

PDF of linear transformations

- The joint PDF of Z can be found directly in terms of the joint PDF of X by finding the equivalent events of infinitesimal rectangles.

$$\left. \begin{array}{l} V = aX + bY \\ W = cX + eY \end{array} \right\} \Rightarrow \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \triangleq A \begin{bmatrix} X \\ Y \end{bmatrix}$$

A^{-1} exists

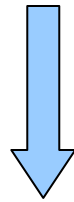
$$\forall (v, w) \exists \text{unique } (x, y) : \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} v \\ w \end{bmatrix}$$



PDF of linear transformations

$$f_{V,W}(v, w) = \frac{f_{X,Y}(x, y)}{\left| \frac{dP}{dxdy} \right|} = \frac{f_{X,Y}(x, y)}{|A|}$$

In general: $\mathbf{Z} = A\mathbf{X}$



$$f_{\mathbf{Z}}(\mathbf{z}) \triangleq f_{z_1, \dots, z_n}(z_1, \dots, z_n) = \frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n)}{|A|} \Bigg|_{\mathbf{x}=A^{-1}\mathbf{z}} = \frac{f_{\mathbf{x}}(A^{-1}\mathbf{z})}{|A|}$$

Ex 4.36 linear trafo of jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}}$$

$$\begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \triangleq A \begin{bmatrix} X \\ Y \end{bmatrix} \quad \longrightarrow \quad |A| = 1$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1} \begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} V - W \\ V + W \end{bmatrix}$$

$$f_{V,W}(v,w) = f_{X,Y}\left(\frac{v-w}{\sqrt{2}}, \frac{v+w}{\sqrt{2}}\right)$$

Ex 4.36 linear trafo of jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}$$

$$\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)} \rightarrow \frac{\frac{(v-w)^2}{2} - 2\rho \frac{(v-w)(v+w)}{2} + \frac{(v+w)^2}{2}}{2(1-\rho^2)}$$

$$= \frac{v^2(1-\rho) + w^2(1+\rho)}{2(1-\rho)(1+\rho)}$$

$$f_{V,W}(v,w) = f_{X,Y}\left(\frac{v-w}{\sqrt{2}}, \frac{v+w}{\sqrt{2}}\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left\{\frac{v^2}{2(1+\rho)} + \frac{w^2}{2(1-\rho)}\right\}}$$

V and W are independent, Gaussian, zero-mean, r.v.'s with variance $1+\rho$ and $1-\rho$ respectively



PDF of general transformations

$$V = g_1(X, Y) \text{ and } W = g_2(X, Y)$$

assume \downarrow invertibility

$$x = h_1(v, w) \text{ and } y = h_2(v, w)$$

find equivalent event of infinitesimal rectangles

$$\begin{aligned} f_{V,W}(v, w) &= \frac{f_{X,Y}(h_1(v, w), h_2(v, w))}{\left| \frac{dP}{dx dy} \right|} = \frac{f_{X,Y}(h_1(v, w), h_2(v, w))}{\underbrace{|J(x, y)|}_{\text{Jacobian of trafo}}} \\ &= f_{X,Y}(h_1(v, w), h_2(v, w)) |J(v, w)| \end{aligned}$$



Jacobian definition & property

$$J(x, y) = \det \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \quad J(v, w) = \det \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix}$$

$$|J(v, w)| = \frac{1}{|J(x, y)|}$$

$$\begin{aligned} f_{V,W}(v, w) &= \frac{f_{X,Y}(h_1(v, w), h_2(v, w))}{|J(x, y)|} \\ &= f_{X,Y}(h_1(v, w), h_2(v, w)) |J(v, w)| \end{aligned}$$

sum of such terms
when there are
multiple solutions

Ex 4.37 Rayleigh r.v.

- Let X and Y be zero-mean, unit-variance, independent Gaussian r.v.'s. Find joint PDF of V and W , when $V = (X^2 + Y^2)^{\frac{1}{2}}$; $W = \angle(X, Y)$

inverse trafo Cartesian \rightarrow polar

$$\begin{array}{l}
 x = v \cos w \\
 y = v \sin w
 \end{array}
 \longrightarrow
 J(v, w) = \det \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix} = \begin{vmatrix} \cos w & -v \sin w \\ \sin w & v \cos w \end{vmatrix} = v$$

$$\begin{aligned}
 f_{V,W}(v, w) &= f_{X,Y}(h_1(v, w), h_2(v, w)) |J(v, w)| \\
 &= \frac{v}{2\pi} e^{-\frac{v^2 \cos^2(w) + v^2 \sin^2(w)}{2}} = \frac{1}{2\pi} v e^{-v^2/2} ; v \geq 0, 0 \leq w < 2\pi
 \end{aligned}$$

independent $U[0, 2\pi)$ Rayleigh

$$\mathbf{X} = (X_1, \dots, X_n)$$

Finding joint PDF for n functions of n r.v.'s

$$Z_1 = g_1(\mathbf{X}), Z_2 = g_2(\mathbf{X}), \dots, Z_n = g_n(\mathbf{X})$$

$$z_1 = g_1(\mathbf{x}), z_2 = g_2(\mathbf{x}), \dots, z_n = g_n(\mathbf{x})$$

unique \downarrow inverse

$$x_1 = h_1(\mathbf{z}), x_2 = h_2(\mathbf{z}), \dots, x_n = h_n(\mathbf{z})$$

$$f_{Z_1, \dots, Z_n}(z_1, \dots, z_n) = \frac{f_{X_1, \dots, X_n}(h_1(\mathbf{z}), \dots, h_n(\mathbf{z}))}{|J(x_1, \dots, x_n)|}$$

$$= f_{X_1, \dots, X_n}(h_1(\mathbf{z}), \dots, h_n(\mathbf{z})) |J(z_1, \dots, z_n)|$$

$$J(x_1, \dots, x_n) = \det \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \dots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}; J(z_1, \dots, z_n) = \det \begin{bmatrix} \frac{\partial h_1}{\partial z_1} & \dots & \frac{\partial h_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial z_1} & \dots & \frac{\partial h_n}{\partial z_n} \end{bmatrix}$$



Expected value of functions of r.v.'s

similar to expected value of a function of one r.v.

$$E[Z] = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy & X, Y \text{ jointly continuous} \\ \sum_i \sum_n g(x_i, y_n) p_{X,Y}(x_i, y_n) & X, Y \text{ discrete} \end{cases}$$

above generalizes to function of n r.v.'s

in the obvious way: add more integrals, sums, and arguments



Ex 4.39 expected value of sum of r.v.'s

$$Z = X + Y$$

$$\downarrow$$
$$E[Z] = E[X + Y]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x' + y') f_{X,Y}(x', y') dx' dy'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x' f_{X,Y}(x', y') dx' dy' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y' f_{X,Y}(x', y') dx' dy'$$

$$= \int_{-\infty}^{\infty} x' f_X(x') dx' + \int_{-\infty}^{\infty} y' f_Y(y') dy' = E[X] + E[Y]$$

generalized: E sum of r.v.'s is sum of E 's

no independence needed



Ex 4.40

X and Y are **independent** r.v.'s

} 2 assumptions!

$$g(X, Y) = g_1(X)g_2(Y)$$

$$E[g(X, Y)] = E[g_1(X)g_2(Y)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x')g_2(y')f_X(x')f_Y(y')dx'dy'$$

$$= \left\{ \int_{-\infty}^{\infty} g_1(x')f_X(x')dx' \right\} \left\{ \int_{-\infty}^{\infty} g_2(y')f_Y(y')dy' \right\}$$

$$= E[g_1(X)]E[g_2(Y)]$$

generalizes directly



Correlation of 2 r.v.'s

jk^{th} joint moment of X and Y

$$E[X^j Y^k] = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x, y) dx dy & X, Y \text{ jointly continuous} \\ \sum_i \sum_n x_i^j y_n^k p_{X,Y}(x_i, y_n) & X, Y \text{ discrete} \end{cases}$$

$j=0 \rightarrow$ moments of Y

$k=0 \rightarrow$ moments of X

$j=1, k=1 \rightarrow E[XY]$: correlation of X and Y

if $E[XY] = 0$ we say that X and Y are orthogonal



Covariance of 2 r.v.'s

j k^{th} central moment of X and Y : $E \left[\underbrace{\left(X - E[X] \right)^j}_{\text{centered r.v.'s}} \underbrace{\left(Y - E[Y] \right)^k}_{\text{centered r.v.'s}} \right]$

$j=2, k=0 \rightarrow \text{VAR}(X)$

$j=0, k=2 \rightarrow \text{VAR}(Y)$

$j=1, k=1 \rightarrow E \left[\left(X - E[X] \right) \left(Y - E[Y] \right) \right] \triangleq \text{COV}(X, Y)$

$$\begin{aligned} \text{COV}(X, Y) &= E \left[XY - XE[Y] - YE[X] + E[X]E[Y] \right] \\ &= E[XY] - 2E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \\ &= E[XY] \text{ if } E[X]=0 \text{ \&/or } E[Y]=0 \end{aligned}$$



Ex 4.41 COV of independent r.v.'s

$$\begin{aligned} \text{COV}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[(X - E[X])]E[(Y - E[Y])] && \text{independence} \\ &= 0 \end{aligned}$$

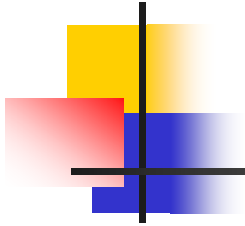
pairs of independent r.v.'s have COV zero



Correlation coefficient of X and Y

$$\begin{aligned}\rho_{X,Y} &\triangleq \frac{COV(X,Y)}{\sigma_X \sigma_Y} \\ &= \frac{E[(X - E[X])(Y - E[Y])]}{\sigma_X \sigma_Y} \\ &= E\left[\left(\frac{X - E[X]}{\sigma_X}\right)\left(\frac{Y - E[Y]}{\sigma_Y}\right)\right]\end{aligned}$$

correlation of zero-mean, unit-variance normalized r.v.'s



X and Y are said to be **uncorrelated** if $\rho_{X,Y} = 0$

X and Y independent \Rightarrow $\text{COV}(X, Y) = 0 \Rightarrow \rho_{X,Y} = 0 \Rightarrow$ uncorrelated

Ex 4.18

$\rho_{X,Y} = 0$, with X and Y jointly Gaussian \Rightarrow X and Y independent



Keep them straight

$E[XY] = 0$: X and Y are orthogonal
correlation is zero: orthogonal

$\rho_{X,Y} = 0$: X and Y are uncorrelated
covariance is zero: uncorrelated

independence \Rightarrow uncorrelated

Gaussian + uncorrelated \Rightarrow independence

correlation \sim linear dependence
dependence can be other than linear



Jointly Gaussian r.v.'s

$$f_{X,Y}(x,y) =$$

$$= \frac{\exp \left\{ \frac{-1}{2(1-\rho_{X,Y}^2)} \left[\left(\frac{x-m_x}{\sigma_x} \right)^2 - 2\rho_{X,Y} \left(\frac{x-m_x}{\sigma_x} \right) \left(\frac{y-m_y}{\sigma_y} \right) + \left(\frac{y-m_y}{\sigma_y} \right)^2 \right] \right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{X,Y}^2}}$$

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

PDF is centered at (m_x, m_y)

PDF bell-shape depends on $\sigma_x, \sigma_y, \rho_{X,Y}$



Ellipses of constant-PDF

PDF is constant for ellipse described by:

$$\left[\left(\frac{x - m_x}{\sigma_x} \right)^2 - 2\rho_{X,Y} \left(\frac{x - m_x}{\sigma_x} \right) \left(\frac{y - m_y}{\sigma_y} \right) + \left(\frac{y - m_y}{\sigma_y} \right)^2 \right] = \text{constant}$$

$\rho_{X,Y} = 0$: principal axes along x and y directions

$\rho_{X,Y} \neq 0$: major axis angle $\theta = \frac{1}{2} \arctan \left(\frac{2\rho_{X,Y}\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2} \right)$

$$\sigma_X^2 = \sigma_Y^2 \Rightarrow \theta = 45^\circ$$

DEMO Gaussian.m



Sums of r.v.'s

Let X_1, X_2, \dots, X_n be a sequence of r.v.'s

$$S_n = \sum_{i=1}^n X_i \longrightarrow E[S_n] = \sum_{i=1}^n E[X_i]$$

Ex 5.1 $Z = X + Y \rightarrow E[Z] = E[X] + E[Y]$

$$VAR(Z) = E\left[(Z - E[Z])^2\right] = E\left[(X + Y - E[X] - E[Y])^2\right]$$

$$= E\left[\{(X - E[X]) + (Y - E[Y])\}^2\right]$$

$$= E\left[(X - E[X])^2 + (Y - E[Y])^2 + 2(X - E[X])(Y - E[Y])\right]$$

$$= VAR(X) + VAR(Y) + 2COV(X, Y) \text{ not sum of individual } VARs$$



Variance of sum of r.v.'s

variance of sum of r.v.'s, in general:

$$VAR\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n VAR(X_i) + \sum_{i=1}^n \sum_{j=1, \neq i}^n COV(X_i, X_j)$$

variance of sum \neq sum of variances

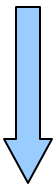
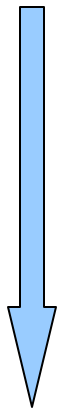
special case X_i, X_j independent r.v.'s $\Rightarrow COV(X_i, X_j) = 0$

$\rightarrow VAR\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n VAR(X_i)$ if X_i are independent r.v.'s



Ex 5.2 sum of n i.i.d. r.v.'s

$$S_n = \sum_{i=1}^n X_i ; \text{ with } X_i \text{ independent, identically distributed r.v.'s}$$
$$X_i \sim (\mu, \sigma^2)$$



$$E[S_n] = \sum_{i=1}^n E[X_i] = n\mu$$

$$VAR(S_n) = \sum_{i=1}^n VAR(X_i) = nVAR(X_i) = n\sigma^2$$



Central Limit Theorem

Let X_1, X_2, \dots be a sequence of iid r.v.'s with finite mean $E[X] = \mu$

and finite variance σ^2 , and let $S_n = \sum_{j=1}^n X_j$; this $\Rightarrow S_n \sim (n\mu, n\sigma^2)$

- as n becomes large...the CDF of a properly normalized S_n approaches the CDF of a Gaussian r.v. (Central Limit Theorem)



CLT

Let S_n be the sum of n iid r.v.'s with finite mean $E[X] = \mu$ and finite variance σ^2 , and let Z_n be the zero-mean, unit variance r.v.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \text{ then } \lim_{n \rightarrow \infty} P[Z_n \leq z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$$

$$S_n = \sum_{j=1}^n X_j \quad \text{the } X_j \text{ can have any distribution}$$



Ex 5.11

- Orders in a restaurant are iid r.v.'s $\sim(\mu=\$8, \sigma=\$2)$. Estimate the probability that the first 100 customers spend a total of more than \$840. Estimate the probability that they spend a total of between \$780 and \$820.

Let X_k be the expenditure of the k^{th} customer,

$$\text{then } S_{100} = \sum_{j=1}^{100} X_j \sim (n\mu = 800, n\sigma^2 = 400)$$

$$\longrightarrow Z_{100} = \frac{S_{100} - n\mu}{\sigma\sqrt{n}} = \frac{S_{100} - 800}{2(10)}$$



Ex 5.11 cont'd

$$Z_{100} = \frac{S_{100} - n\mu}{\sigma\sqrt{n}} = \frac{S_{100} - 800}{2(10)}$$

$$P[S_{100} > 840] = P\left[Z_{100} > \frac{840 - 800}{2(10)}\right] = Q(2) \approx 2.28(10^{-2})$$

$$\begin{aligned} P[780 \leq S_{100} \leq 820] &= P\left[\frac{780 - 800}{2(10)} \leq Z_{100} \leq \frac{820 - 800}{2(10)}\right] \\ &= 1 - 2Q(1) \approx 1 - 3.18(10^{-1}) = 0.682 \end{aligned}$$

Note: these are estimated probabilities