
autocorrelation and covariance depend only on time difference

Wide-sense stationary (WSS) SP

- Often we cannot determine whether a process is stationary, but we can determine whether
  - *the mean is constant* $m_X(t) = m\,;\forall t$
  - *the autocovariance (equivalently the autocorrelation) is a function of time difference only* $C_X(t_1, t_2) = C_X(t_1 - t_2)\,;\forall t_1, t_2$
- SP is WSS if both conditions hold

Jointly WSS SPs

- *X*(*t*) and *Y*(*t*) are jointly WSS SPs if
  - $X(t)$ is WSS
  - $Y(t)$ is WSS
  - $C_{XY}(t_1,t_2)=C_{XY}(t_1-t_2)=C_{XY}(\tau)$
- All stationary SPs are WSS



Autocorrelation function of WSS SP

- Plays a crucial role in the design of linear signal processing algorithms

i $R_X(0) = E\left[X^2(t)\right]$ average power of the process

ii $R_X(\tau) = E\left[X(t+\tau)X(t)\right] = E\left[X(t)X(t+\tau)\right] = R_X(-\tau)$

an even function of τ

iii $$E^2[XY] \le E\left[X^2\right]E\left[Y^2\right] \forall X, Y$$

$$\Downarrow$$

$$R_X^2(\tau) = E^2\left[X(t+\tau)X(t)\right] \le E\left[X^2(t+\tau)\right]E\left[X^2(t)\right] = R_X^2(0)$$

$$\Downarrow$$

$$\left|R_X(\tau)\right| \le R_X(0)$$

autocorrelation is max at $\tau=0$

- If $R_x(d)=R_x(0)$ → $R_x(\tau)$ is periodic with period equal to d.
- Let $X(t) = m + N(t)$

  $N(t)$ is a zero-mean process $\ni R_N(\tau) \underset{\tau \to \infty}{\to} 0$

$$R_X(\tau) = E\left[\{m + N(t+\tau)\}\{m + N(t)\}\right]$$
$$= m^2 + 2mE\left[N(t)\right] + R_N(\tau)$$
$$= m^2 + R_N(\tau) \underset{\tau \to \infty}{\to} m^2$$

$R_X(\tau)$ approaches m^2 (the square of the mean) as $\tau \to \infty$

Autocorrelation function of WSS SP

- Can have three types of component:
- A component that vanishes as $\tau \rightarrow \infty$
- A periodic component
- A component due to a nonzero mean

Ex 6.30

$$R_X(\tau) = e^{-2\alpha|\tau|}$$

$X(t)$ is zero-mean and $R_X(\tau) \underset{\tau\to\infty}{\to} 0$

$$R_Y(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau)$$

random phase sinusoid

$Y(t)$ is zero-mean and $R_X(\tau)$ has period f_0^{-1}

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + m^2$$

$Z(t) = X(t) + Y(t) + m$; $X(t)$ and $Y(t)$ independent

Note that it is not possible to determine which fraction of the mean is contributed by the individual components

WSS Gaussian SPs

- Mean is not a function of *t*
- Autocovariance is a function of time difference only
- Joint PDF depends on mean and autocovariance only
- A WSS Gaussian SP is stationary

WSS --> SS

all we need is m and $C_X(\tau)$

Ex 6.31

- X_n is iid sequence of $N(0,\sigma^2)$ r.v.'s. Y_n is the average of two consecutive values

$$Y_n = \frac{X_n + X_{n-1}}{2} \Longrightarrow m_Y = 0.5E\left[X_n + X_{n-1}\right] = 0$$

$$C_Y(i,j) = E\left[Y_iY_j\right] = 0.25E\left[\left(X_i + X_{i-1}\right)\left(X_j + X_{j-1}\right)\right]$$

$$= 0.25\left\{E\left[X_iX_j\right] + E\left[X_iX_{j-1}\right] + E\left[X_{i-1}X_j\right] + E\left[X_{i-1}X_{j-1}\right]\right\}$$

$$= \frac{1}{2}\sigma^2\delta_{i-j} + \frac{1}{4}\sigma^2\delta_{i-j+1} + \frac{1}{4}\sigma^2\delta_{i-j-1}$$

Y_n is WSS

Y_n is a linear trafo of Gaussian r.v.'s, so Y_n is Gaussian

joint PDF is Gaussian, specified by m_Y and $C_Y(i,j)$

Time averages & ergodicity

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i)$$

ensemble averaging
repeating the experiment
from many realizations

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\zeta)\,dt$$

time averaging
based on single realization

Ergodic ⟶ Stationary but not Vice versa

an ergodic theorem states conditions under which a time average converges when the observation interval becomes large

we're interested in ergodic theorems that state when time averages converge to the ensemble average or expected value

$$X(t) = A\cos(2\pi f_c t + \vartheta)$$

ϑ is U. RV from $0 \longrightarrow 2\pi$

Is X(t) mean and autocorrelation ergodic?

Mean, Autocorrelation of sine wave with random phase

$$X(t) = A\cos(2\pi f_c t + \theta)$$

A and f_c are constants and θ is a RV that is uniformly distributed over range of 0 and 2π

$$m_x(t) = \int_0^{2\pi} A\cos(2\pi f_c t + \theta)\frac{1}{2\pi}d\theta$$

$$m_x(t) = A\left[\sin(2\pi f_c t + \theta)\right]_0^{2\pi} = zero$$

$$< X(t)_T >= \int_0^{T} A\cos(2\pi f_c t + \theta)dt$$

$$< X(t)_T >= A\left[\frac{\sin(2\pi f_c t + \theta)}{2\pi f}\right]_0^{T} = zero$$

Ergodic

Mean, Autocorrelation of sine wave with random phase

$$X(t) = A\cos(2\pi f_c t + \theta)$$

A and f_c are constants and θ is a RV that is uniformly distributed over range of 0 and 2π

$$R_x(t_1,t_2) = E[X(t_1)X(t_2)] = A^2 E[Cos(2\pi f_c t_1 + \theta)Cos(2\pi f_c t_2 + \theta)]$$

$$R_x(t_1,t_2) = \frac{A^2}{2} E[Cos(2\pi f_c(t_1 - t_2) + Cos(2\pi f_c(t_1 + t_2) + 2\theta)]$$

$$R_x(t_1,t_2) = \frac{A^2}{2} Cos(2\pi f_c(t_1 - t_2) = R_x(\tau) = \frac{A^2}{2} Cos(2\pi f_c(\tau)) \quad \text{W.S Stationary}$$

$$< X(t)X(t+\tau) >= A^2 \frac{1}{2T} \int_{-T}^{T} Cos(2\pi f_c t + \theta)Cos(2\pi f_c(t+\tau) + \theta)dt$$

$$< X(t)X(t+\tau) >= A^2 \frac{1}{2T} \int_{-T}^{T} [Cos(2\pi f_c \tau) + Cos(2\pi f_c(2t+\tau) + 2\theta)]dt$$

$$< X(t)X(t+\tau) >= \frac{A^2}{2} Cos(2\pi f_c(\tau) = R_x(\tau) \qquad Ergodic$$

Properties of Gaussian RP

1. If a Gaussian process X(t) is applied to a LTIS then the output is also a Gaussian Process.
2. Gaussain Random process defined at a set of time instants is completely defined by the vector mean M and the covariance matrix C
3. If X(t) is a Gaussian WSS RP then it is a SS RP
4. If x(t) is a Gaussian RP with uncorrelated RV then they are also independent.

Cross Correlation function

- Consider two random processes X(t) and Y(t) with autocorrelation functions $R_x(t_1,t_2)$, $R_y(t_1,t_2)$ respectively
- The cross correlation function of X(t) and Y(t) is defined by $R_{xy}(t_1,t_2)=E[X(t_1)Y(t_2)]$ and $R_{yx}(t_1,t_2)=E[Y(t_1)X(t_2)]$.
- If X(t) and Y(t) are WSS then $R_{xy}(t_1,t_2)= R_{xy}(\tau)$
- $R_{xy}(\tau)= R_{yx}(-\tau)$

Transmission of a random process through a linear filter

- Suppose RP X(t) is applied to a LTIS with impulse response h(t), producing RP Y(t).
  - What is the mean and autocorrelation of Y(t) w.r.t those of X(t) Assuming X(t) is WSS RP.

$$m_y(t) = E[y(t)] = E[h(t) \otimes X(t)]$$

$$m_y(t) = E[\int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau] = \int_{-\infty}^{\infty} E[h(\tau) X(t-\tau)] d\tau$$

$$m_y(t) = \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)] d\tau = h(t) \otimes m_x(t)$$

Since X(t) is WSS RP, $m_x(t) = m$

$$m_y(t) = m \int_{-\infty}^{\infty} h(\tau) d\tau = mH(0)$$

$H(0)$ is zero frequency response of the system

$$R_y(t_1, t_2) = E[y(t_1) y(t_2)] =$$

$$E[(h(t_1) \otimes X(t_1))(h(t_2) \otimes X(t_2))]$$

$$= E[\int_{-\infty}^{\infty} h(\lambda) X(t_1 - \lambda) d\lambda \int_{-\infty}^{\infty} h(\upsilon) X(t_2 - \upsilon) d\upsilon] =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda) h(\upsilon) E[X(t_1 - \lambda) X(t_2 - \upsilon)] d\lambda d\upsilon$$

$$R_y(t_1, t_2) = h(t) \otimes h(t) \otimes R_x(t_1, t_2)$$

Since X(t) is WSS RP, $R_x(t_1, t_2) = R_x(t_1 - t_2)$

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\lambda) h(\upsilon) R_x(\tau - \lambda + \upsilon) d\lambda d\upsilon$$

Power Spectral Density

- From Communication Theory we know that Autocorrelation is the IFT of PSD
- IF X(t) is WSS RP

$$R_X(\tau) \overset{F.T}{\longleftrightarrow} S_X(f)$$

- Properties of PSD

1. $S_X(0) = \int_{-\infty}^{\infty} R_x(\tau) d\tau$
2. $E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$
3. $S_X(f) \geq 0 \quad \text{Non Negative}$
4. $S_X(f) = S_X(-f) \quad \text{for real valued random process}$

Ex. PSD of sine wave with random phase

$$R_x(\tau) = \frac{A^2}{2} Cos(2\pi f_c(\tau)) \text{ W.S Stationary}$$

- Using the F.T

$$S_x(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)]$$

Ex.

$$R_x(\tau) = \begin{cases} A^2(1-\frac{|\tau|}{T}) & |\tau| \le T \\ 0 & Otherwise \end{cases}$$

- Using the F.T

$$S_x(f) = A^2 T \sin c^2(fT)$$

Ex.

$$Y(t) = X(t)\cos(2\pi f_c t + \theta)$$

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(2\pi f_c t + \theta)X(t+\tau)\cos(2\pi f_c(t+\tau)+\theta)]$$

$$R_Y(\tau) = E[X(t)X(t+\tau)]E[\cos(2\pi f_c t + \theta)\cos(2\pi f_c(t+\tau)+\theta)]$$

$$R_Y(\tau) = R_X(\tau)\cos(2\pi f \tau)$$

$$S_Y(f) = \frac{1}{2}[S_X(f - f_c) + S_X(f - f_c)]$$

Relation among PSD of the input and output RP of LTIS

- Suppose RP X(t) is applied to a LTIS with impulse response h(t), producing RP Y(t).
  - What is the PSD of Y(t) w.r.t PSD of X(t) Assuming X(t) is WSS RP.

$$S_Y(f) = S_X(f)\left|H(f)\right|^2$$

Ex.

$$H(f) = 1 - \exp(-j2\pi fT)$$

$$\left|H(f)\right|^2 = H(f)H^*(f)$$

$$\left|H(f)\right|^2 = (1 - \exp(-j2\pi fT))(1 - \exp(+j2\pi fT))$$

$$\left|H(f)\right|^2 = (1 + 1 - \exp(-j2\pi fT) - \exp(+j2\pi fT))$$

$$\left|H(f)\right|^2 = 2(1 - \cos(2\pi fT)) = 4\sin^2(\pi fT)$$

$$S_Y(f) = 4\sin^2(\pi fT)S_X(f)$$

Cross Spectral Density

- Provides a measure of the frequency interrelationship between two random processes.
- X(t) and Y(t) are jointly WSS RP with $R_{XY}(\tau)$ and $R_{YX}(\tau)$
- Thus they have as a FT. $S_{XY}(f)$ and $S_{YX}(f)$
- $S_{XY}(f) = S_{YX}(-f)$

Noise

- Unwanted signal that tend to disturb the transmission and processing of signals in communication systems.
- Thermal Noise → random motion of electrons in a conductor.
- Shot noise → arises in electronic devices, sudden change in voltage or current.

White Noise

- White → occupies all frequencies→ PSD is independent on the operating frequency
- Dimensions of N_o is watt per Hertz, N_o=KT

$$S_N(f) = \frac{N_o}{2}$$

$$R_N(\tau) = \frac{N_o}{2}\delta(\tau)$$

- Any two different samples of white noise no matter how close they are will be uncorrelated.
- If white noise is Gaussian then they will also be independent

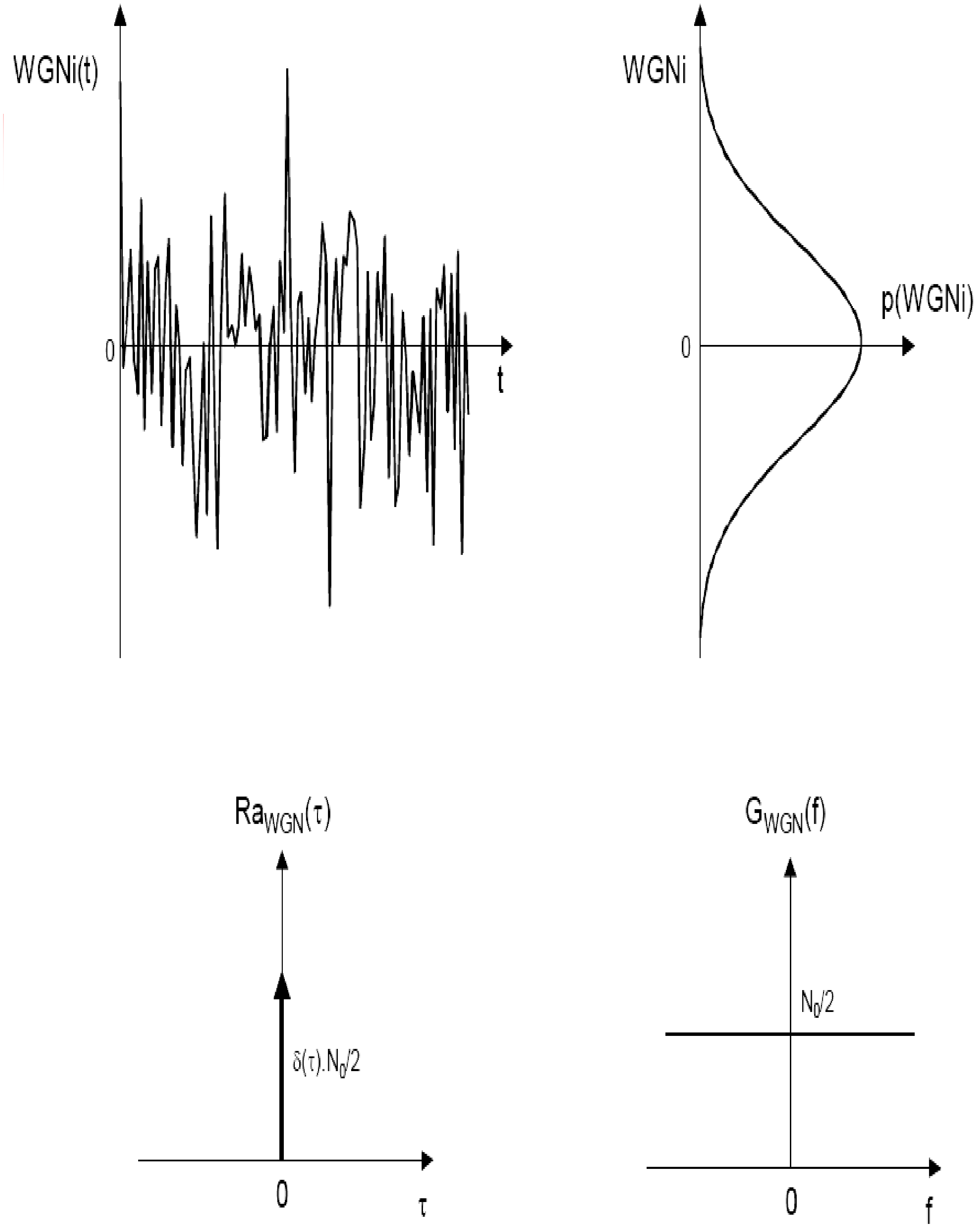
Example

- X(t) and Y(t) have zero mean and they are individually stationary in the wide sense.
- Z(t)=X(t)+Y(t)   find $S_Z(f)$

$$R_Z(t_1,t_2) = E[Z(t_1)Z(t_2)]$$
$$= E[(X(t_1)+Y(t_1))(X(t_2)+Y(t_2))]$$
$$= E[X(t_1)X(t_2)] + E[X(t_2)Y(t_1)] + E[X(t_1)Y(t_2)] + E[Y(t_1)Y(t_2)]$$
$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$
$$S_Z(f) = S_X(f) + S_Y(f) + S_{XY}(f) + S_{YX}(f)$$

If X and Y are uncorrelated

$$S_Z(f) = S_X(f) + S_Y(f)$$



WGNi(t)
0
t
WGNi
p(WGNi)
0
$Ra_{WGN}(\tau)$
$\delta(\tau).N_0/2$
0
τ
$G_{WGN}(f)$
$N_0/2$
0
f

Ideal low pass filtered white noise

- A white Gaussian noise with zero mean and variance $N_0/2$ is applied to an ideal low pass filter of bandwidth B and amplitude response of one.
- PSD of output Y(t) is

$$S_Y(f) = \begin{cases} \frac{N_o}{2} & -B \leq f \leq B \\ 0 & otherwise \end{cases}$$

- The autocorrelation is $R_Y(\tau) = N_0 B \sin c(2B\tau)$
- Autocorrelation maximum at τ equal zero equal $N_o B$ and passes through zero at $\tau = n/2B$ for n= integer values and variance $N_o B$
- If noise is sampled at rate 2B then they are uncorrelated and being Gaussian then statistically independent

RC low pass filtered white noise

- The H(f) of RC filter is $H(f) = \dfrac{1}{1 + j2\pi fRC}$
- The PSD of the o/p is $S_Y(f) = \dfrac{1}{1 + (2\pi fRC)^2}$
- The autocorrelation of the output is $R_Y(\tau) = \dfrac{N_o}{4RC}\exp(-\dfrac{|\tau|}{RC})$
- If noise is samples at rate 0.217/RC then they are uncorrelated and being Gaussian then statistically independent

Ex sine wave plus white noise

$$X(t) = A\cos(2\pi f_c t + \theta) + N(t)$$

θ is Uniformely distributed, N(t) is WGN

$$\mathrm{R_X}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau)$$

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{N_0}{2}$$

Noise Equivalent Bandwidth

- Output average power of ILPF→ N_oB
- Output average power of RCLPF→ $N_o/4RC$

- Output average power of any filter → $N_oBH^2(0)$
- Equivalent Bandwidth =

$$B = \frac{\int_0^\infty |H(f)|^2 df}{H^2(0)}$$