Geometric r.v.

- # of independent Bernoulli trials until first occurrence of "success"

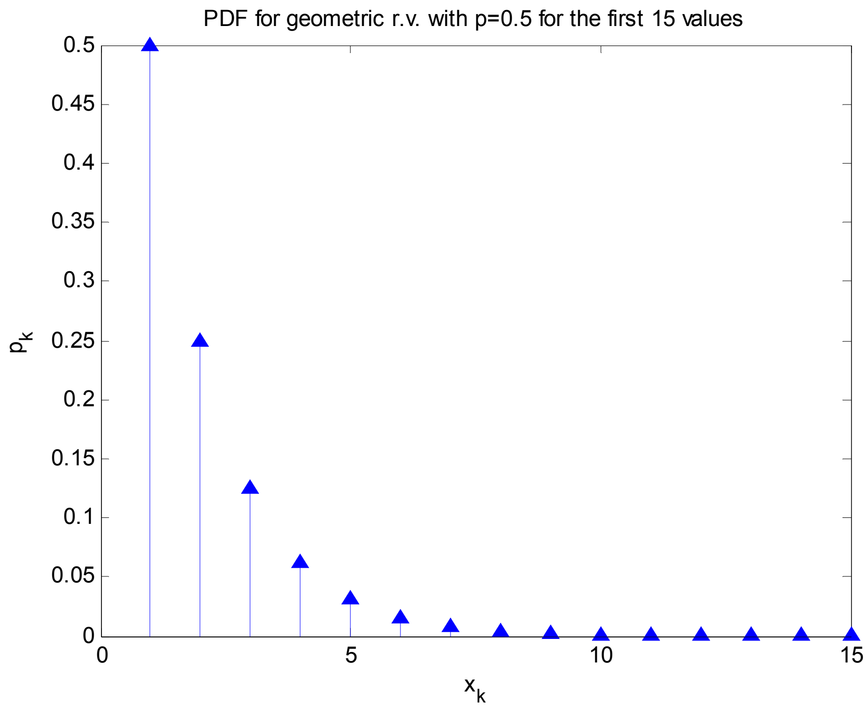
$$S_X = \{1, 2, \cdots\}$$

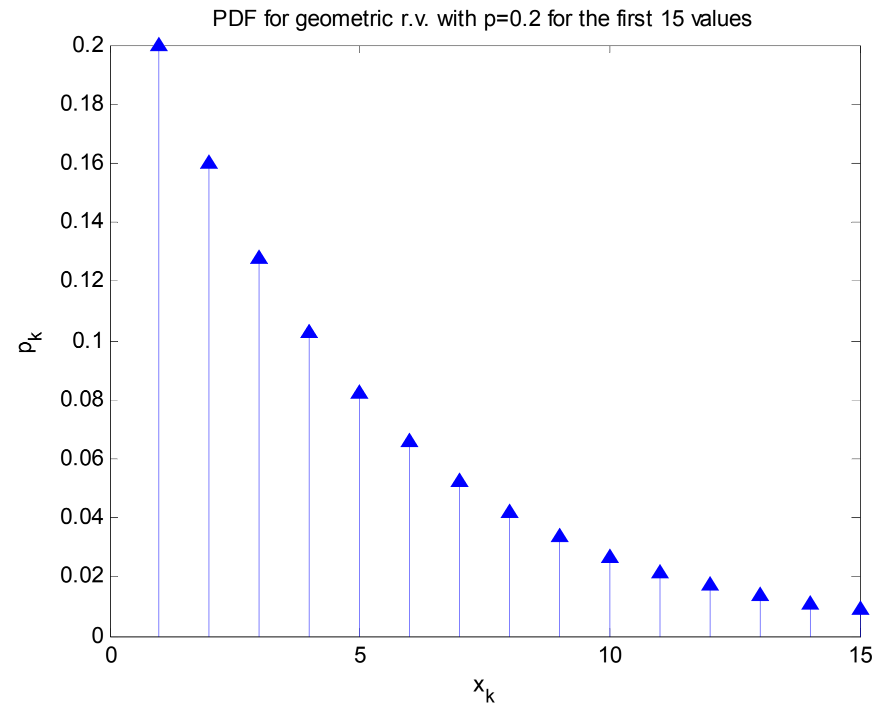
$$p = P[A] = P["success"] \text{ in each Bernoulli trial}$$

$$P[M = k] = (1-p)^{k-1} p \quad \text{for } k = 1, 2, \cdots$$

geometric decay

Geometric PDF

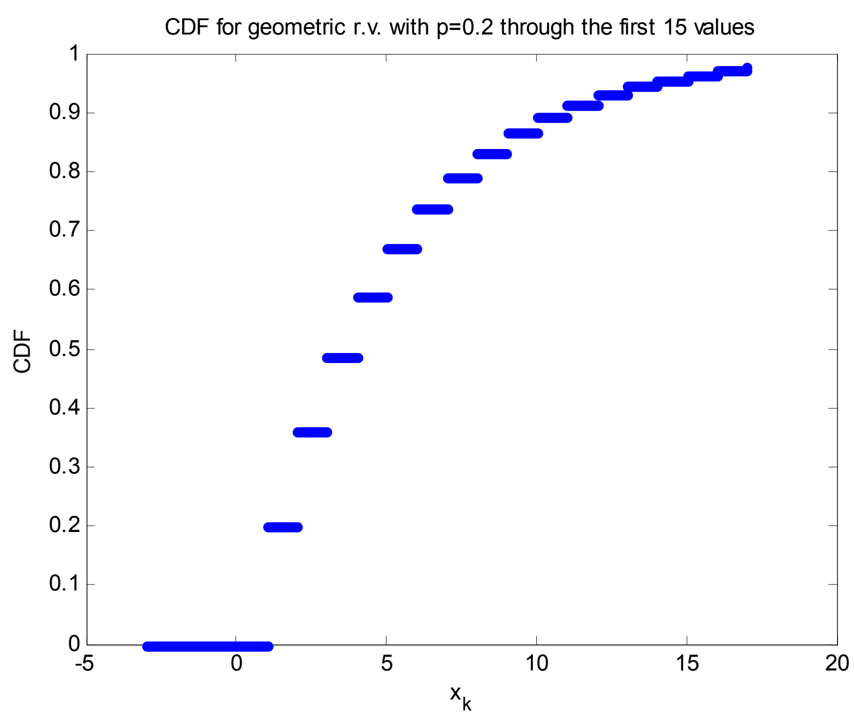




decay like 0.5^k and 0.8^k respectively

Geometric CDF

$$P\left[M \le k\right] = \sum_{j=1}^{k} \left(1-p\right)^{j-1} p = \frac{1-\left(1-p\right)^{k}}{1-\left(1-p\right)} p = 1-\left(1-p\right)^{k} = 1-q^{k}$$

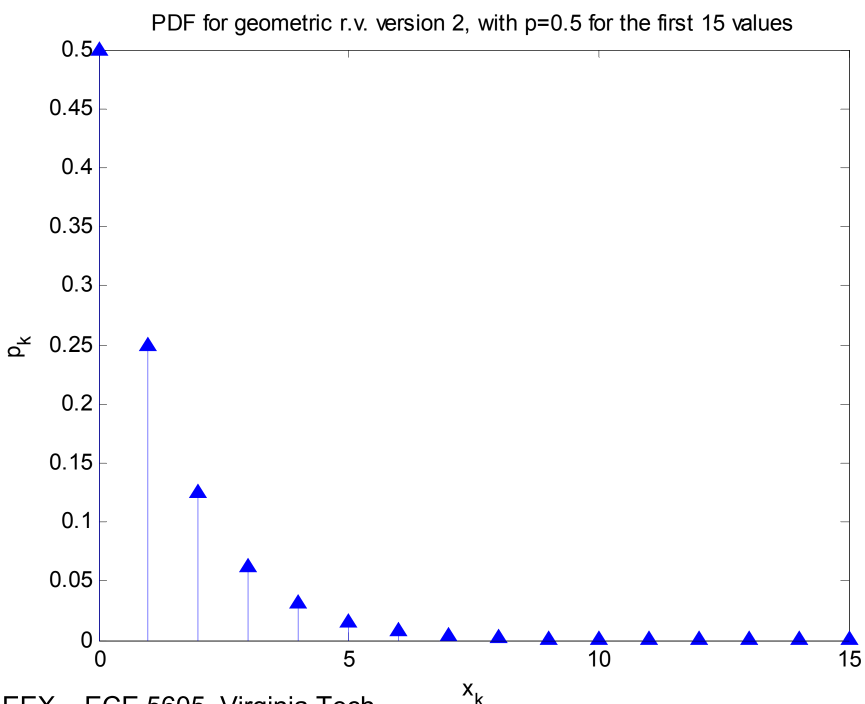


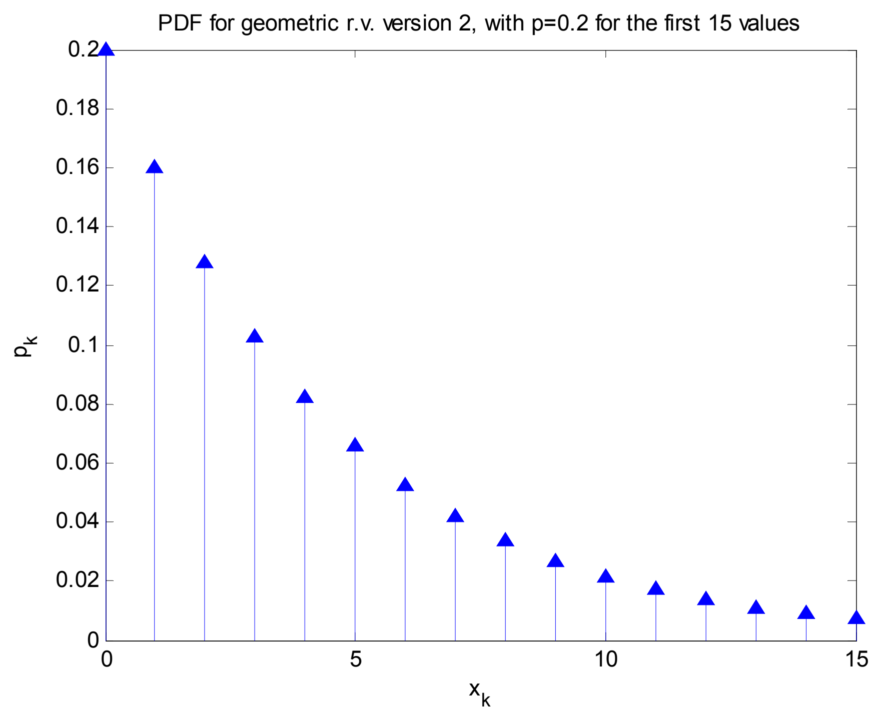
Geometric r.v. – version 2

If our interest is in the # of failures before a success occurs:

$$P\left[M' = k\right] = P\left[M = k+1\right] = \left(1-p\right)^{k} p \quad \text{for } k = 0,1,2,\cdots$$

M′ is also a geometric r.v.





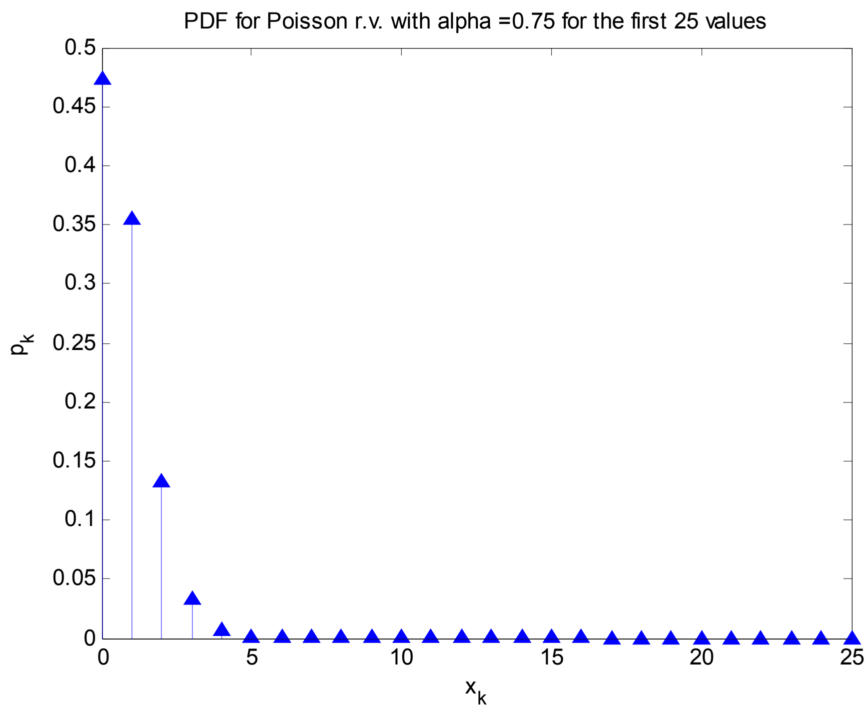
Poisson r.v.

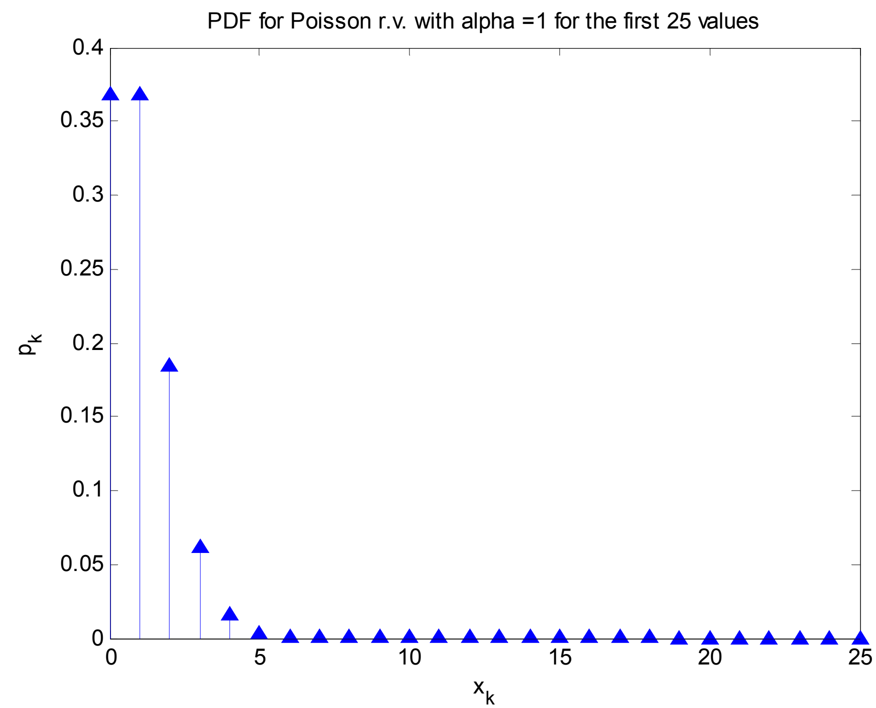
- Interested in counting the # occurrences of an event in a certain time period or in a certain region in space
  - *Events occur completely "at random"*
    - Emissions from radioactive substances
    - Counts of demands for telephone connections
    - Counts of defects in a semiconductor chip

$$P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha} \quad \text{for } k=0,1,2,\cdots$$

α is the average # of event occurrences in a specified interval or region in space

Poisson PDF



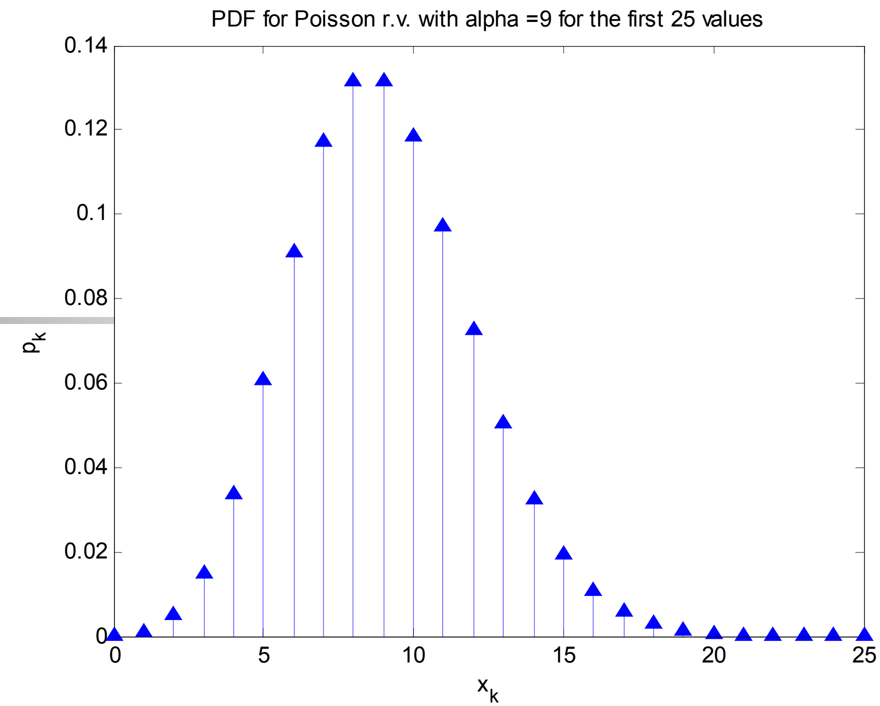


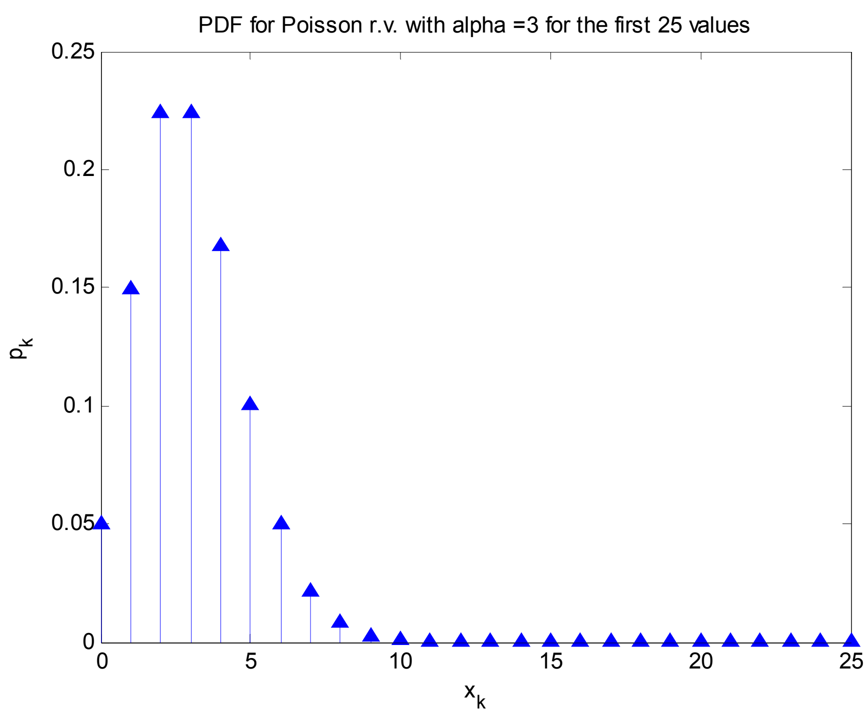
$\max P[N=k]$ is at 0 for $\alpha < 1$

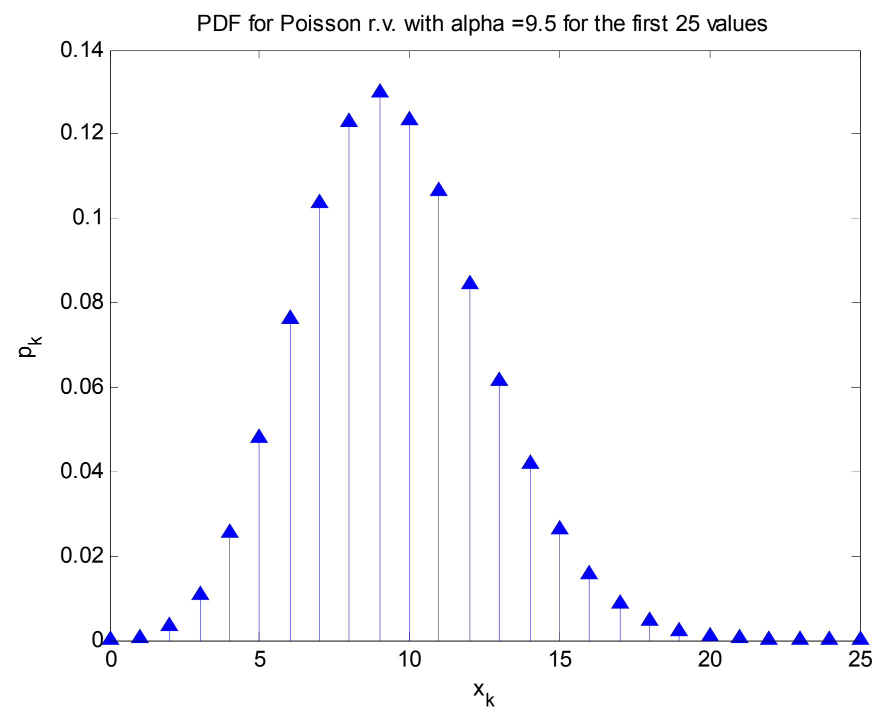
$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$

Poisson PDF

$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$







$\max P[N=k]$ is at $\lfloor \alpha \rfloor$ for $\alpha > 1$

Poisson PDF

$$P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha} \text{ for } k=0,1,2,\cdots$$

$$\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}e^{-\alpha}=e^{-\alpha}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}=e^{-\alpha}e^{\alpha}=1$$

$P[S]=1$

if n is large and p is small, then for $\alpha=np$

$$p_k=\binom{n}{k}p^k(1-p)^{n-k}\simeq\frac{\alpha^k}{k!}e^{-\alpha} \quad k=0,1,\cdots$$

Poisson PMF is the limiting form of the binomial PMF when the number of Bernoulli trials is made very large and the probability of success is kept small, so that $\alpha=np$

recall: numerical problems in calculating binomial coefficients

Ex probability of bit error in comms

- P[bit error]=10^{-3}. P[≥5 bit errors in block of 10^3 bits]

Bernoulli trials with “success” corresponding to bit error

$$p_k = \binom{1000}{k} 10^{-3k} \left(1-10^{-3}\right)^{1000-k} \simeq \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1, \cdots$$

$$\alpha = np = 10^3 10^{-3} = 1$$

$$P[N \geq 5] = 1 - P[N < 5] \simeq 1 - \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= 1 - e^{-1}\left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right\} = 0.00366$$

Ex

- Requests for telephone connections arrive at a switching office at the rate of λ calls per second. It is known that the number of requests follows a Poisson r.v. What is *P*[no call requests in *t* sec]? What is *P*[≥*n* call requests in *t* sec]?

average # requests in a *t*-sec period is $\alpha=\lambda t$

⇓

$N(t)$, the # requests in *t* sec, is Poisson with $\alpha=\lambda t$

$$P\left[N(t)=0\right]=\frac{(\lambda t)^0}{0!}e^{-\lambda t}=e^{-\lambda t}$$

$$P\left[N(t)\geq n\right]=1-P\left[N(t)<n\right]=1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

Exponential r.v.

- Arises in modeling of the time between occurrence of events, and in modeling lifetime of devices and systems; λ is the rate at which events occur

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$$F_X(x) = \left(1 - e^{-\lambda x}\right) u(x)$$

shown earlier

- For a Poisson r.v., the time between events is an exponentially distributed r.v. with parameter $\lambda = \frac{\alpha}{T}$ events per second

- Binomial $\rightarrow$ Poisson
- Geometric $\rightarrow$ exponential

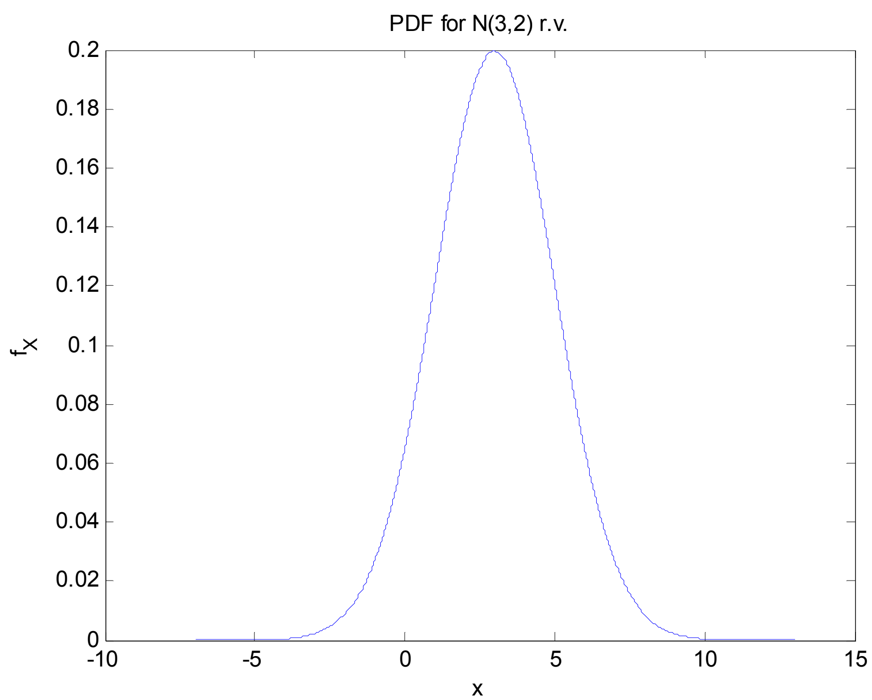
Gaussian (normal) r.v.

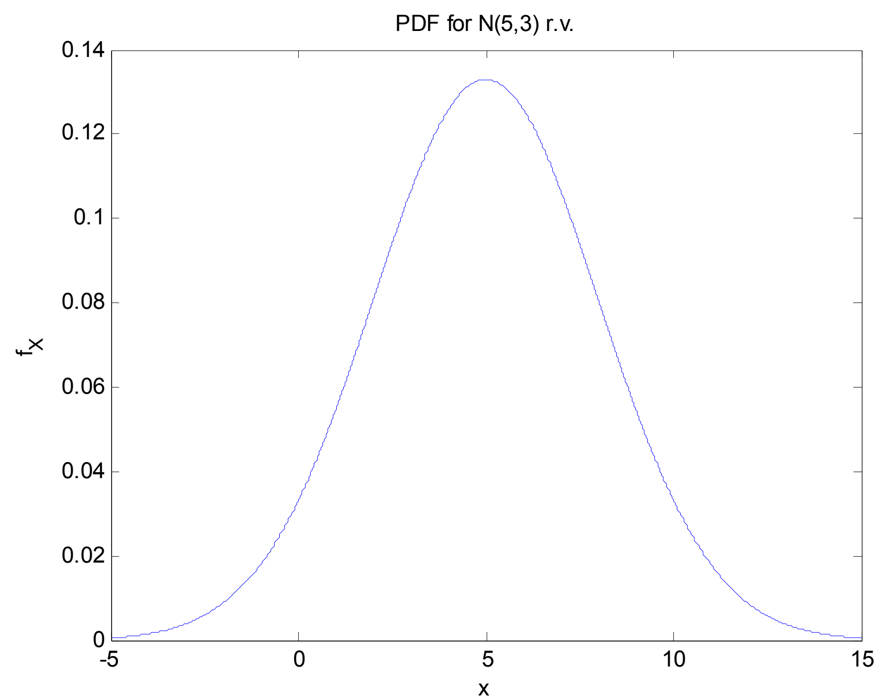
- In many situations in man-made and natural phenomena one deals with a r.v. *X* that consists of a large sum of "small" r.v.'s
  - *Exact PDF becomes complex and unwieldy*
- Under fairly general conditions, as the number of components becomes large (CLT), the CDF approaches that of the normal r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$= N(m,\sigma)$$

Gaussian (normal) r.v. - PDF





the "bell-shaped" curve

Ex normal PDF integrates to 1

$$\left[\int_{-\infty}^{\infty} f_X(x)dx\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}dx \int_{-\infty}^{\infty} e^{-y^2/2}dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)/2}\,dxdy$$

Cartesian

$COV: x = r\cos\theta$

$COV: y = r\sin\theta$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2/2}\,rdrd\theta$$

polar

$$= \int_{0}^{\infty} e^{-r^2/2}\,rdr = e^{-r^2/2}\Big|_{0}^{\infty} = 1$$

Gaussian (normal) r.v. - CDF

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(s-m)^2 / 2\sigma^2} \, ds$$

$\Downarrow$ $COV : t = \frac{s-m}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} \, dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right) \text{ where } \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt$$

$\Uparrow$

CDF for $N(0,1)$ r.v.

"standard normal"

Q-function

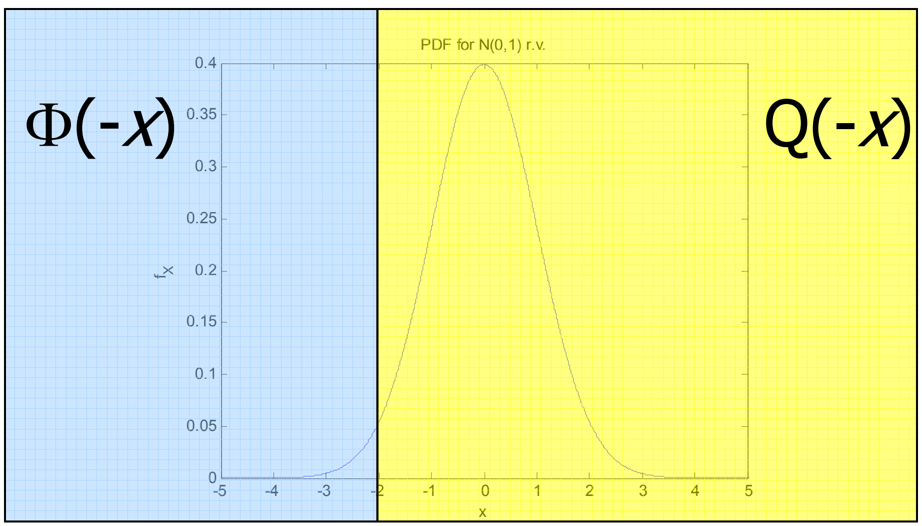
used by EE as error probability

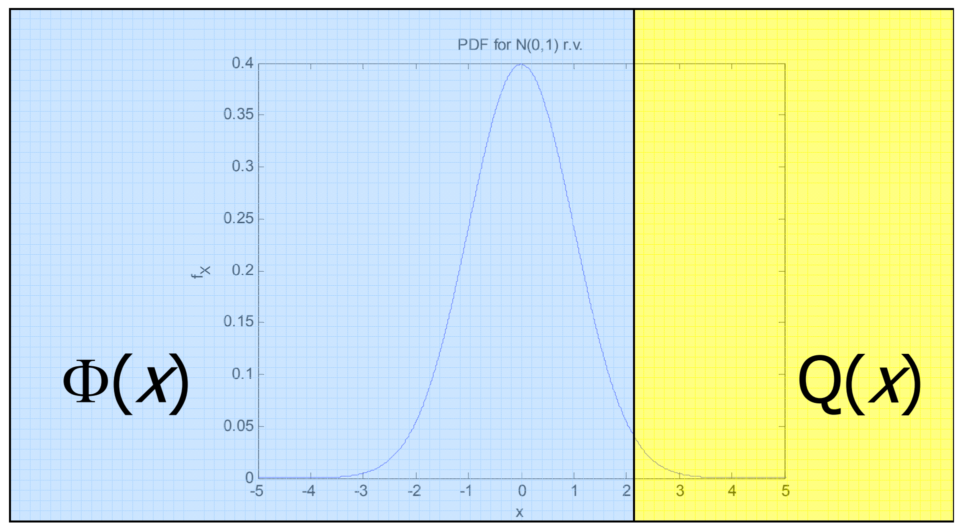
$$Q(x) \triangleq 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

probability of the "tail"

$$Q(0) = 0.5$$

$$Q(-x) = 1 - \Phi(-x) = 1 - Q(x)$$





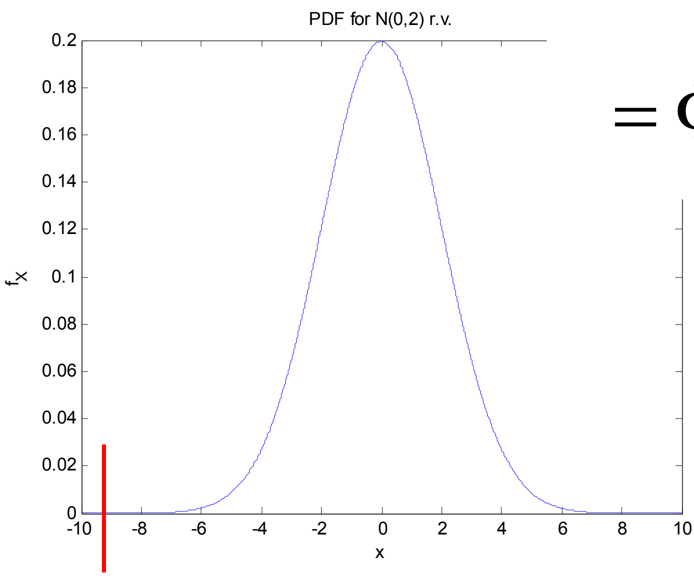
Gaussian (normal) r.v.

- Plays an important role in communication systems, where transmission of signals is subject to noise
  - *Noise resulting from the thermal motion of electrons, can – from physical principles – be shown to have a Gaussian PDF*

Ex

- A communication system accepts a positive voltage *V* as input and outputs a voltage $Y=\alpha V+N$, where $\alpha=10^{-2}$ and *N* is ~$N(0,2)$. Find $V \ni P[Y<0]=10^{-6}$

$$P[Y<0]=P[\alpha V+N<0]=P[N<-\alpha V]$$



$$=\Phi\left(\frac{-\alpha V}{\sigma}\right)=Q\left(\frac{\alpha V}{\sigma}\right)=10^{-6}$$

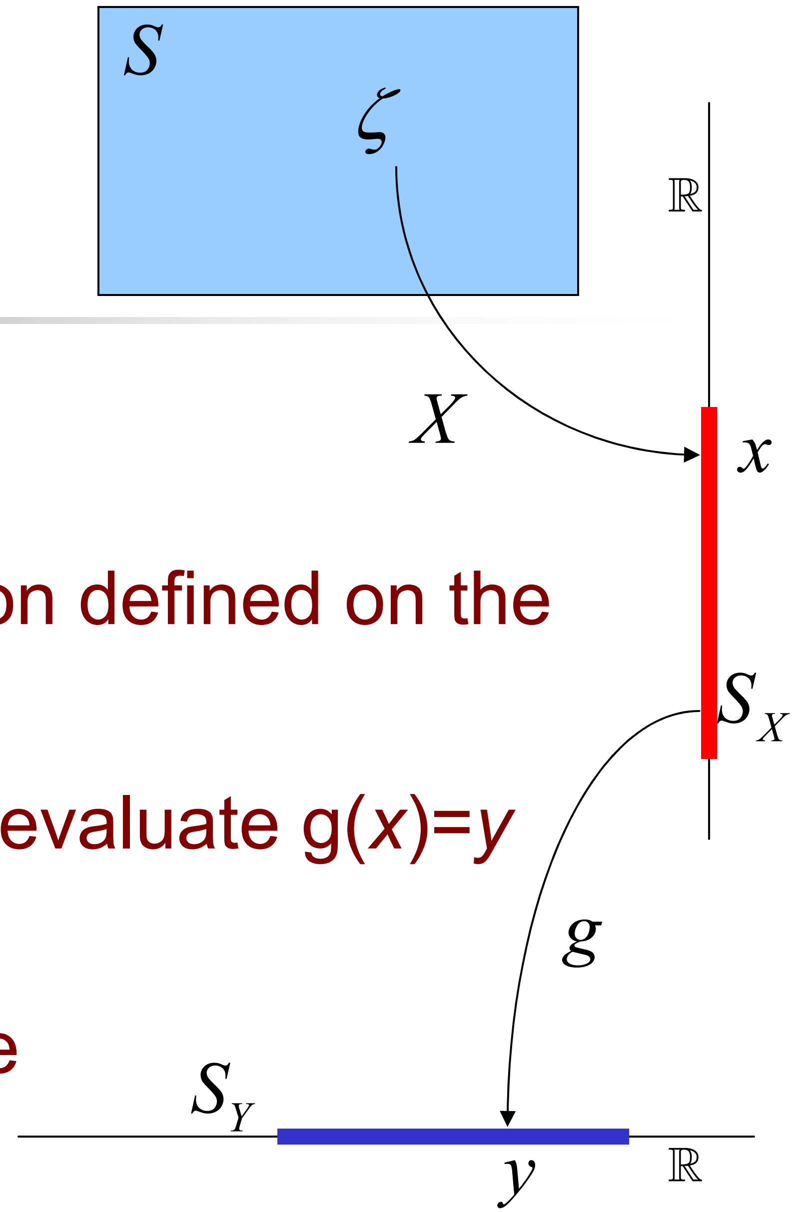
Table

$$\frac{\alpha V}{\sigma}=4.7535$$

$$V=4.7535\frac{\sigma}{\alpha}=950.6$$

Functions of a r.v.

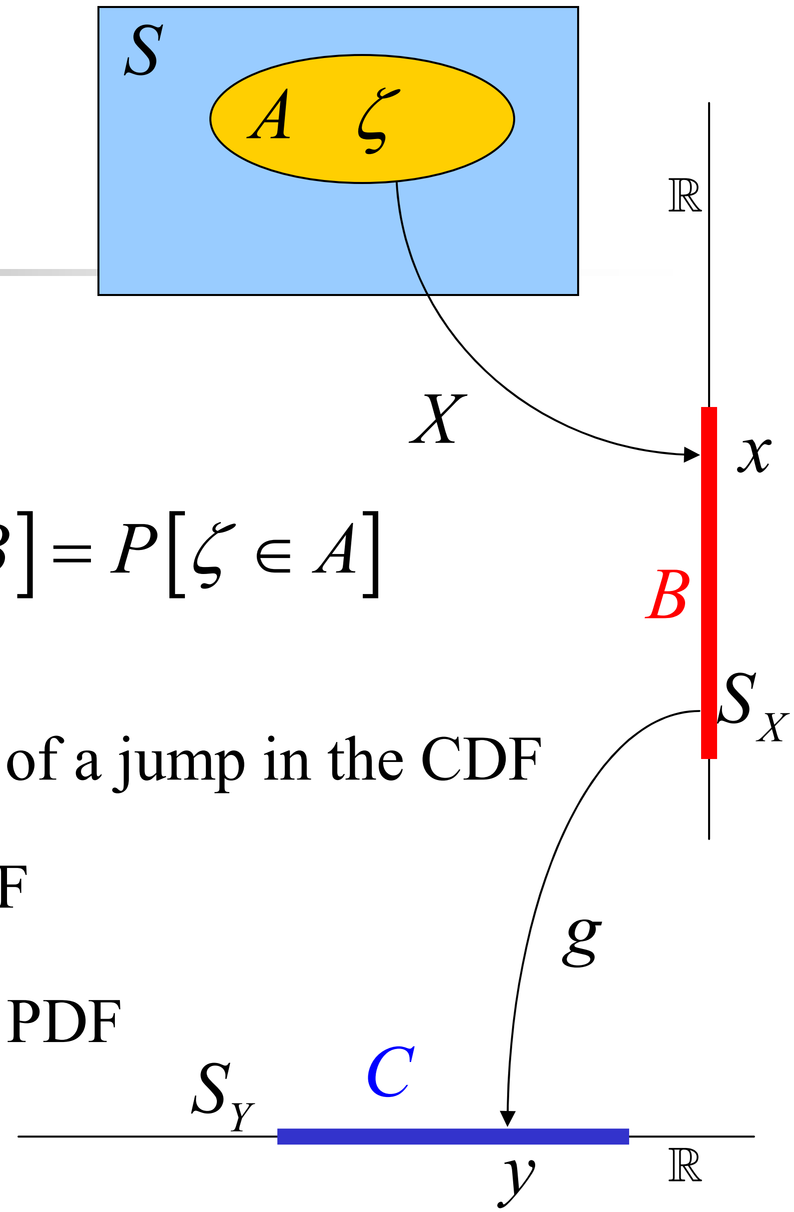
- *X* is a random variable
- g(*x*) is a real-valued function defined on the real line
- *Y*=g(*X*), i.e. for every *X*=*x*, evaluate g(*x*)=*y* and assign it to *Y*
- *Y* is also a random variable
- Find CDF and PDF of *Y*



ultimately probabilities are induced by the underlying experiment

Induced probability

equivalent events



$$P[Y \in C] = P[g(X) \in C] = P[X \in B] = P[\zeta \in A]$$

useful events:

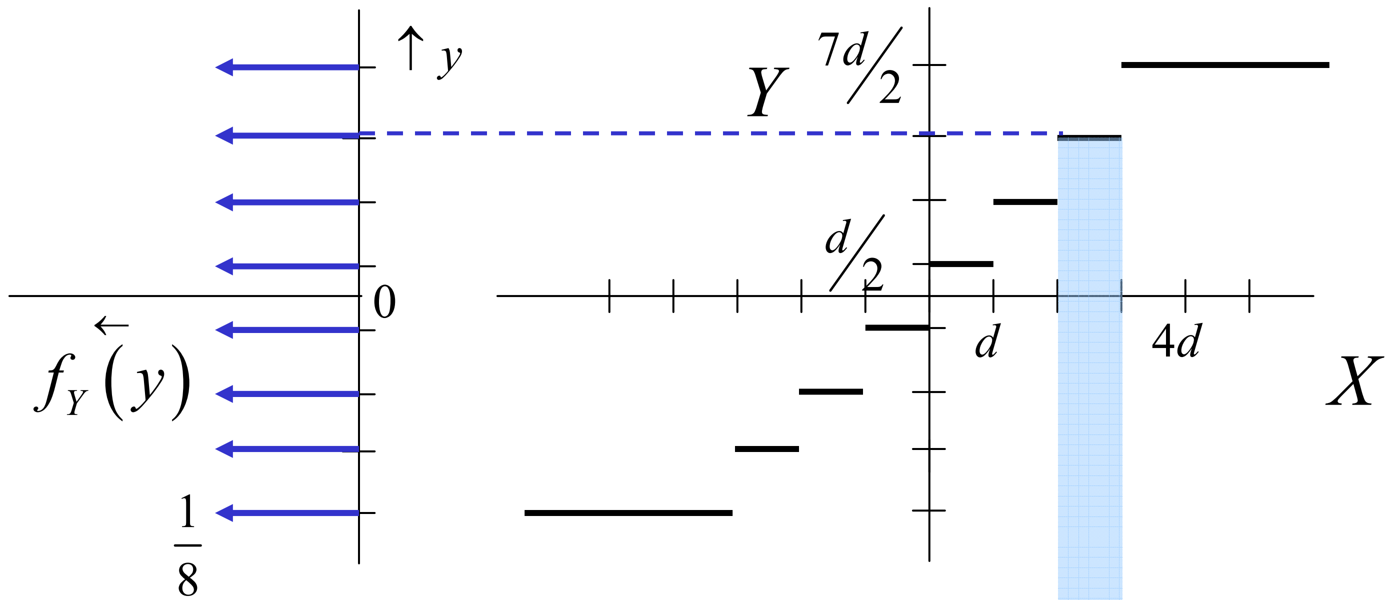
$\{g(X) = y_k\}$ is used to find the magnitude of a jump in the CDF

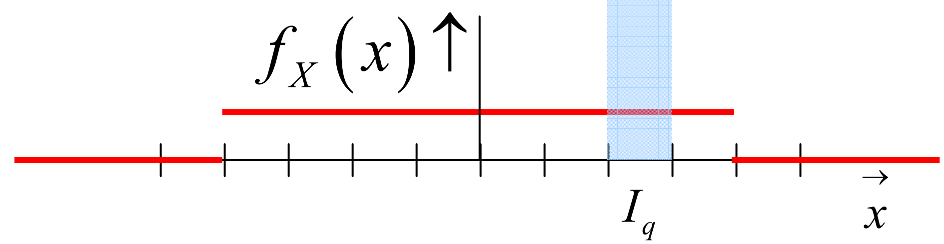
$\{g(X) \le y\}$ is used to directly find the CDF

$\{y < g(X) \le y + h\}$ is useful in finding the PDF

Ex 3.22 8-level uniform quantizer

- Let *X* be a sample voltage of a speech waveform; assume *X* is uniform over [-4d,4d]

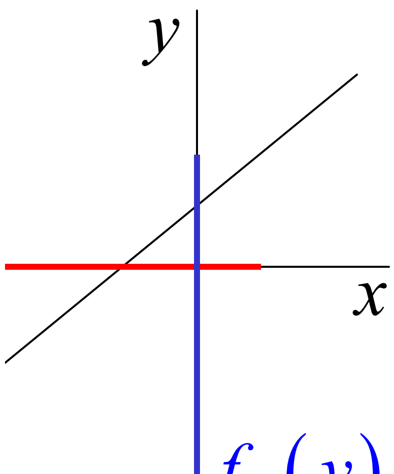




$$P[Y = q] = \int_{I_q} f_X(t)\,dt$$

Ex 3.23 a linear function $Y = aX + b \quad a \neq 0$

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = P[aX \le y - b]$$

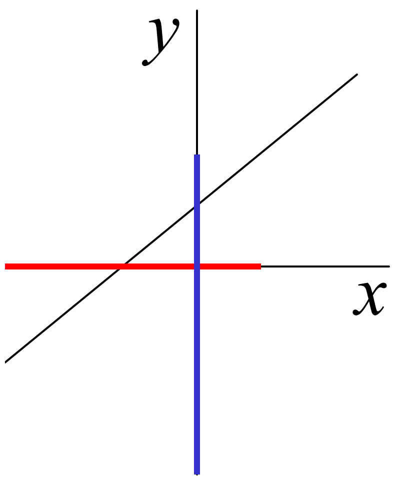


$$= \begin{cases} P\left[X \le \dfrac{y-b}{a}\right] & a > 0 \\ P\left[X \ge \dfrac{y-b}{a}\right] & a < 0 \end{cases} = \begin{cases} F_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ -\dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Ex 3.24 linear function of Gaussian r.v.



$$Y = aX + b \quad a \neq 0$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

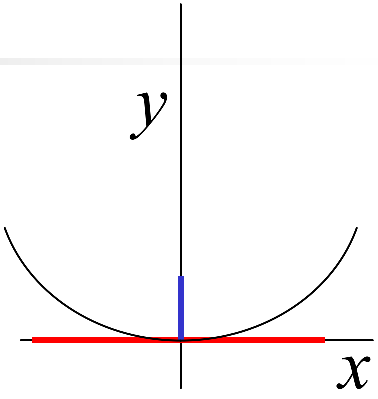
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{|a\sigma|\sqrt{2\pi}} e^{-(y-b-am)^2 / 2(a\sigma)^2}$$

linear function of a Gaussian r.v. is also a Gaussian r.v.

Ex 3.25 square law device

$$Y = X^2$$



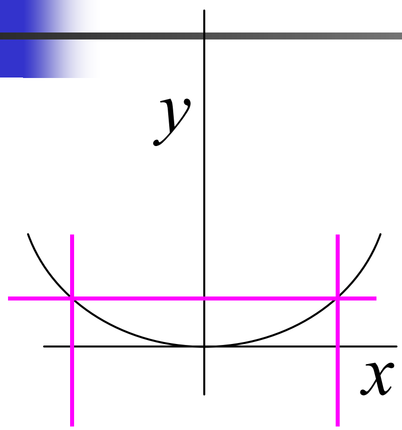
$$F_Y(y) = P[Y \le y] = P\left[X^2 \le y\right]$$

$$= P\left[-\sqrt{y} \le X \le \sqrt{y}\right] = \left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right] u(y)$$

$$f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right] u(y)$$

$$= \left[\frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}}\right] u(y)$$

from Ex 3.26



$$y_0 = g(x) \leftarrow x_0, x_1$$

produces 2 terms in PDF

$$f_Y(y) = \frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}}$$

Redo Ex 3.27

for $y < 0$: $\quad y = x^2$ has no solutions $\Rightarrow f_Y(y) = 0$

for $y \geq 0$: $\quad y = x^2$ has two solutions: $x_0 = \sqrt{y}$; $x_1 = -\sqrt{y}$

$dy/dx = 2x$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| dy/dx \right|} \right]_{x=x_k} = \left[\frac{f_X(x)}{|2x|} \right]_{x=x_0} + \left[\frac{f_X(x)}{|2x|} \right]_{x=x_1}$$

$$= \left[\frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}} \right] u(y)$$

Nonlinear function $Y = g(X)$

$$P\left[C_y\right] = P\left[B_y\right]$$

equivalent events ⇓ induce equal probabilities

$$f_Y(y)\left|dy\right| = f_X(x_1)\left|dx_1\right| + f_X(x_2)\left|dx_2\right| + f_X(x_3)\left|dx_3\right|$$

⇓

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| dy \big/ dx \right|} \right]_{x=x_k} = \sum_k \left[f_X(x) \left| \frac{dx}{dy} \right| \right]_{x=x_k}$$

function of y

Ex 3.28

$Y = \cos(X)$

$X \sim U(0, 2\pi]$

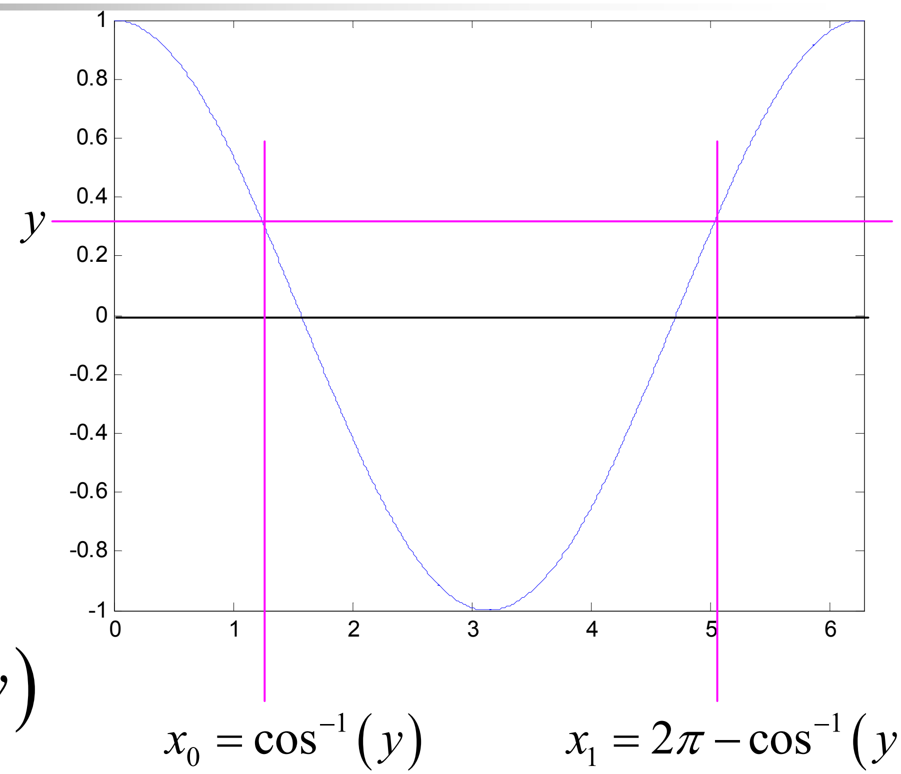
$\Big\downarrow \; Y = \cos(X)$

for $y < -1$ or $y > 1$: no sol^s

$$f_Y(y) = 0$$

for $-1 \le y \le 1$:

$$x_0 = \cos^{-1}(y);\; x_1 = 2\pi - \cos^{-1}(y)$$



$$\left.\frac{dy}{dx}\right|_{x=x_0} = -\sin(x_0) = -\sin\{\cos^{-1}(y)\} = -\sqrt{1-y^2}$$

$$Y = \cos(X)$$

$$X \sim U(0, 2\pi] \rightarrow f_X(x) = \frac{1}{2\pi}\left[u(x) - u(x - 2\pi)\right]$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left|dy/dx\right|}\right]_{x=x_k} = \frac{1}{2\pi\left|-\sqrt{1-y^2}\right|} + \frac{1}{2\pi\left|\sqrt{1-y^2}\right|} = \frac{1}{\pi\sqrt{1-y^2}}$$

$$-1 \le y \le 1$$

$$\int_{-\infty}^{y}$$

$$F_Y(y) = \frac{1}{2} + \frac{\sin^{-1}(y)}{\pi} \text{ for } -1 \le y \le 1$$

Y has the arcsine distribution

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$