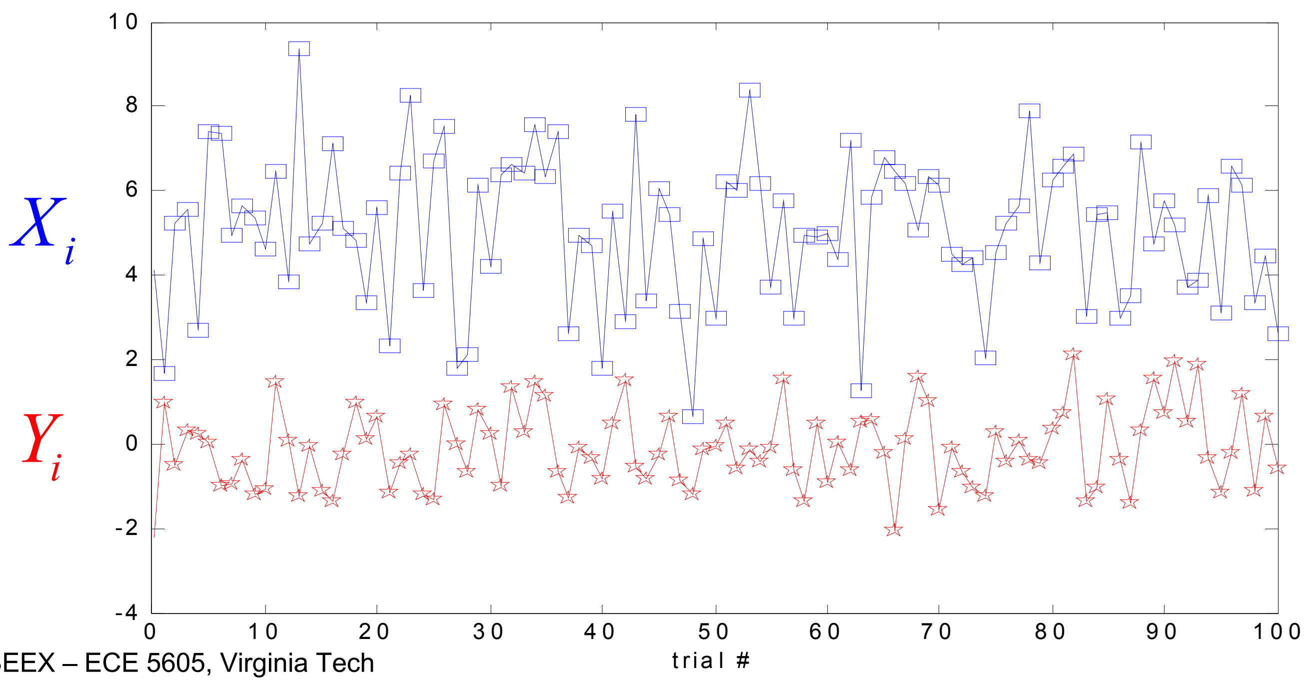
Expected value of r.v.'s

- CDF or PDF are complete (probabilistic) descriptions of the behavior of a random variable. Sometimes we are interested in less information; in a partial characterization.



different center
different spread

Expected value or MEAN

$$E[X] \triangleq \int_{-\infty}^{\infty} t f_X(t) dt \qquad E[X] \triangleq \sum_k x_k p_X(x_k)$$

defined if integral or sum converges absolutely

$$E[|X|] = \int_{-\infty}^{\infty} |t| f_X(t) dt < \infty \qquad E[|X|] = \sum_k |x_k| p_X(x_k) < \infty$$

there are random variables for which the above do not converge
we then say "the mean does not exist"

$E[X]$ represents the "center of mass"

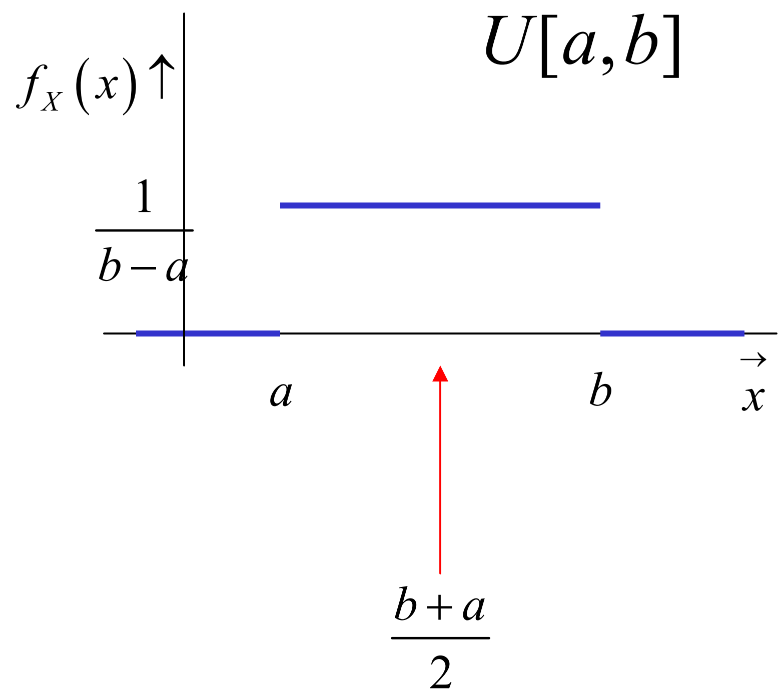
arithmetic average of large # independent observations of a r.v. will tend to the mean; it's like the "average of *X*"

Ex 3.29 mean of U[a,b]

$$E[X] \triangleq \int_{-\infty}^{\infty} t f_X(t) dt$$

$$= \int_a^b t \frac{1}{b-a} dt$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$



midpoint of interval [a,b]

Ex 3.31 mean of exponential r.v.

- Time X between customer arrivals at a service station has an exponential PDF with parameter λ. Find the mean inter-arrival time.

$$\int u\,dv = uv - \int v\,du$$

$$E[X] = \int_0^\infty \underbrace{t}_{u}\,\overbrace{\lambda e^{-\lambda t}dt}^{dv} = t\,e^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}dt$$

$$= \lim_{t\to\infty} te^{-\lambda t} - 0 + \frac{e^{-\lambda t}}{-\lambda}\Bigg|_0^\infty = \frac{1}{\lambda}$$

As λ is customer arrival rate in customers per second, the mean inter-arrival time of λ^{-1} seconds/customer makes sense!

Ex 3.32 mean of geometric r.v.

- *N* is the number of times a computer polls a terminal until the terminal has a message ready for transmission. Assuming the terminal produces messages according to a sequence of independent Bernoulli trials, *N* has a geometric distribution. Find the mean of *N*.

$$E[N] = \sum_{k=1}^{\infty} k p q^{k-1}$$

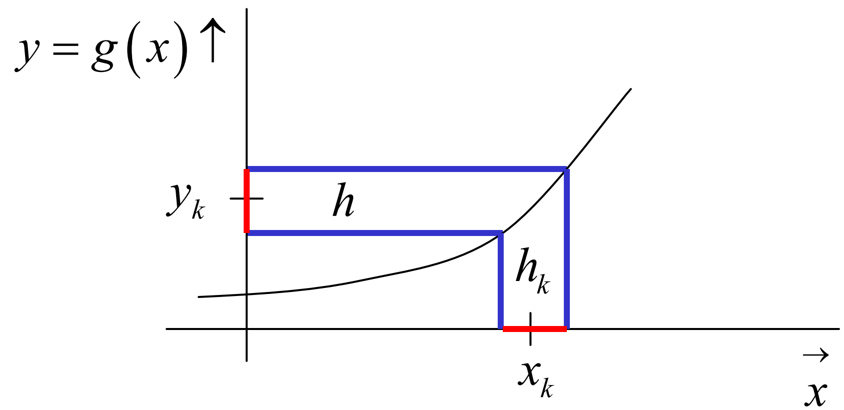
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \xrightarrow{d/dx} \sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

$$= p \frac{1}{(1-q)^2} = \frac{1}{p}$$

if probability of "success" is p, then it makes sense that on average it takes p^{-1} trials to hit "success"

Expected value of a function of a r.v.

$$E[Y] \triangleq \int_{-\infty}^{\infty} t f_Y(t)\,dt$$



alternative in terms of *X*

$$E[Y] \simeq \sum_k y_k f_Y(y_k) h$$

equivalent events

$$= \sum_k g(x_k) f_X(x_k) h_k$$

limit h→0

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$\Theta \sim U(0, 2\pi]$$

Ex 3.33

$$Y = a\cos(\omega t + \Theta)$$

- Sampling a sinusoid with random phase. Find the expected value of *Y* and of *Y*², the power of *Y*.

$$E[Y] = E[a\cos(\omega t + \Theta)]$$
$$= \int_0^{2\pi} a\cos(\omega t + \theta)\frac{1}{2\pi}d\theta$$
$$= \frac{-a}{2\pi}\sin(\omega t + \theta)\Big|_0^{2\pi}$$
$$= \frac{-a}{2\pi}\left[\sin(\omega t + 2\pi) - \sin(\omega t)\right] = 0$$

$$E\left[Y^2\right] = E\left[a^2\cos^2(\omega t + \Theta)\right]$$
$$= \frac{a^2}{2}E\left[1 + \cos(2\omega t + 2\Theta)\right]$$
$$= \frac{a^2}{2}\left[1 + \int_0^{2\pi}\cos(2\omega t + 2\theta)\frac{1}{2\pi}d\theta\right]$$
$$= \frac{a^2}{2}$$

agreement with time-averages: "DC" value of 0, power $a^2/2$

Variance of *X*

- Provides information about a random variable – in addition to its mean value – regarding its deviations from the mean

$$D \triangleq X - E[X]$$ deviation from the mean

measures of "width/spread"

$$VAR[X] \triangleq E\left[\left(X - E[X]\right)^2\right]$$ variance of r.v.

$$STD[X] \triangleq \sqrt{VAR[X]}$$ standard deviation of r.v.

$$\begin{aligned} VAR[X] &= E\left[X^2 - 2E[X]X + E^2[X]\right] \\ &= E\left[X^2\right] - 2E[X]E[X] + E^2[X] \\ &= E\left[X^2\right] - E^2[X] \end{aligned}$$

Ex 3.36 variance of X~U[a,b]

$$E[X]=\frac{b+a}{2}$$

$$VAR[X]=\frac{1}{b-a}\int_a^b\left(x-\frac{b+a}{2}\right)^2dx$$

$$=\frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2}y^2dy=\frac{y^3}{3(b-a)}\Bigg|_{-(b-a)/2}^{(b-a)/2}=\frac{(b-a)^2}{12}$$

$$U[-2,4]\to E[X]=\frac{4+(-2)}{2}=1;VAR[X]=\frac{(4-(-2))^2}{12}=3$$

3.38 variance of Gaussian r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$\int_{-\infty}^{\infty}$ ↓

$$\sigma\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-(x-m)^2 / 2\sigma^2} dx$$

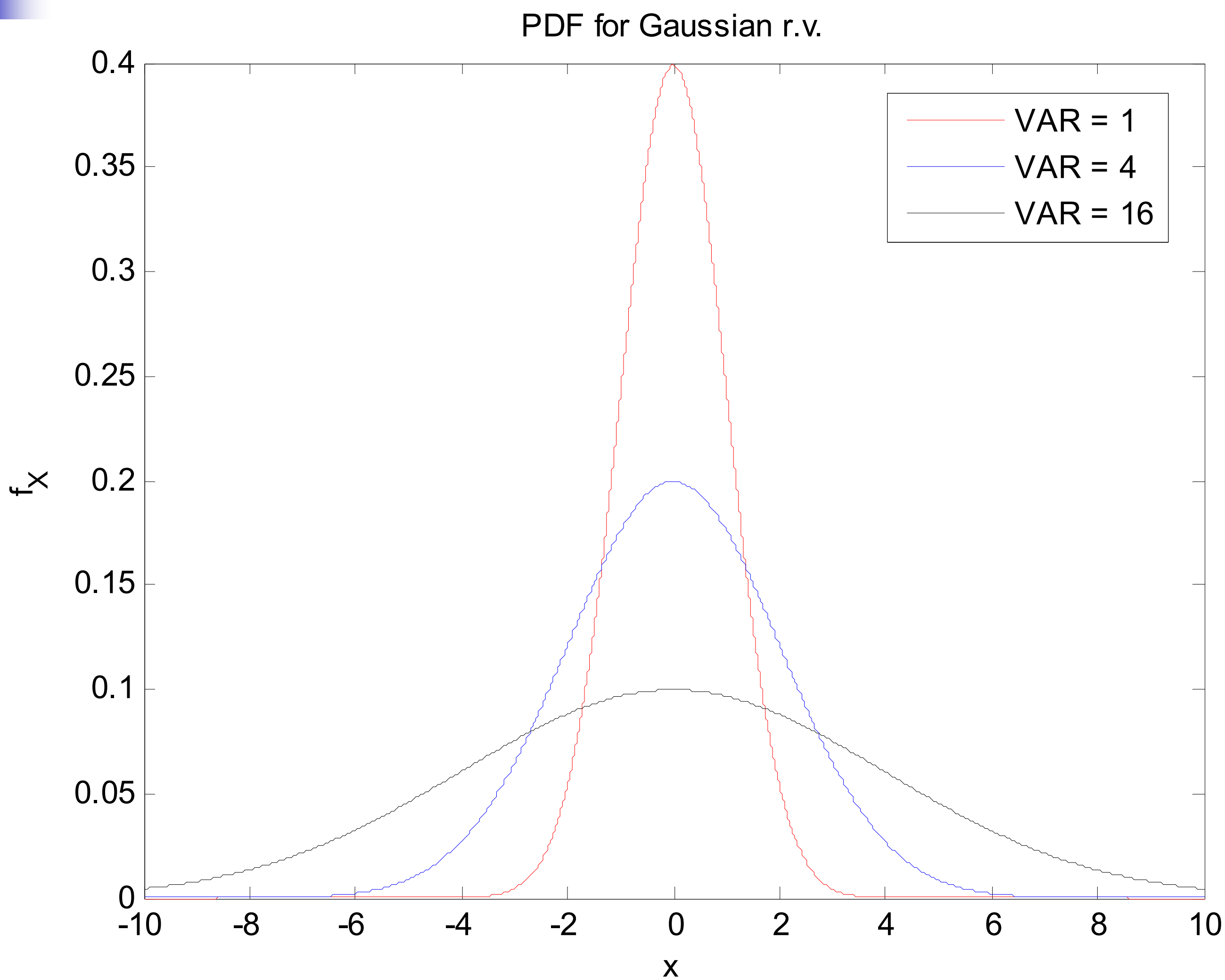
$\frac{d}{d\sigma}$ ↓

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} \frac{(x-m)^2}{\sigma^3} e^{-(x-m)^2 / 2\sigma^2} dx$$

↓

$$VAR[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^2 e^{-(x-m)^2 / 2\sigma^2} dx = \sigma^2$$

Effect of VAR on the PDF of a Gaussian r.v.



Mean & variance

- Are the two most important parameters for (partially) characterizing the PDF of a r.v.
- Others are sometimes used, e.g.

$$skewness \triangleq \frac{E\left[\left(X-E[X]\right)^3\right]}{\sigma^3}$$

  which measures the degree of asymmetry about the mean
  - *Skewness is zero for a symmetric PDF*
- Each involves n^{th} moment of r.v. X: $E\left[X^n\right]=\int_{-\infty}^{\infty} x^n f_X(x)\,dx$
- $VAR[X]=E\left[X^2\right]-E^2[X]$
- Under certain conditions, a PDF is completely specified if all moments are known (more on that later)

$$VAR[c]=0 \quad VAR[X+c]=VAR[X] \quad VAR[cX]=c^2VAR[X]$$

Multiple r.v.'s

- Several r.v.'s at a time
  - *Measuring different quantities simultaneously*
    - Engine oil pressure, RPM, generator voltage
  - *Repeated measurement of the same quantity*
    - Sampling a waveform, such as EEG, speech

↓

- Joint behavior of two or more r.v.'s
  - *Independence of sets of r.v.'s*
  - *Correlation if not independent*

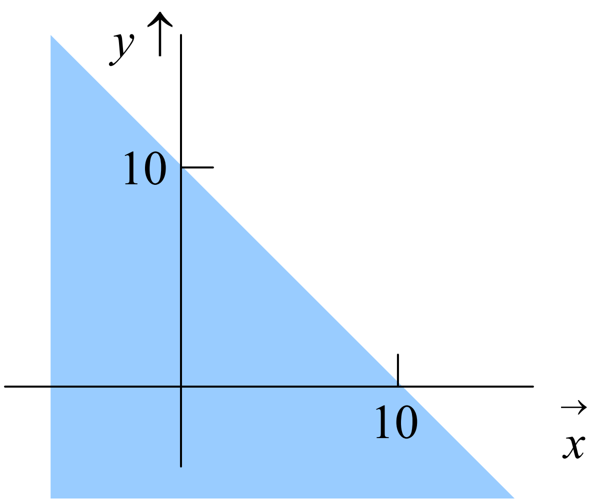
Ex 4.1

- Let a random experiment be the selection of a student's name from an urn: outcome ζ.
- Define the following three functions:
  - *$H(\zeta)$ = student's height, in inches*
  - *$W(\zeta)$ = student's weight, in pounds*
  - *$A(\zeta)$ = student's age, in years*
- ($H(\zeta)$, $W(\zeta)$, $A(\zeta)$) is a vector r.v.

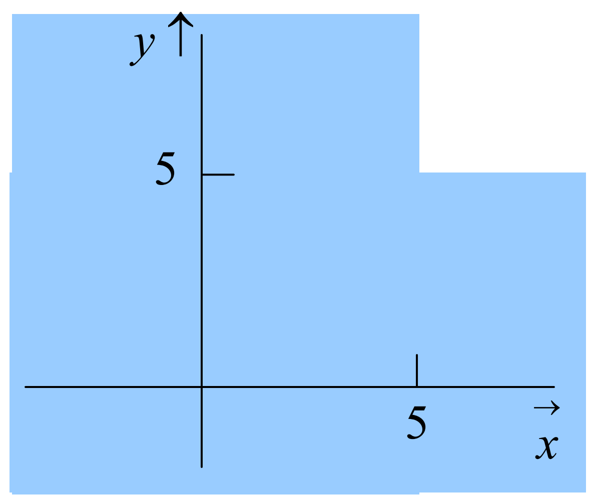
Events & probabilities

- Each event involving an *n*-dimensional r.v. **X**=(X_1, X_2,…, X_n) has a corresponding region in an *n*-dimensional real space
  - *E.g. 2-D r.v.* $\mathbf{X}=(X,Y)$

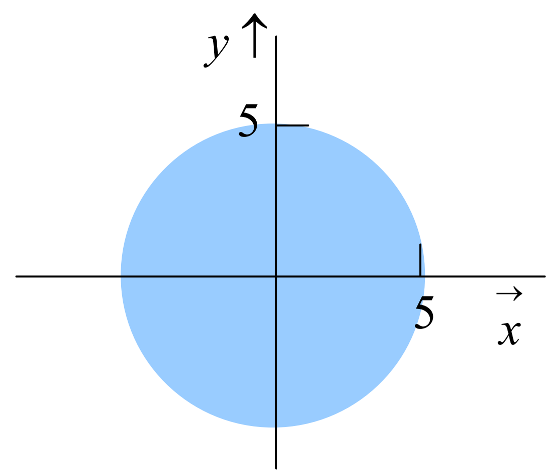
$$A=\{X+Y\le 10\}$$



$$B=\{\min(X,Y)\le 5\}$$



$$C=\{X^2+Y^2\le 25\}$$



Independence

- Intuitively, if r.v.'s X and Y are "independent," then events that involve only X should be independent of events that involve only Y. In other words, if A_1 is any event that involves X only and A_2 is any event that involves Y only, then

$$P[X \in A_1, Y \in A_2] = P[X \in A_1]P[Y \in A_2]$$

- In general, n r.v.'s are independent if

$$P[X_1 \in A_1, \cdots, X_n \in A_n] = P[X_1 \in A_1] \cdots P[X_n \in A_n]$$

where A_k is an event that involves X_k only

if r.v.'s are independent, knowing the probabilities of the r.v.'s in isolation suffices to specify probabilities of joint events

Pairs of discrete r.v.'s

vector r.v. $\mathbf{X} = (X, Y)$

$$S = \left\{ \left(x_j, y_k \right), \; j = 1, 2, \cdots, k = 1, 2, \cdots \right\}$$

joint probability mass function:

$$p_{X,Y}\left(x_j, y_k\right) = P\left[\left\{X = x_j\right\} \cap \left\{Y = y_k\right\}\right]$$

$$\triangleq P\left[X = x_j, Y = y_k\right] \qquad j = 1, 2, \cdots \quad k = 1, 2, \cdots$$

gives probability of occurrence of pairs (x_j, y_k)

For any event A: $P\left[\mathbf{X} \in A\right] = \sum_{\left(x_j, y_k\right)} \sum_{\in A} p_{X,Y}\left(x_j, y_k\right)$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}\left(x_j, y_k\right) = P[S] = 1$$

Marginal PMF's

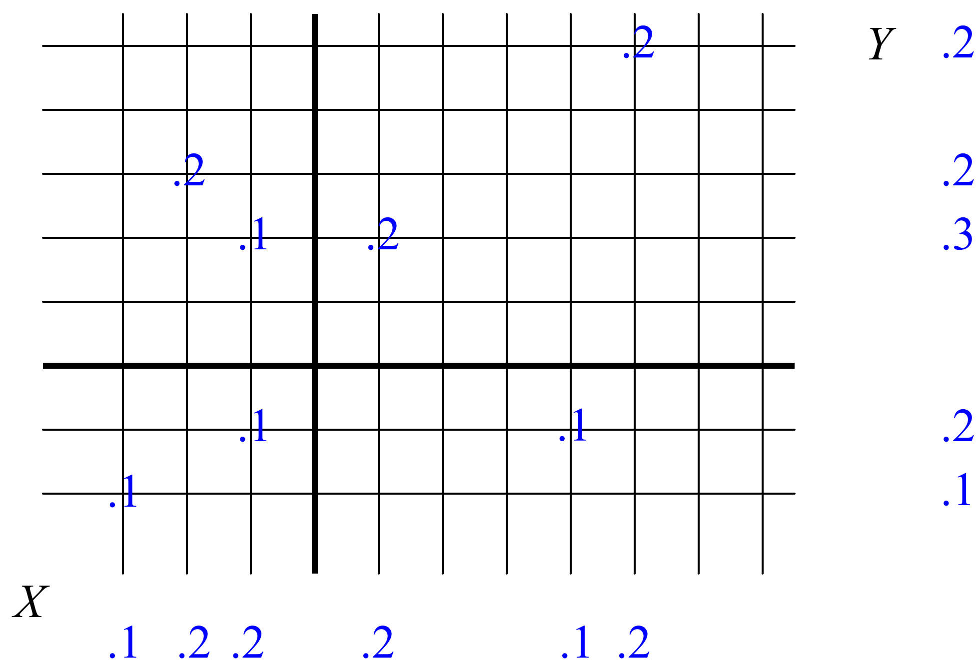
$$p_X(x_j) = p_{X,Y}(x_j, \text{any } y_k)$$

$$= P\left[\left\{\{X = x_j\} \cap \{Y = y_1\}\right\} \cup \left\{\{X = x_j\} \cap \{Y = y_2\}\right\} \cup \cdots\right]$$

$$= \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k)$$

$$p_Y(y_k) = \sum_{j=1}^{\infty} p_{X,Y}(x_j, y_k)$$

marginal PMF's are insufficient – in general - for specifying joint PMF



Ex 4.6 tossing loaded dice

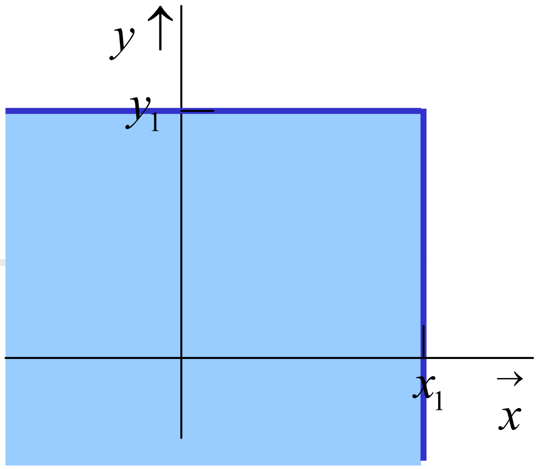
joint PMF in units of $\frac{1}{42}$

$j \backslash k$	1	2	3	4	5	6		
1	2	1	1	1	1	1	7	1
2	1	2	1	1	1	1	7	1
3	1	1	2	1	1	1	7	1
4	1	1	1	2	1	1	7	1
5	1	1	1	1	2	1	7	1
6	1	1	1	1	1	2	7	1
	7	7	7	7	7	7	$/42$	
	1	1	1	1	1	1		$/6$

marginals give no clue

Joint CDF of *X* and *Y*

- Basic building block:



semi-infinite rectangle: $\{(x, y) : \{x \leq x_1\} \cap \{y \leq y_1\}\}$

joint cumulative distribution function of *X* and *Y*:

$$F_{X,Y}(x_1, y_1) = P\left[\{x \leq x_1\} \cap \{y \leq y_1\}\right]$$

product-form event

$$= P\left[X \leq x_1, Y \leq y_1\right]$$

long-term proportion of times in which the outcome (*x*,*y*) falls in the blue rectangle, or the probability mass in it

properties of the joint CDF

i. $F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2) \quad x_1 \le x_2, y_1 \le y_2$

non-decreasing in NE direction

ii. $F_{X,Y}(-\infty, y_1) = F_{X,Y}(x_2, -\infty) = 0$

neither *X* nor *Y* takes on values <-∞

iii. $F_{X,Y}(\infty, \infty) = 1$

X and *Y* can take on only values <∞

iv. $F_X(x) = F_{X,Y}(x, \infty) = P[X \le x, Y < \infty] = P[X \le x]$

$F_Y(y) = F_{X,Y}(\infty, y) = P[X < \infty, Y \le y] = P[Y \le y]$

marginal cumulative distribution functions

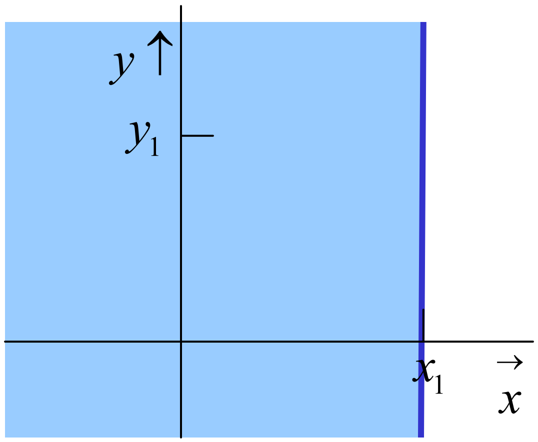
v. $\lim_{x \to a^+} F_{X,Y}(x, y) = F_{X,Y}(a, y)$

$\lim_{y \to b^+} F_{X,Y}(x, y) = F_{X,Y}(x, b)$

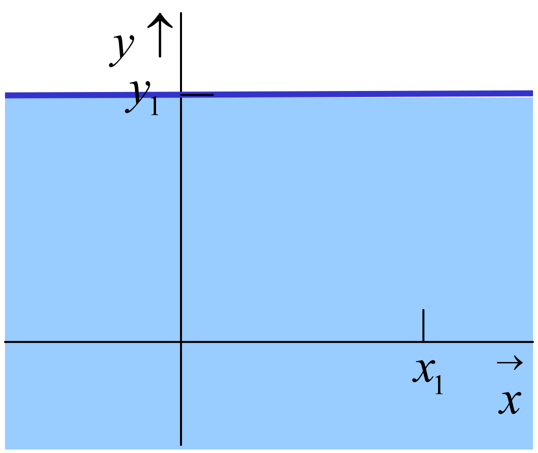
continuous from N & E

Ex 4.8 marginals for continuous joint r.v.'s

$$\text{Given } F_{X,Y}(x,y) = \left(1-e^{-\alpha x}\right)\left(1-e^{-\beta y}\right)u(x)u(y)$$



$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \left(1-e^{-\alpha x}\right)u(x)$$



$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = \left(1-e^{-\beta y}\right)u(y)$$

marginal CDFs are exponential distributions with parameters α and β respectively

Joint CDF in terms of joint PDF & v.v.

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x',y')\,dx'\,dy'$$

↓ if jointly continuous r.v.'s

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

if the CDF is discontinuous
- or the partial derivatives are discontinuous -
then the joint PDF does not exist

$$A = \{(x, y) : a_1 < x \le b_1 ; a_2 < y \le b_2\}$$

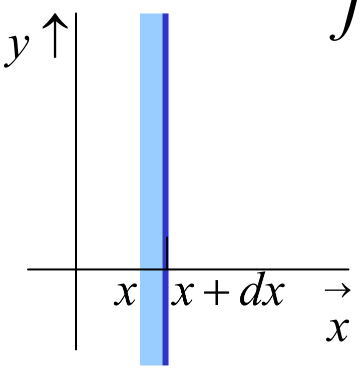
$$P[a_1 < X \le b_1, a_2 < Y \le b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}(x', y') dx' dy'$$

↓

$$P[x < X \le x + dx, y < Y \le y + dy] = \int_{x}^{x+dx} \int_{y}^{y+dy} f_{X,Y}(x', y') dx' dy'$$

$$\simeq f_{X,Y}(x, y) dxdy$$

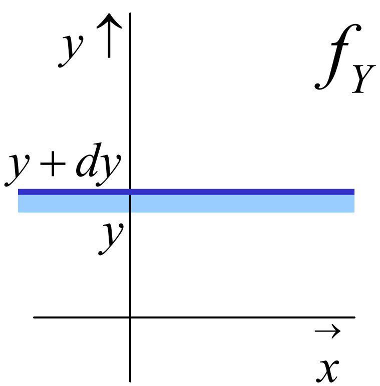
Marginal PDF's from joint PDF



$$f_X(x) = \frac{d}{dx} F_{X,Y}(x,\infty) = \frac{d}{dx} \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{X,Y}(x',y')dy' \right\} dx'$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,y')dy'$$

concentrating mass on x-axis



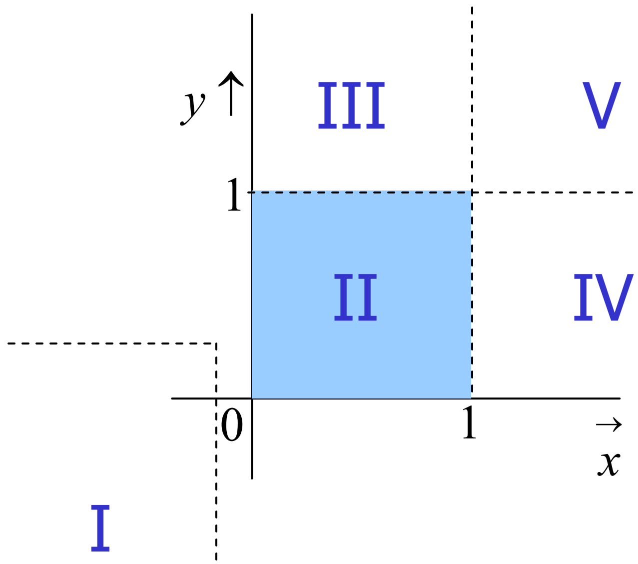
$$f_Y(y) = \frac{d}{dy} F_{X,Y}(\infty,y) = \int_{-\infty}^{\infty} f_{X,Y}(x',y)dx'$$

concentrating mass on y-axis

marginal PDF's are obtained by integrating out the other variable(s)

$$f_{X,Y}(x,y) = \begin{cases} 1 & (x,y) \in [0,1]\times[0,1] \\ 0 & o.w. \end{cases}$$

Ex 4.10 CDF for jointly uniform r.v.'s



$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x',y')dx'dy'$$

$$= 0 \quad \text{for } (x,y) \in I$$

$$= \int_0^x\int_0^y 1dx'dy' = xy \quad \text{for } (x,y) \in II$$

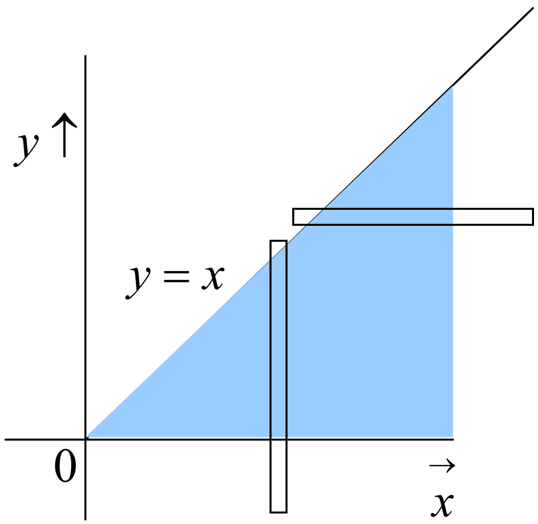
$$= \int_0^x\int_0^1 1dx'dy' = x \quad \text{for } (x,y) \in III$$

$$= \int_0^1\int_0^y 1dx'dy' = y \quad \text{for } (x,y) \in IV$$

$$= \int_0^1\int_0^1 1dx'dy' = 1 \quad \text{for } (x,y) \in V$$

Ex 4.11 finding marginal PDF's

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & o.w. \end{cases}$$



$$1 = \int_0^\infty \int_0^x ce^{-x}e^{-y}\,dydx = \int_0^\infty ce^{-x} \int_0^x e^{-y}\,dydx$$

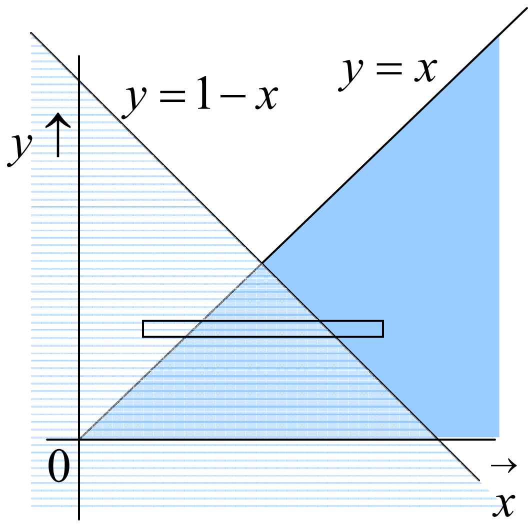
$$= \int_0^\infty ce^{-x} \frac{e^{-x} - 1}{-1}\,dx = \frac{c}{2} \Rightarrow c = 2$$

$$f_X(x) = \int_0^x 2e^{-x}e^{-y}\,dy = 2e^{-x}\left(1 - e^{-x}\right) \quad 0 \le x < \infty$$

$$f_Y(y) = \int_y^\infty 2e^{-x}e^{-y}\,dx = 2e^{-y}\left(e^{-y}\right) = 2e^{-2y} \quad 0 \le y < \infty$$

are these PDF's?

Ex 4.12 Find P[*X*+ *Y*≤1] in EX 4.11



$$P\left[X+Y\le 1\right]=\int_0^{0.5}\int_y^{1-y}2e^{-x}e^{-y}dxdy$$

$$=\int_0^{0.5}2e^{-y}\int_y^{1-y}e^{-x}dxdy$$

$$=\int_0^{0.5}2e^{-y}\left[e^{-y}-e^{-(1-y)}\right]dy$$

$$=-e^{-1}+1-e^{-1}=1-2e^{-1}$$

Ex 4.13 Find marginal PDFs of jointly Gaussian PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(x^2-2\rho xy+y^2\right) / 2\left(1-\rho^2\right)}$$

$$f_X(x) = \frac{e^{-x^2 / 2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left(-2\rho xy+y^2\right) / 2\left(1-\rho^2\right)} dy$$

completing the square

$$= \frac{e^{-x^2\left(1-\rho^2\right) / 2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left(\rho^2 x^2-2\rho xy+y^2\right) / 2\left(1-\rho^2\right)} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-\left(y-\rho x\right)^2 / 2\left(1-\rho^2\right)} dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = N(0,1)$$

it's a PDF

symmetry

$$\Downarrow$$

$$f_Y(y) = f_X(x)$$

Independence of two r.v.'s

discrete r.v.'s X and Y are independent

⇕ iff

joint PMF is product of marginal PMFs

$\forall x_j, y_k$

Ex 4.15 another look at Ex 4.6 loaded dice

joint PMF in units of $\frac{1}{42}$

marginals give no clue

$j \backslash k$	1	2	3	4	5	6	
1	2	1	1	1	1	1	1
2	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
4	1	1	1	2	1	1	1
5	1	1	1	1	2	1	1
6	1	1	1	1	1	2	1
	1	1	1	1	1	1	

marginals in units of $/6$

not independent

$$p_{X,Y}\left(x_j, y_k\right) \neq p_X\left(x_j\right) p_Y\left(y_k\right) \;\; \forall x_j, y_k$$

Independence of two r.v.'s in general

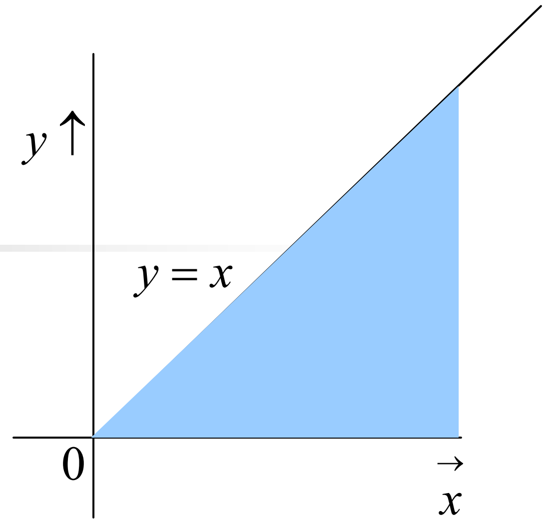
⇕ IFF

$$p_{X,Y}\left(x_j, y_k\right) = p_X\left(x_j\right) p_Y\left(y_k\right) \ \forall x_j, y_k$$ discrete

$$F_{X,Y}\left(x, y\right) = F_X\left(x\right) F_Y\left(y\right) \ \forall x, y$$

$$f_{X,Y}\left(x, y\right) = f_X\left(x\right) f_Y\left(y\right) \ \forall x, y$$ jointly continuous

Ex 4.17 back to Ex 4.11



joint $f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & o.w. \end{cases}$

marginal PDFs

$$f_X(x) = \int_0^x 2e^{-x}e^{-y}\,dy = 2e^{-x}\left(1-e^{-x}\right) \quad 0 \le x < \infty$$

$$f_Y(y) = \int_y^\infty 2e^{-x}e^{-y}\,dx = 2e^{-y}\left(e^{-y}\right) = 2e^{-2y} \quad 0 \le y < \infty$$

product? $2e^{-x}e^{-y} \neq 2e^{-x}\left(1-e^{-x}\right)2e^{-2y}$

X and *Y* are not independent

Ex 4.18 back to Ex 4.13

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\left(x^2-2\rho xy+y^2\right)/2\left(1-\rho^2\right)}$$

$$f_X(x)=\frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad f_Y(y)=\frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$f_X(x)f_Y(y)=\frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}=\frac{e^{-\left(x^2+y^2\right)/2}}{2\pi}$$

$\equiv$ iff $\rho=0$

Conditional probability

- Many r.v.'s are not independent, such as when measuring one variable to learn about another one, or when sampling slowly relative to the rate of change of a signal.
- We're then interested in the probability of an event related to Y, given the knowledge of $X=x$ (a measurement)
- We'll see that $E[Y|X=x]$ is of significance also

Conditional probability

$$P[Y \in A \mid X = x] = \frac{P[Y \in A, X = x]}{P[X = x]}$$

X discrete ↓

$$F_Y(y \mid x_k) = \frac{P[Y \le y, X = x_k]}{P[X = x_k]} \text{ for } P[X = x_k] > 0$$

if derivative exists ↓

$$f_Y(y \mid x_k) = \frac{d}{dy} F_Y(y \mid x_k)$$

↓

$$P[Y \in A \mid X = x_k] = \int_{y \in A} f_Y(y \mid x_k)\, dy$$

independence →

$$F_Y(y \mid x) = F_Y(y)$$

$$f_Y(y \mid x) = f_Y(y)$$

Conditional probability

$$P[Y \in A \mid X = x] = \frac{P[Y \in A, X = x]}{P[X = x]}$$

X and *Y* discrete

$$p_Y(y_j \mid x_k) = P[Y = y_j \mid X = x_k] = \frac{P[X = x_k, Y = y_j]}{P[X = x_k]}$$

δ functions
~ PMF weights

$$= \begin{cases} \dfrac{p_{X,Y}(x_k, y_j)}{p_X(x_k)} \text{ for } P[X = x_k] = p_X(x_k) > 0 \\ 0 \text{ for } P[X = x_k] = p_X(x_k) = 0 \end{cases}$$

in general

$$P[Y \in A \mid X = x_k] = \sum_{y_j \in A} p_Y(y_j \mid x_k) = \sum_{y_j \in A} p_Y(y_j)$$

if also independent

Ex 4.20 back to Ex 4.14

- *X* is the input and *Y* the output of a communication channel. *P*[*Y*<0|*X*=+1]?

$$f_Y\left(y\,|\,1\right) \sim U\left[-1,3\right] \Longrightarrow P\left[Y<0\,|\,X=+1\right] = \int_{-1}^{0} \frac{1}{4} dy = \frac{1}{4}$$

integrate over event

Conditional CDF of *Y* given *X*=x

$$F_Y(y \mid x) \triangleq \lim_{h \to 0} F_Y(y \mid x < X \le x+h)$$

$$= \frac{P[Y \le y, x < X \le x+h]}{P[x < X \le x+h]} = \frac{\int_{-\infty}^{y} \int_{x}^{x+h} f_{X,Y}(x', y') dx' dy'}{\int_{x}^{x+h} f_X(x') dx'}$$

$$\simeq \frac{h \int_{-\infty}^{y} f_{X,Y}(x, y') dy'}{h f_X(x)} = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y') dy'}{f_X(x)}$$

if independent

$$= \int_{-\infty}^{y} f_Y(y') dy' = F_Y(y)$$

Conditional PDF of Y given $X=x$

$$F_Y(y \mid x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')\, dy'}{f_X(x)}$$

$$f_Y(y \mid x) = \frac{d}{dy} F_Y(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

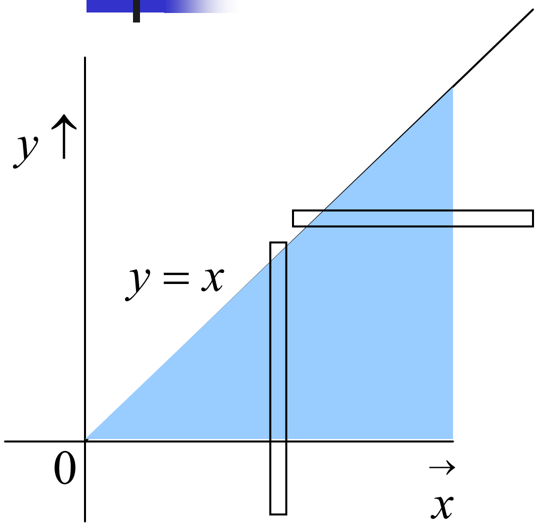
almost Bayes' rule

$$f_Y(y \mid x)\, dy = \frac{f_{X,Y}(x, y)\, dx dy}{f_X(x)\, dx}$$

if independent

$$f_Y(y \mid x) = f_Y(y)$$

Ex 4.21 back to Ex 4.11



$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & o.w. \end{cases}$$

$$f_X(x) = 2e^{-x}\left(1 - e^{-x}\right) \quad 0 \le x < \infty$$

$$f_Y(y) = 2e^{-2y} \quad 0 \le y < \infty$$

$$f_X(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x}e^{-y}}{2e^{-2y}} = e^{-(x-y)} \; ; y \le x < \infty$$

$$f_Y(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2e^{-x}\left(1 - e^{-x}\right)} = \frac{e^{-y}}{1 - e^{-x}} \; ; 0 \le y < x$$