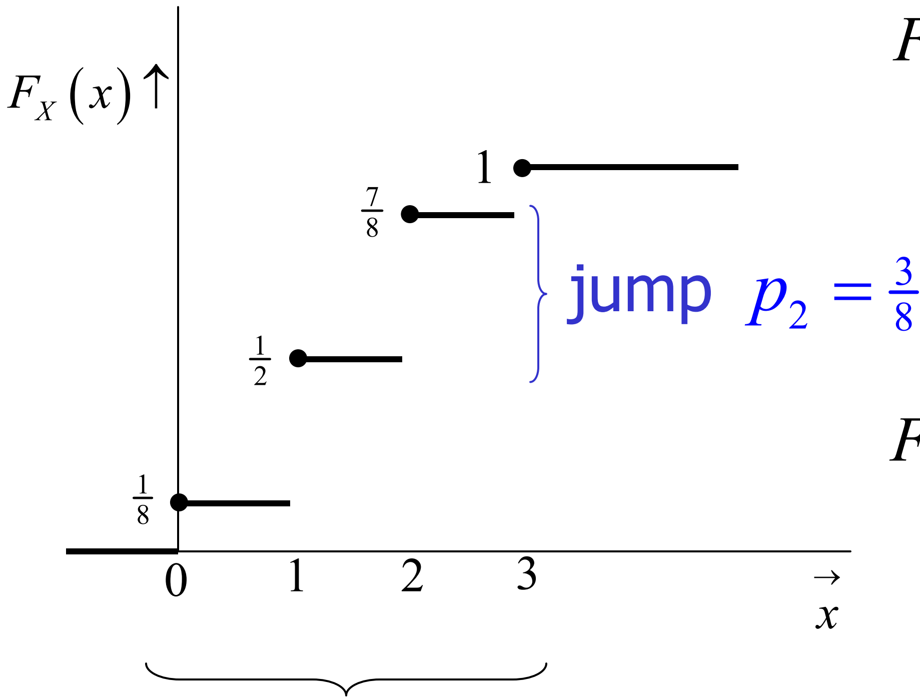
Ex CDF # heads in three coin tosses



$$F_X(2-\delta) = P[X \le 2-\delta]$$
$$= P[\{0 \; or \; 1 \; heads\}]$$
$$= p_0 + p_1$$

$$\Delta$$

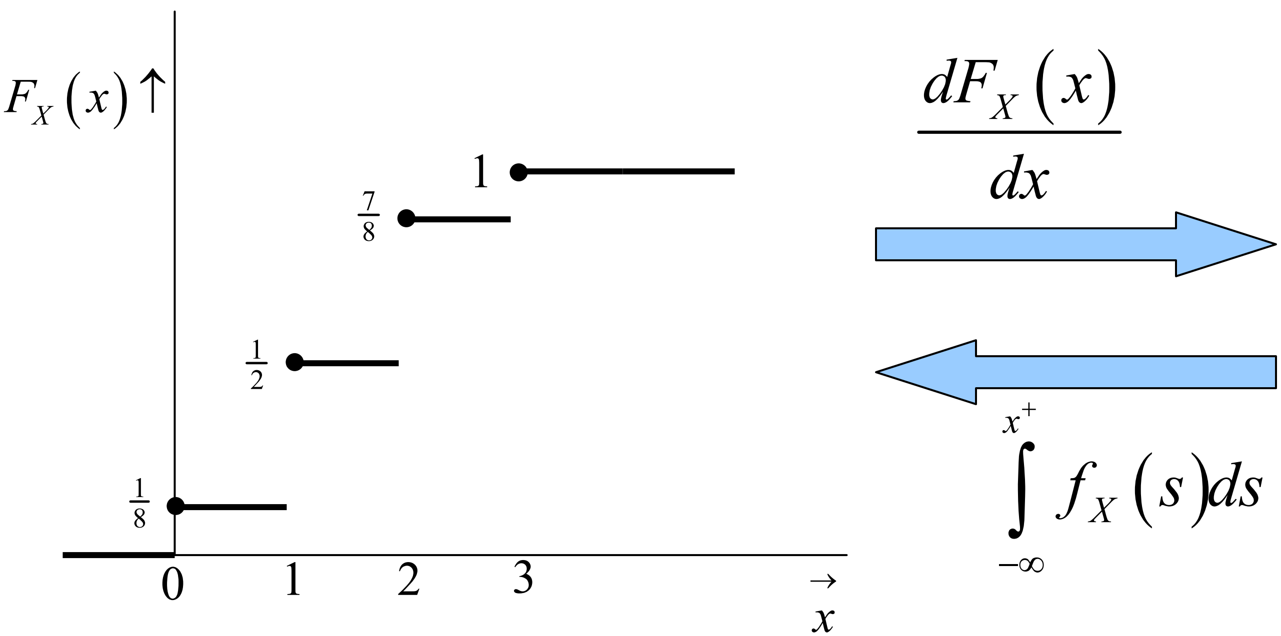
$$F_X(2+\delta) = P[X \le 2+\delta]$$
$$= P[\{0 \; or \; 1 \; or \; 2 \; heads\}]$$
$$= p_0 + p_1 + p_2$$

$$S_X = \{0,1,2,3\}$$

$$p_i = \left\{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right\} \Longrightarrow F_X(x) = \sum_{i:x_i \le x} p_i = \sum_{i=0}^{3} p_i u(x - x_i)$$

Derivative of CDF?

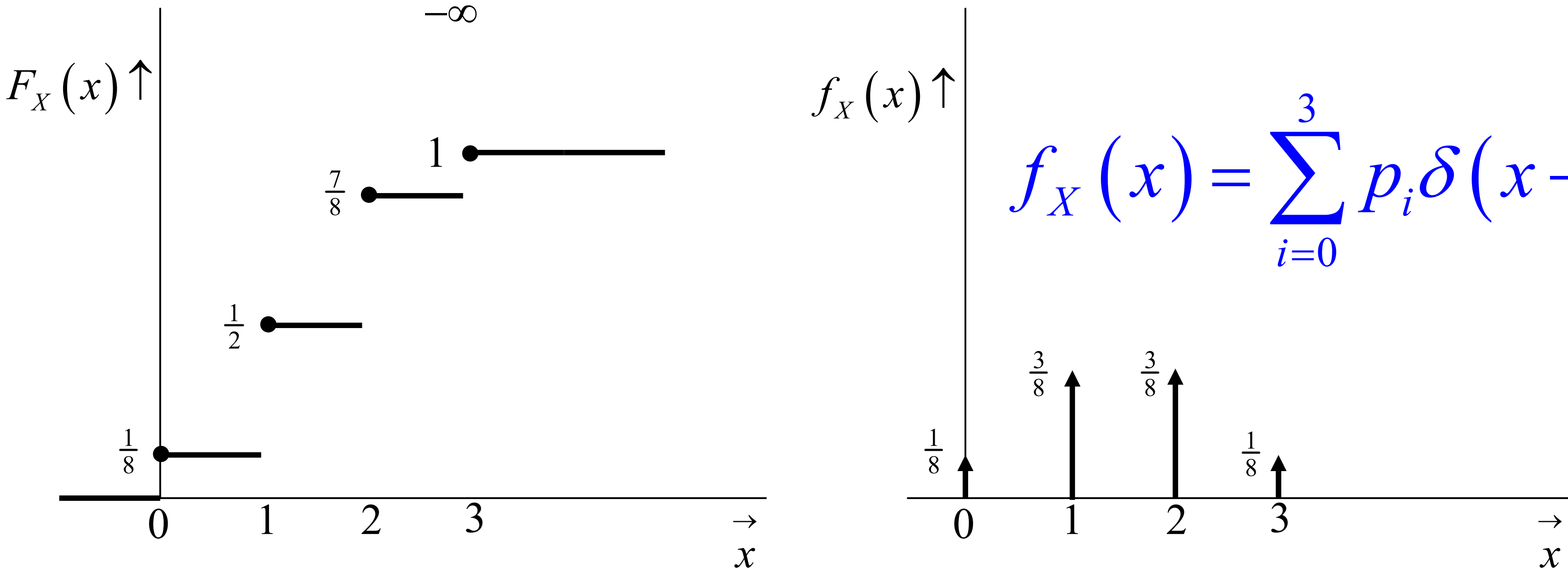
Find $f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X(x^+) - F_X(-\infty) = F_X(x)$



$f_X(x)?$

Derivative of CDF?

$$f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X(x^+) - F_X(-\infty) = F_X(x)$$



$$f_X(x) = \sum_{i=0}^{3} p_i \delta(x - x_i)$$

$$\int_{-\infty}^{x^+} \delta(s-a)\,ds = u(x-a)$$

Ex 3.5 message transmission time

- The transmission time X of messages in a communication system obeys the exponential probability law with parameter λ

$$P[X > x] = e^{-\lambda x} \quad x > 0$$

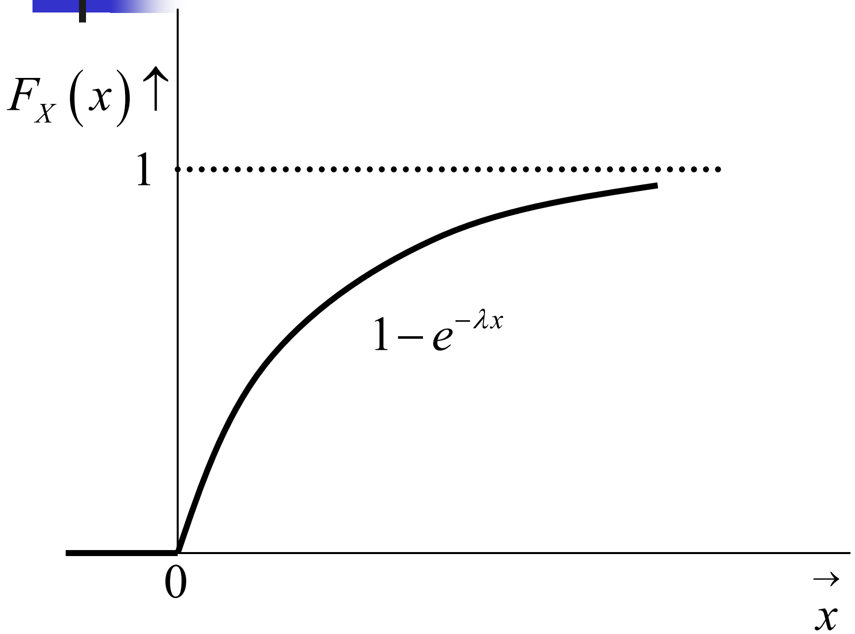
Find cdf $F_X(\mathrm{x})$. Find $P[T < X < 2T]$, where $T = \lambda^{-1}$.

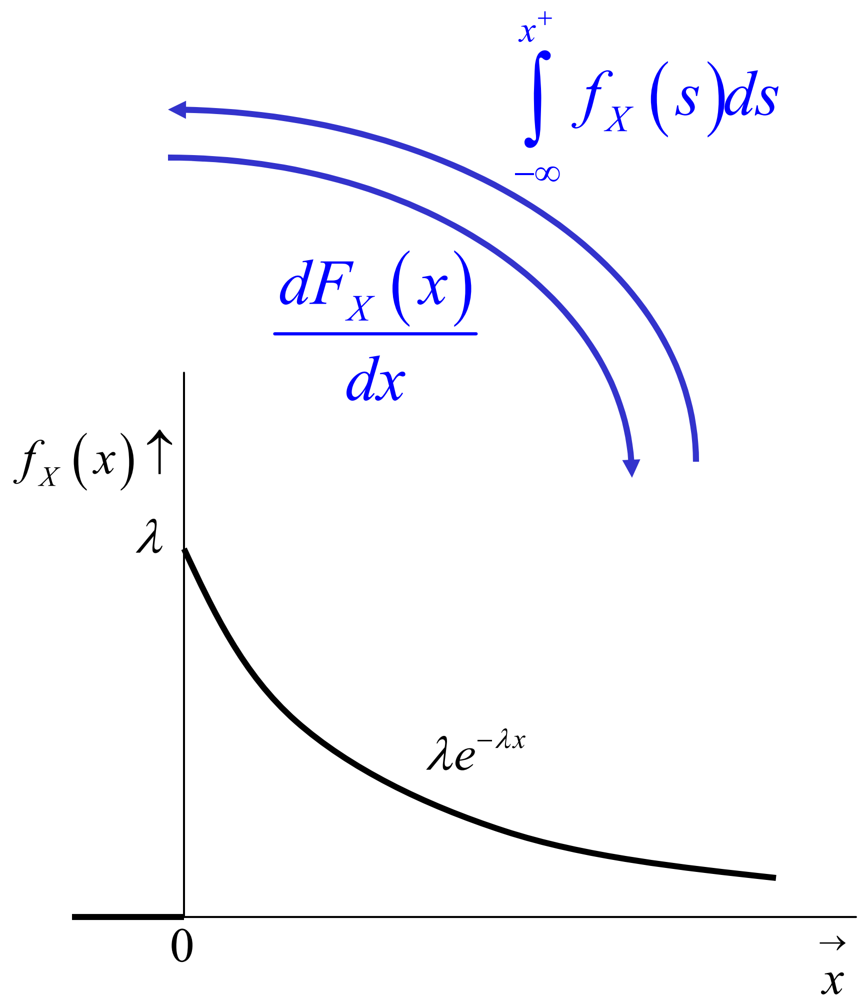
$$F_X(x) = P[X \le x] = 1 - P[X > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$P[T < X \le 2T] = F_X(2T) - F_X(T)$$

$$= 1 - e^{-\lambda 2T} - \left(1 - e^{-\lambda T}\right) = -e^{-2} + e^{-1}$$

Ex 3.5 message transmission time





continuous for all *x*

⇩

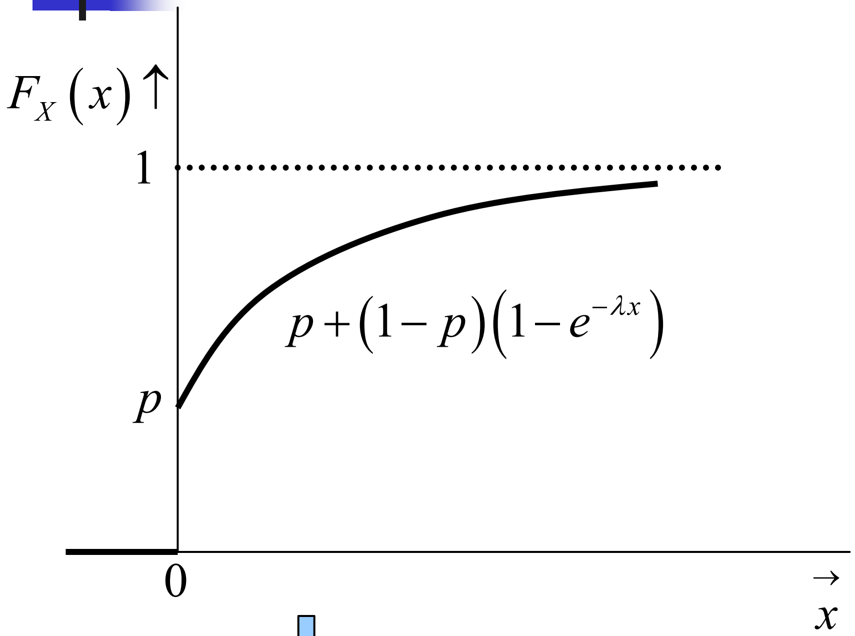
derivative exists everywhere except for *x*=0

Ex 3.6 waiting time in queue

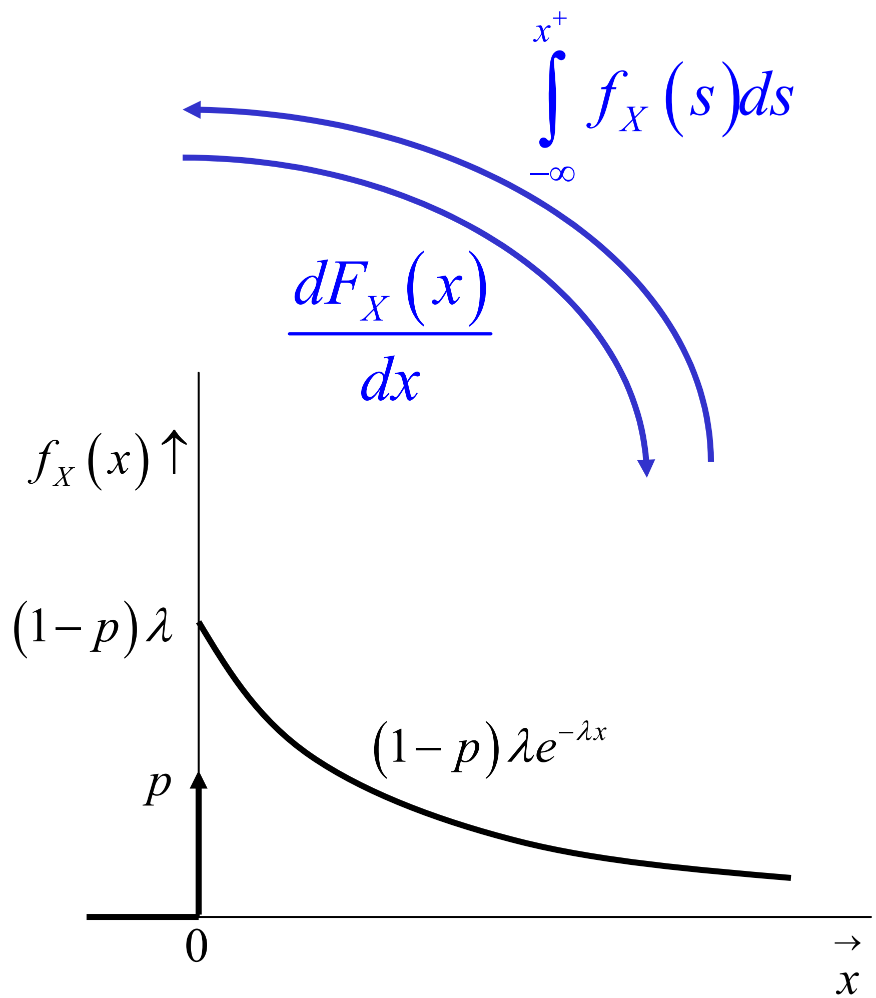
- Customer waiting time *X* is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy; the probability for finding the system idle is *p.*

$$
\begin{aligned}
F_X(x) &= P[X \le x] \\
&= P[X \le x \mid idle]\,p + P[X \le x \mid busy](1-p) \\
&= u(x)\,p + \left(1-e^{-\lambda x}\right)(1-p)\,u(x)
\end{aligned}
$$

Ex 3.6 waiting time in queue



sum of step function
and continuous function



Types of random variables

- Discrete
  - *CDF is right-continuous, staircase function of x, with jumps at a countable set of points* x_k
  - *Probability mass function* $p_X(x_k)$
  - *CDF can be written as weighted sum of unit step functions*
- Continuous
  - *CDF is continuous everywhere*
  - *CDF can be written as the integral of a non-negative function (its derivative)*
  - $P[X=x]=0$ *for all x*

Types of random variables

- Mixed
  - *CDF that has jumps on a countable set of points x_0, x_1, x_2,… and also increases continuously over at least one interval of values of x.*
  - $F_X(x)=pF_D(x)+(1-p)F_C(x)$ with $0<p<1$
  - *Can be viewed as being produced by a two-step process*
    - An unfair coin is tossed: if Heads, a discrete r.v. is generated according to $F_D(x)$; if Tails, a continuous r.v. is generated according to $F_C(x)$

More PDF properties

ii. $P[a \le X \le b] = \int_{a^-}^{b^+} f_X(x)\,dx$

iii. $F_X(x) = \int_{-\infty}^{x^+} f_X(t)\,dt$

the PDF completely specifies the behavior of continuous r.v.'s

iv. $\int_{-\infty}^{\infty} f_X(t)\,dt = F_X(\infty) = 1$ unit probability mass

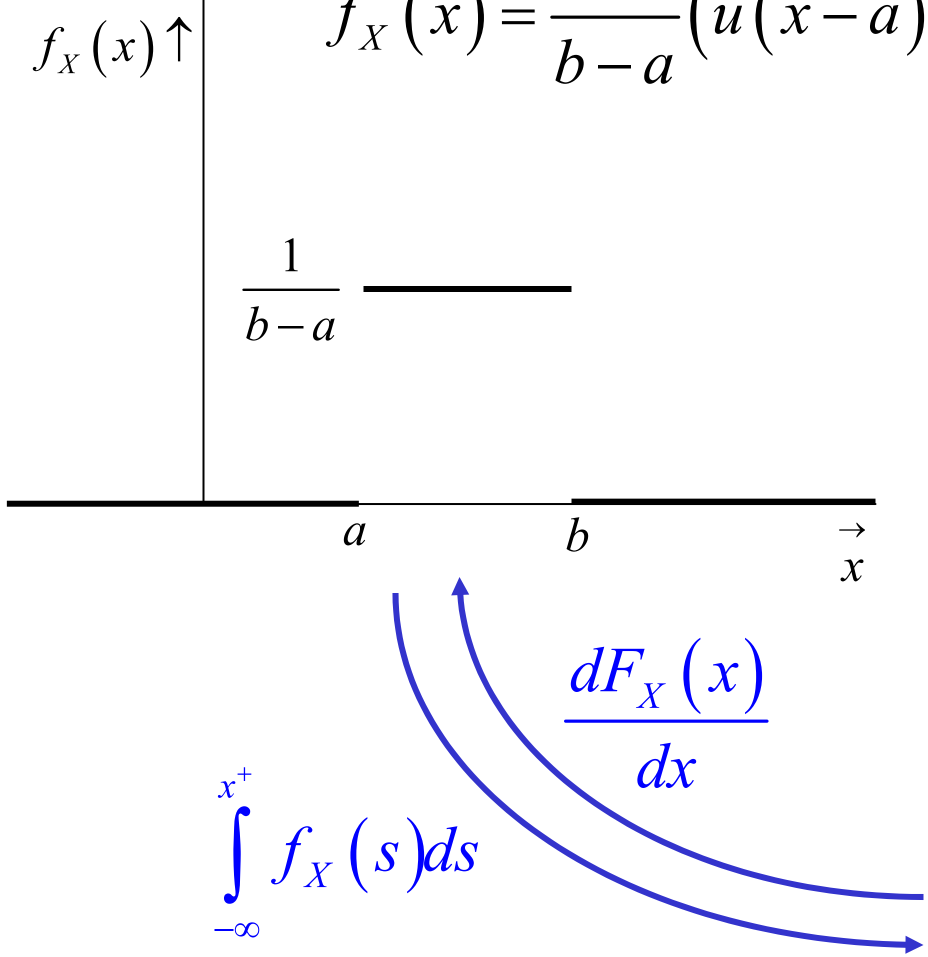
PDF is like "mass" density

a valid PDF can be formed from any nonnegative, piecewise continuous, integrable function

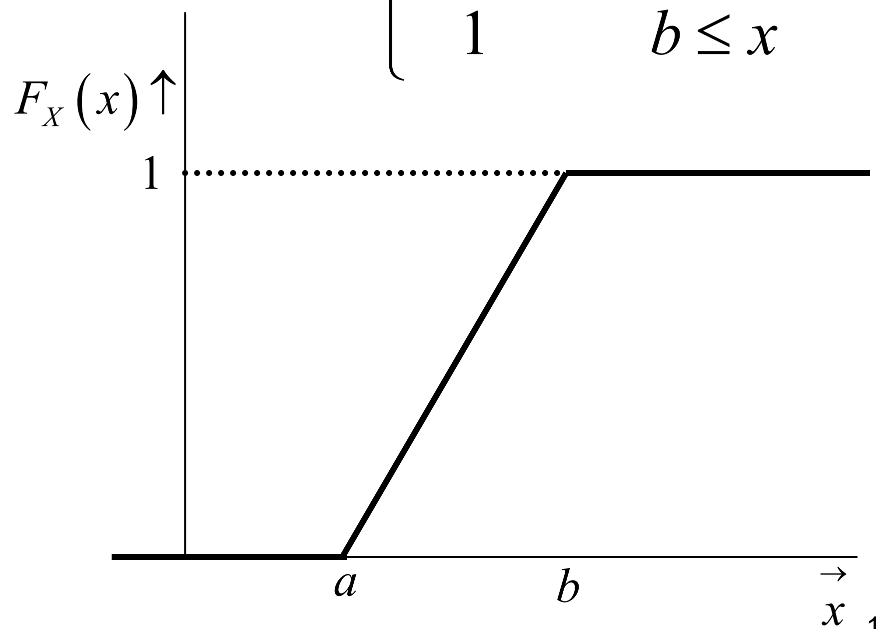
PDF – like CDF – is defined over $(-\infty,\infty)$

Ex 3.7 Uniform random variable

$$f_X(x) = \frac{1}{b-a}\left(u(x-a) - u(x-b)\right)$$



$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases}$$



not magnitude

Ex 3.8 normalization

- PDF of sample values of speech waveforms is found to decay exponentially - at a rate α - for positive and negative values

$$f_X(x) = ce^{-\alpha|x|} \qquad -\infty < x < \infty$$

normalization:

$$\int_{-\infty}^{\infty} ce^{-\alpha|x|}dx = 2\int_{0}^{\infty} ce^{-\alpha|x|}dx = 2c\frac{e^{-\alpha\infty} - e^{-\alpha 0}}{-\alpha} = 1$$

$$\Downarrow$$

$$c = \frac{\alpha}{2}$$

$$P\left[|X| < \upsilon\right] = \frac{\alpha}{2}\int_{-\upsilon}^{\upsilon} e^{-\alpha|x|}dx = \alpha\int_{0}^{\upsilon} e^{-\alpha x}dx = \alpha\frac{e^{-\alpha\upsilon} - e^{-\alpha 0}}{-\alpha} = 1 - e^{-\alpha\upsilon}$$

Conditional CDF's and PDF's

$$F_X(x \mid A) \triangleq \frac{P\left[\{X \le x\} \cap A\right]}{P[A]} \quad \text{if } P[A] > 0$$

$$f_X(x \mid A) \triangleq \frac{d}{dx} F_X(x \mid A)$$

$$f_X(x \mid A) \ni \int_{-\infty}^{x^+} f_X(s \mid A)\, ds = F_X(x \mid A)$$

generalized concept

Ex 3.10 conditional CDF & PDF

- The lifetime *X* of a machine has a continuous CDF. Find the conditional CDF & PDF given the event *A*={*X*>*t*} (machine still working at *t*)

$$F_X(x \mid X>t) = P[X \le x \mid X>t] = \frac{P[\{X \le x\} \cap \{X>t\}]}{P[X>t]}$$

$$= \begin{cases} 0 & x \le t \quad \leftarrow P[\phi] \\ \dfrac{F_X(x)-F_X(t)}{1-F_X(t)} & t<x \end{cases}$$

$\dfrac{d}{dx}$

$$f_X(x \mid X>t) = \begin{cases} 0 & x \le t \\ \dfrac{f_X(x)}{1-F_X(t)} & t<x \end{cases}$$

Bernoulli r.v.

- *A* is an event related to the outcomes of a random experiment

indicator function for *A*: $I_A(\zeta) \triangleq \begin{cases} 0 & \zeta \notin A \\ 1 & \zeta \in A \end{cases}$

assigns # to outcome ⇓ $I_A(\zeta)$ is a r.v.

⇓

$S_X = \{0,1\}$

pmf: $p_I(0) = 1 - p\,; \quad p_I(1) = p = P[A]$

$I_A(\zeta) = 1 \sim \text{"success"} \quad \rightarrow$ Bernoulli r.v.

"tossing of a biased coin": Bernoulli r.v. is a model for this fundamental mechanism for generating randomness"

Binomial r.v.

- Random experiment repeated *n* independent times. Let *X* be the number of times event *A* occurs in these *n* trials.

$$X = I_1 + I_2 + \cdots + I_n$$

sum of Bernoulli r.v.'s (indicator functions for *A* in trial *j*)

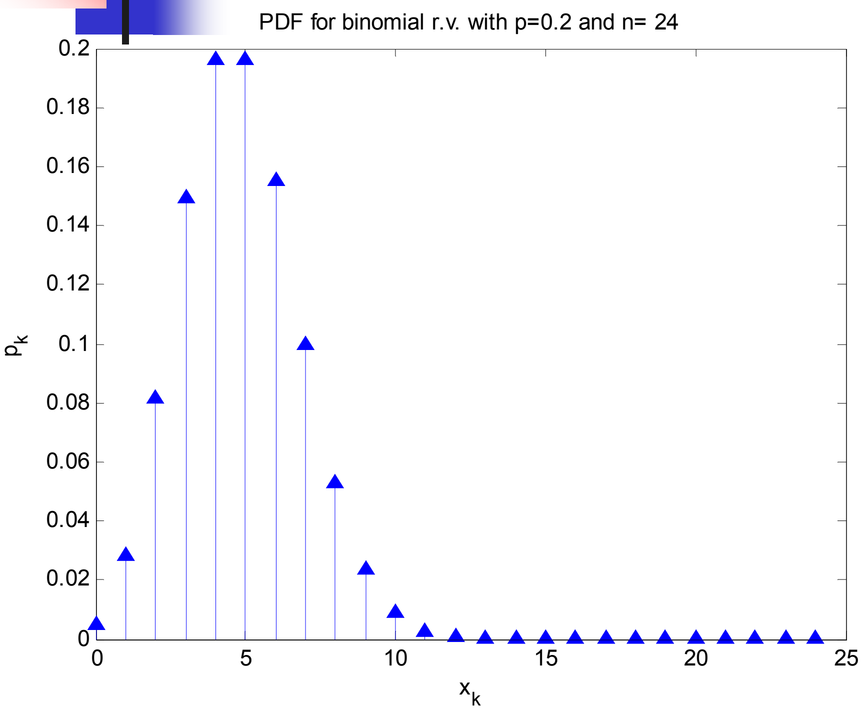
$$S_X = \{0, 1, \cdots, n\}$$

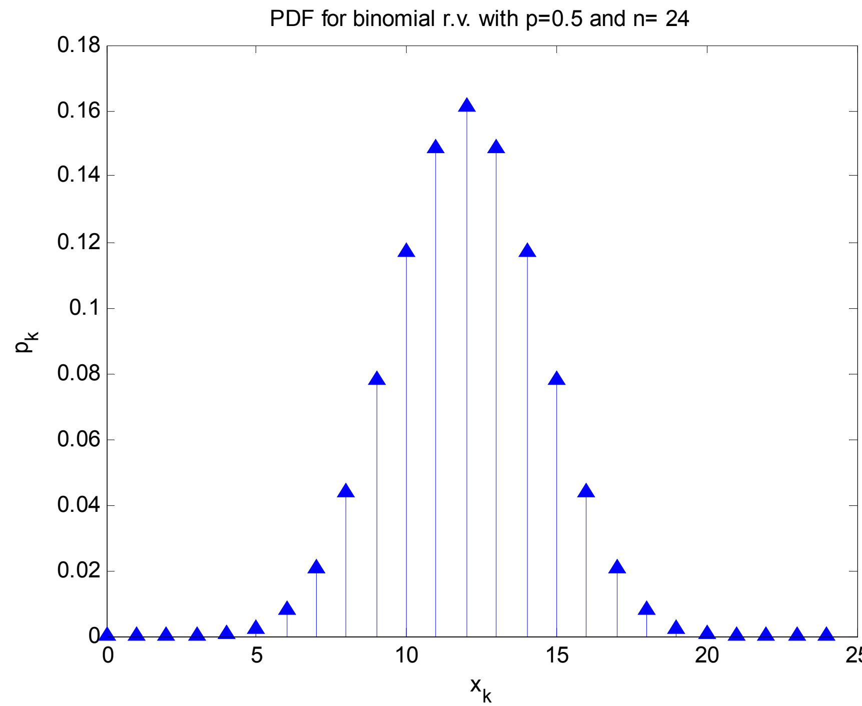
$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \cdots, n$$

Binomial PDF

$$k_{\max} = \arg\max_k P[X = k] = \lfloor (n+1)p \rfloor$$

$$\text{if } \lfloor (n+1)p \rfloor == \lceil (n+1)p \rceil \text{ then also max at } k_{\max-1}$$





Arises in applications where there are two types of object (heads/tails, good/defective, correct/in-error, active/silent), and we're interested in the number of type 1 objects in a randomly selected batch of size *n*, and the type of each object is independent of the types of the other objects in the batch

Geometric r.v.

- # of independent Bernoulli trials until first occurrence of "success"

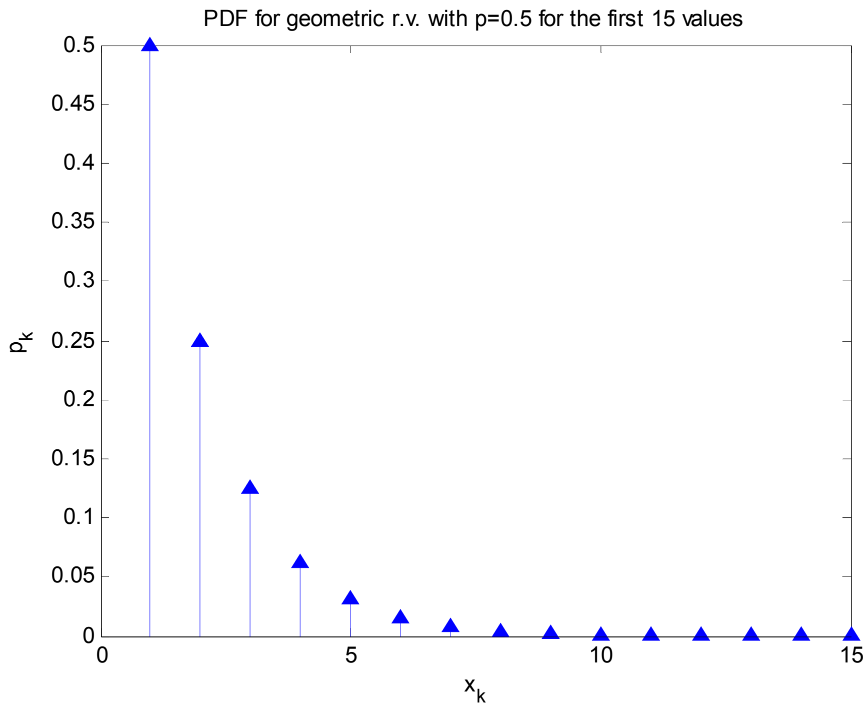
$$S_X = \{1, 2, \cdots\}$$

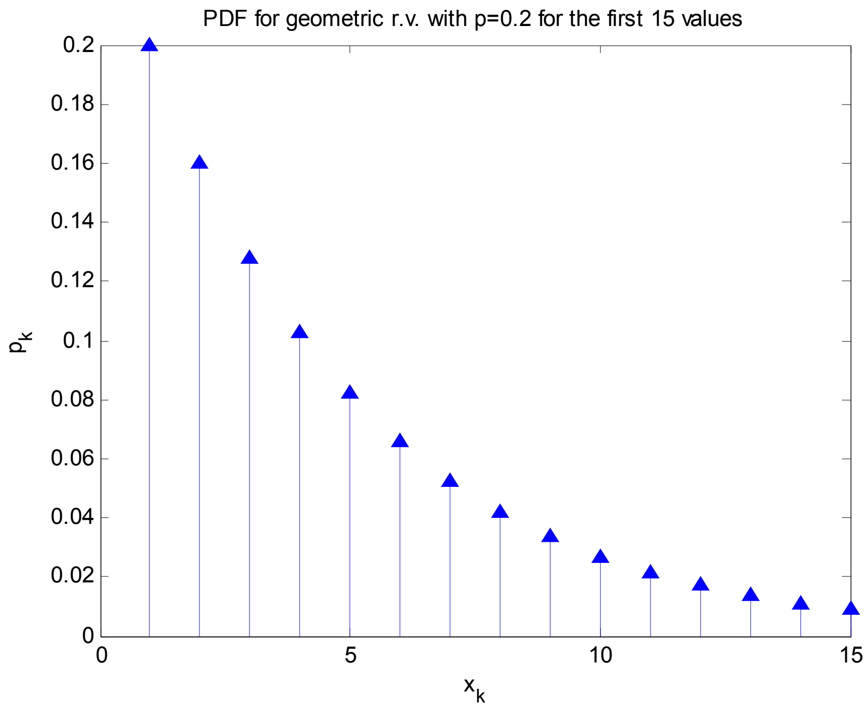
$$p = P[A] = P["success"] \text{ in each Bernoulli trial}$$

$$P[M = k] = (1-p)^{k-1} p \quad \text{for } k = 1, 2, \cdots$$

geometric decay

Geometric PDF

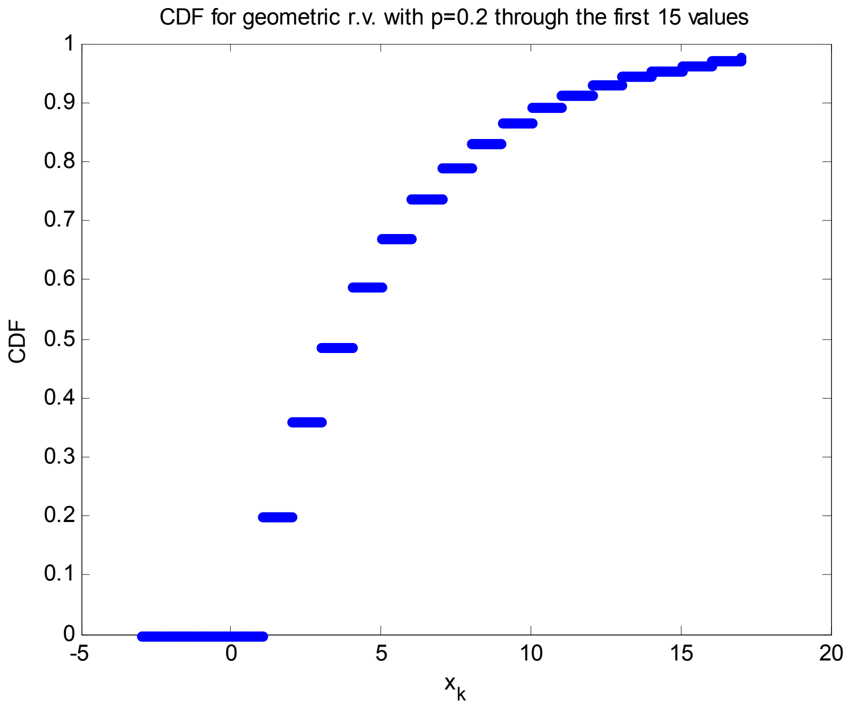




decay like 0.5^k and 0.8^k respectively

Geometric CDF

$$P[M \le k] = \sum_{j=1}^{k} (1-p)^{j-1} p = \frac{1-(1-p)^k}{1-(1-p)} p = 1-(1-p)^k = 1-q^k$$

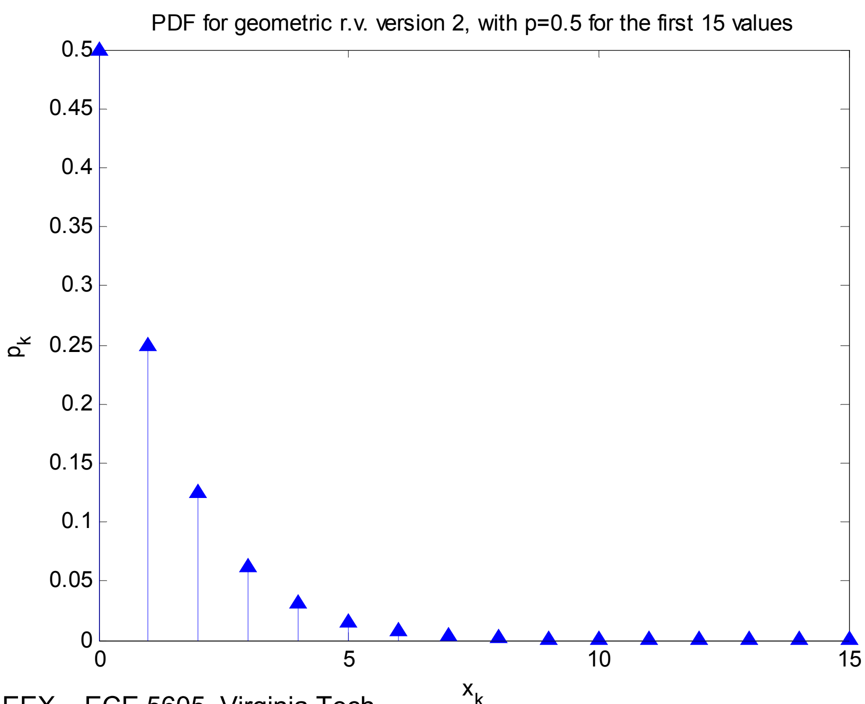


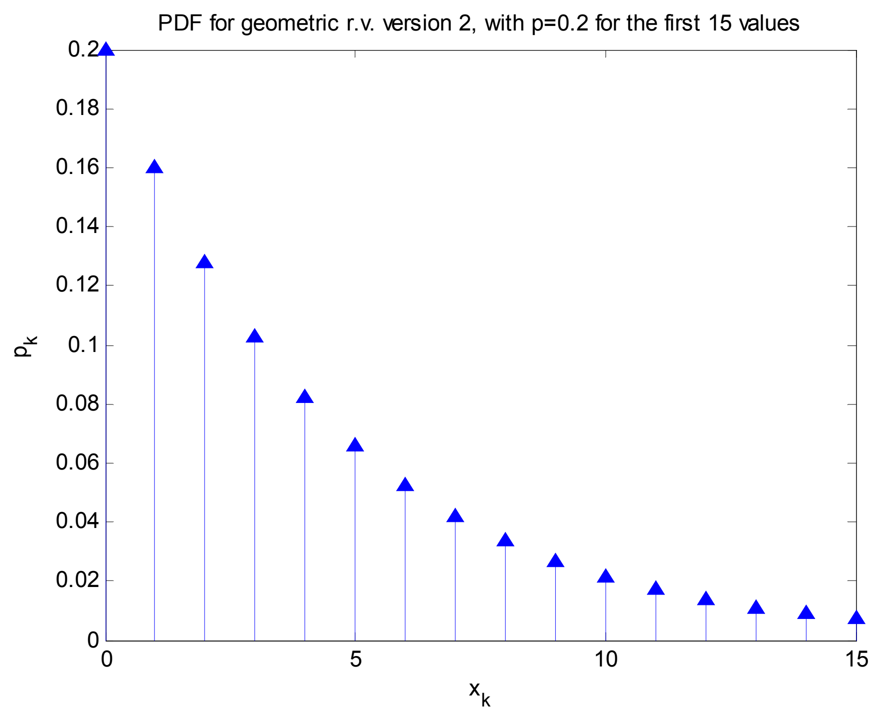
Geometric r.v. – version 2

If our interest is in the # of failures before a success occurs:

$$P[M' = k] = P[M = k+1] = (1-p)^k p \quad \text{for } k = 0, 1, 2, \cdots$$

M' is also a geometric r.v.





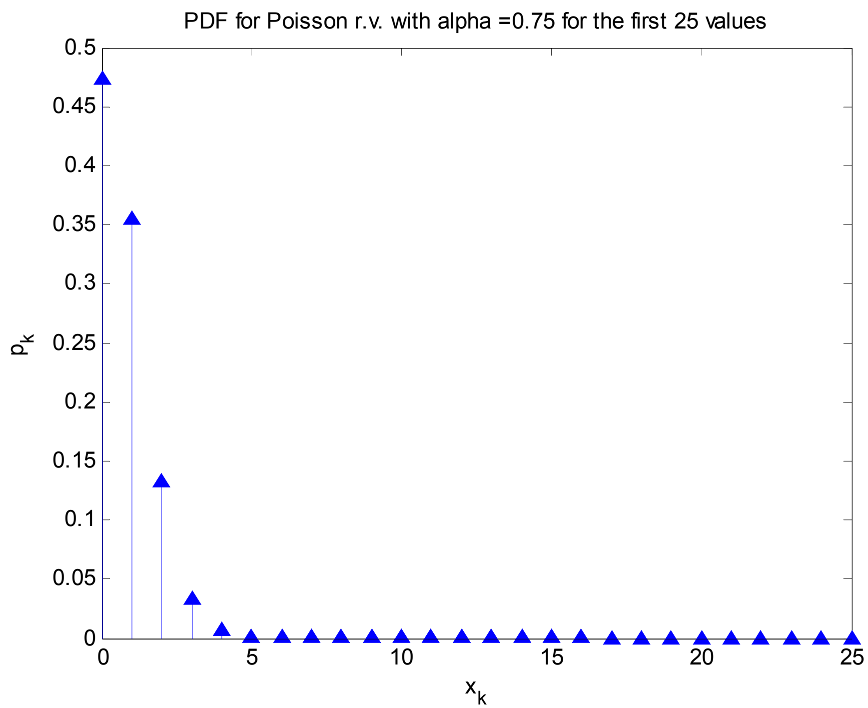
Poisson r.v.

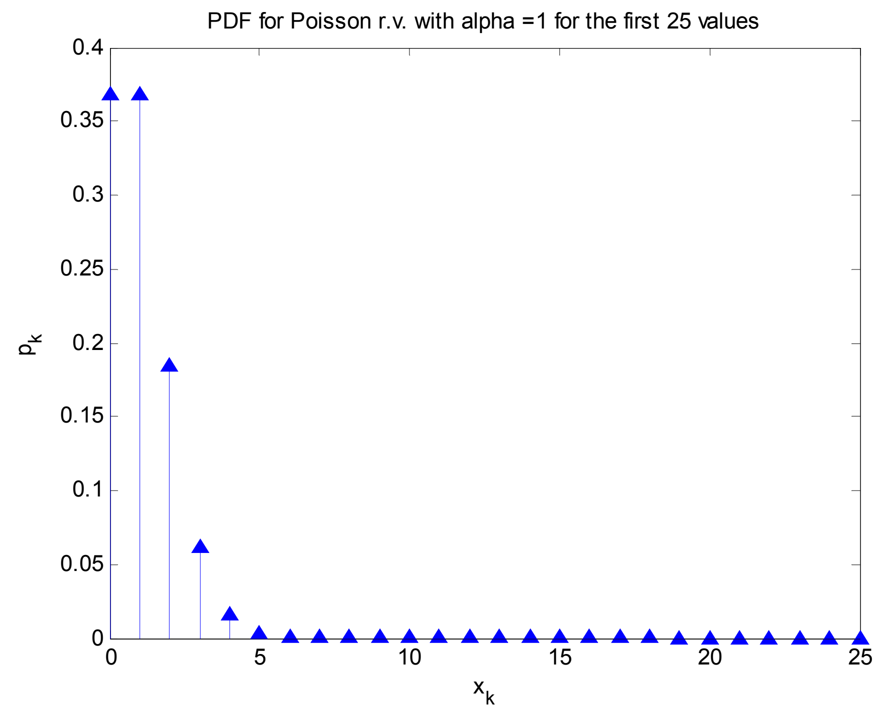
- Interested in counting the # occurrences of an event in a certain time period or in a certain region in space
  - *Events occur completely "at random"*
    - Emissions from radioactive substances
    - Counts of demands for telephone connections
    - Counts of defects in a semiconductor chip

$$P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha} \text{ for } k=0,1,2,\cdots$$

α is the average # of event occurrences in a specified interval or region in space

Poisson PDF



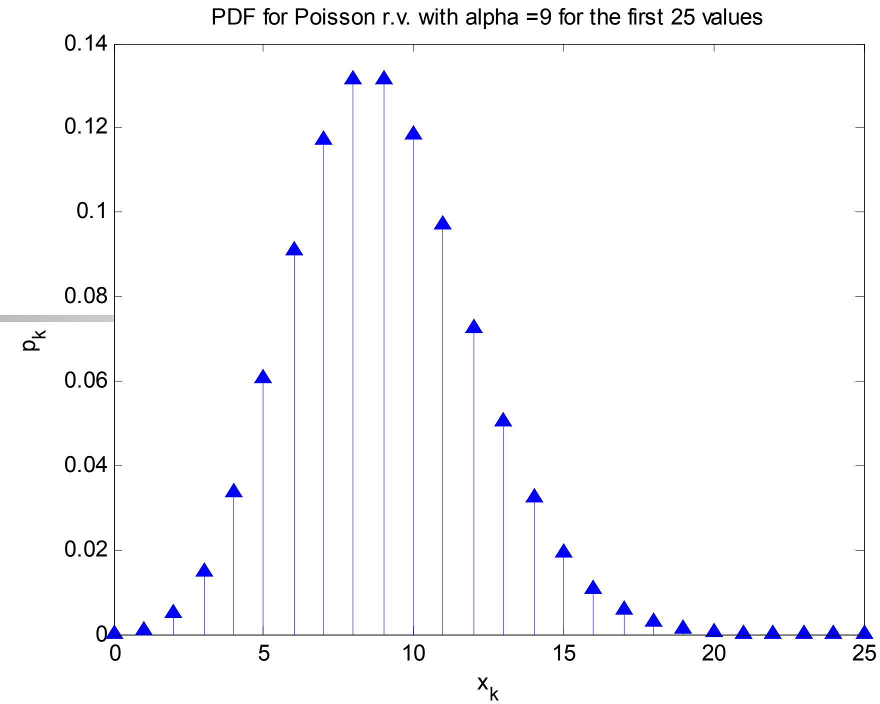


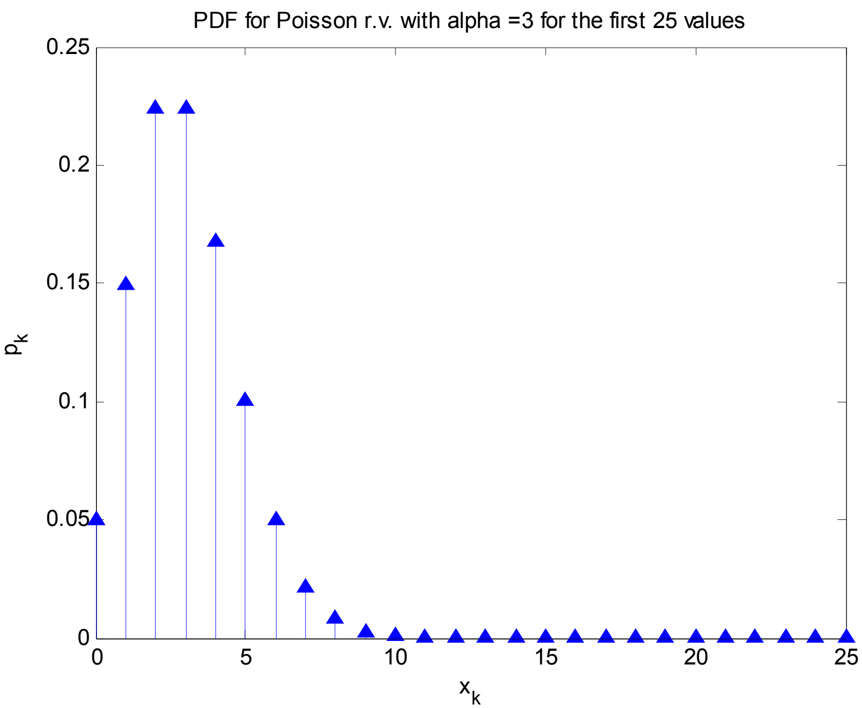
$\max P[N=k]$ is at 0 for $\alpha < 1$

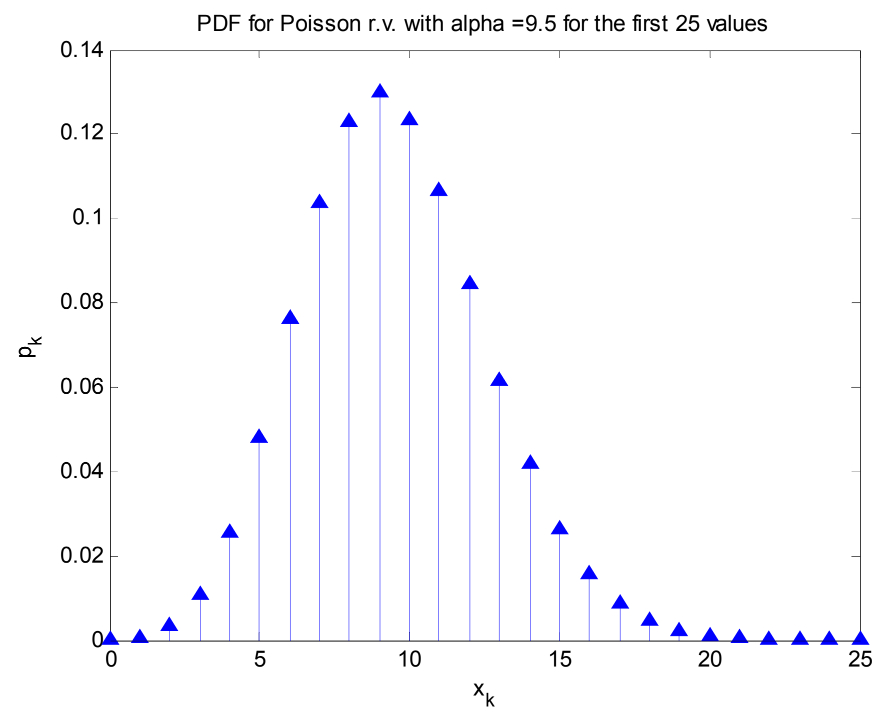
$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$

Poisson PDF

$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$







$\max P[N=k]$ is at $\lfloor \alpha \rfloor$ for $\alpha > 1$

Poisson PDF $P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha}$ for $k=0,1,2,\cdots$

$$\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}e^{-\alpha}=e^{-\alpha}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}=e^{-\alpha}e^{\alpha}=1$$

$P[S]=1$

if n is large and p is small, then for $\alpha = np$

$$p_k=\binom{n}{k}p^k(1-p)^{n-k}\simeq\frac{\alpha^k}{k!}e^{-\alpha}\quad k=0,1,\cdots$$

Poisson PMF is the limiting form of the binomial PMF when the number of Bernoulli trials is made very large and the probability of success is kept small, so that $\alpha = np$

recall: numerical problems in calculating binomial coefficients

Ex probability of bit error in comm^s

- *P*[bit error]=10^{-3}. *P*[≥5 bit errors in block of 10^3 bits]

Bernoulli trials with "success" corresponding to bit error

$$p_k = \binom{1000}{k} 10^{-3k}\left(1-10^{-3}\right)^{1000-k} \simeq \frac{\alpha^k}{k!}e^{-\alpha} \qquad k = 0,1,\cdots$$

$$\alpha = np = 10^3 10^{-3} = 1$$

$$P[N \geq 5] = 1 - P[N < 5] \simeq 1 - \sum_{k=0}^{4} \frac{\alpha^k}{k!}e^{-\alpha}$$

$$= 1 - e^{-1}\left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right\} = 0.00366$$

Ex

- Requests for telephone connections arrive at a switching office at the rate of λ calls per second. It is known that the number of requests follows a Poisson r.v. What is *P*[no call requests in *t* sec]? What is *P*[≥*n* call requests in *t* sec]?

average # requests in a *t*-sec period is $\alpha=\lambda t$

⇩

$N(t)$, the # requests in *t* sec, is Poisson with $\alpha=\lambda t$

$$P\left[N(t)=0\right]=\frac{(\lambda t)^0}{0!}e^{-\lambda t}=e^{-\lambda t}$$

$$P\left[N(t)\geq n\right]=1-P\left[N(t)<n\right]=1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

Exponential r.v.

- Arises in modeling of the time between occurrence of events, and in modeling lifetime of devices and systems; λ is the rate at which events occur

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$$F_X(x) = \left(1 - e^{-\lambda x}\right) u(x)$$

shown earlier

- For a Poisson r.v., the time between events is an exponentially distributed r.v. with parameter $\lambda = \frac{\alpha}{T}$ events per second

- Binomial $\rightarrow$ Poisson
- Geometric $\rightarrow$ exponential

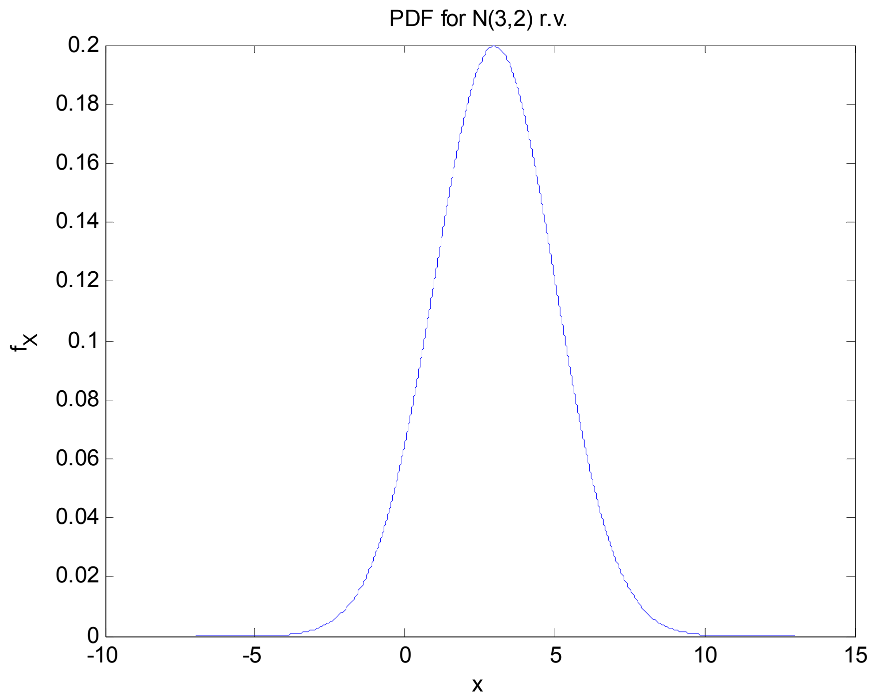
Gaussian (normal) r.v.

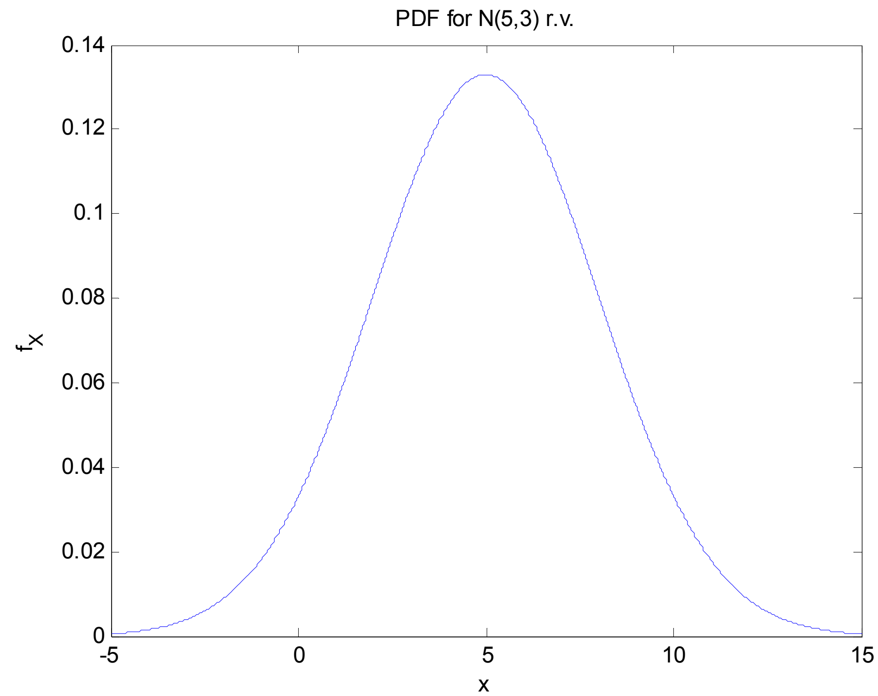
- In many situations in man-made and natural phenomena one deals with a r.v. *X* that consists of a large sum of "small" r.v.'s
  - *Exact PDF becomes complex and unwieldy*
- Under fairly general conditions, as the number of components becomes large (CLT), the CDF approaches that of the normal r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$= N(m, \sigma)$$

Gaussian (normal) r.v. - PDF





the "bell-shaped" curve

Ex normal PDF integrates to 1

$$\left[\int_{-\infty}^{\infty} f_X(x)dx\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}dx \int_{-\infty}^{\infty} e^{-y^2/2}dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dxdy$$ Cartesian

$COV: x = r\cos\theta$

$COV: y = r\sin\theta$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2/2}\,rdrd\theta$$ polar

$$= \int_{0}^{\infty} e^{-r^2/2}\,rdr = e^{-r^2/2}\Big|_{0}^{\infty} = 1$$

Gaussian (normal) r.v. - CDF

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(s-m)^2 / 2\sigma^2} ds$$

$\Downarrow$ $COV: t = \dfrac{s-m}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right) \text{ where } \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

$\Uparrow$

CDF for $\mathcal{N}(0,1)$ r.v.

"standard normal"

Q-function

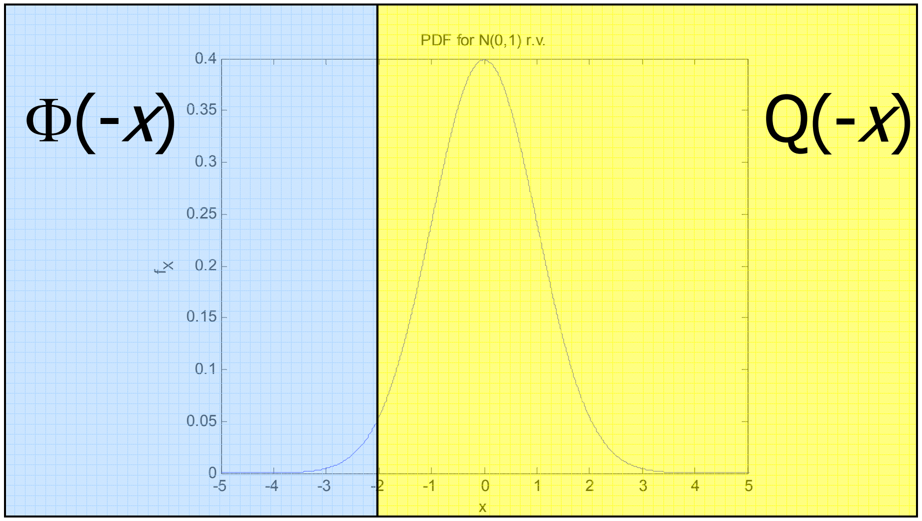
used by EE as error probability

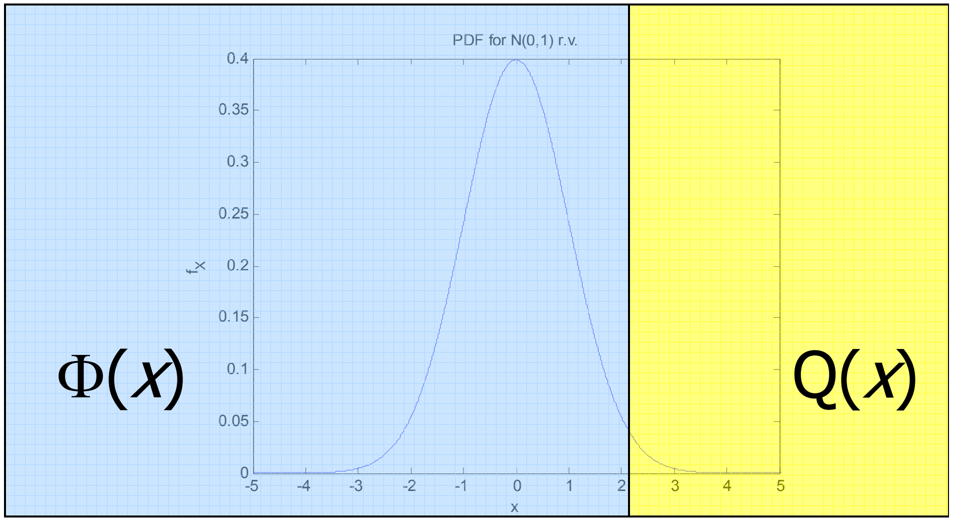
$$Q(x) \triangleq 1-\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt$$

probability of the "tail"

$$Q(0) = 0.5$$

$$Q(-x) = 1-\Phi(-x) = 1-Q(x)$$





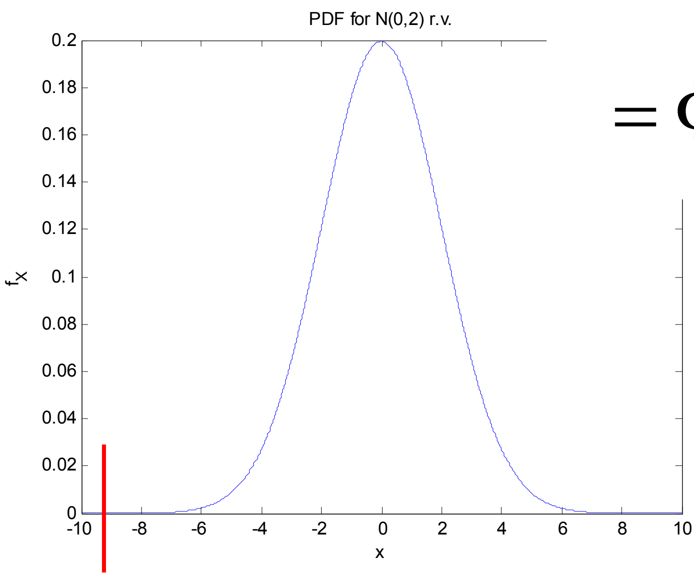
Gaussian (normal) r.v.

- Plays an important role in communication systems, where transmission of signals is subject to noise
  - *Noise resulting from the thermal motion of electrons, can – from physical principles – be shown to have a Gaussian PDF*

Ex

- A communication system accepts a positive voltage *V* as input and outputs a voltage $Y=\alpha V+N$, where $\alpha=10^{-2}$ and *N* is ~$N(0,2)$. Find $V \ni P[Y<0]=10^{-6}$

$$P[Y<0]=P[\alpha V+N<0]=P[N<-\alpha V]$$



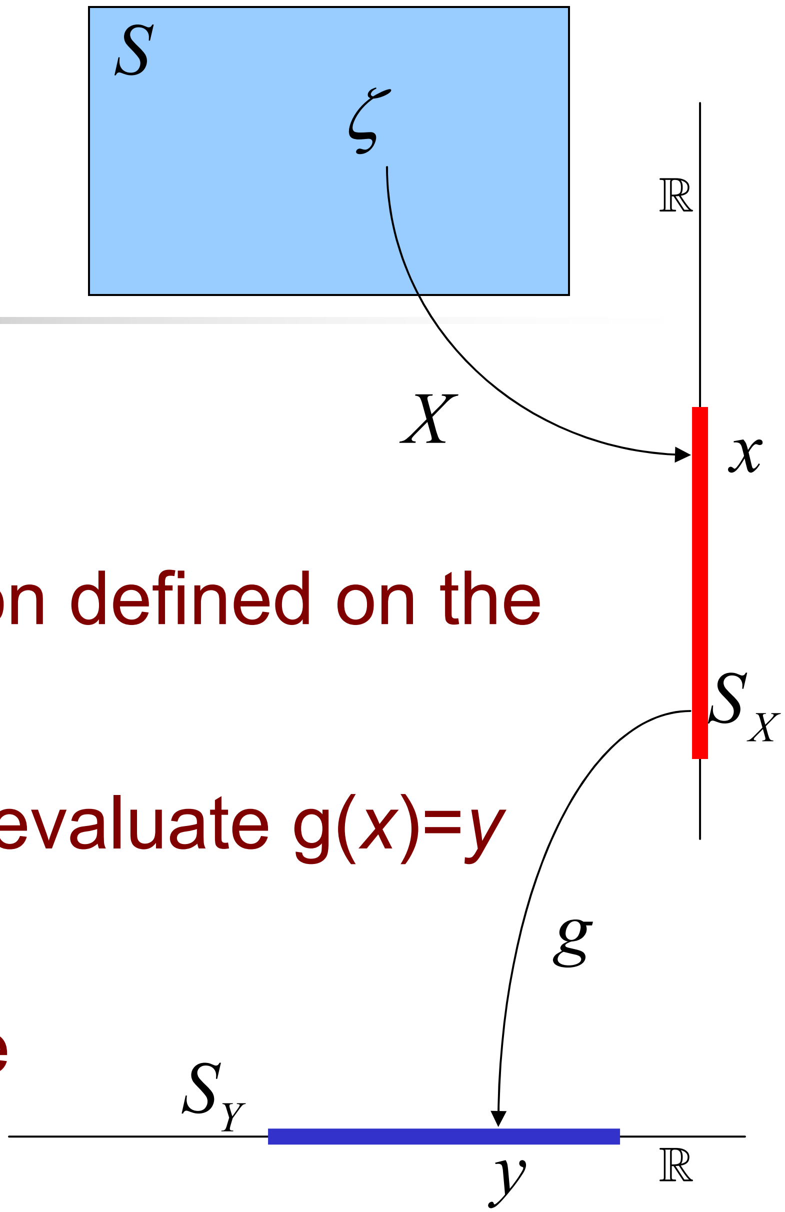
$$=\Phi\left(\frac{-\alpha V}{\sigma}\right)=Q\left(\frac{\alpha V}{\sigma}\right)=10^{-6}$$

Table

$$\frac{\alpha V}{\sigma}=4.7535$$

$$V=4.7535\frac{\sigma}{\alpha}=950.6$$

Functions of a r.v.

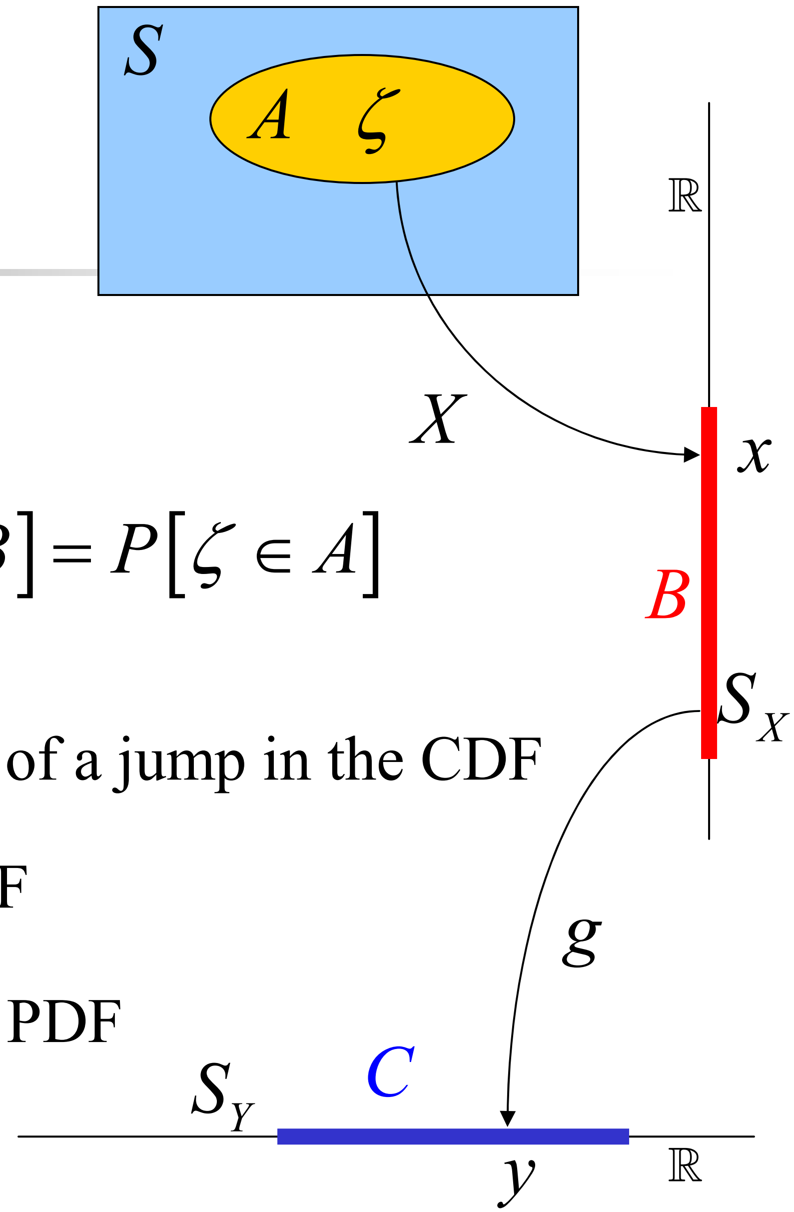


- *X* is a random variable
- g(*x*) is a real-valued function defined on the real line
- *Y*=g(*X*), i.e. for every *X*=*x*, evaluate g(*x*)=*y* and assign it to *Y*
- *Y* is also a random variable
- Find CDF and PDF of *Y*

ultimately probabilities are induced by the underlying experiment

Induced probability

equivalent events



$$P[Y \in C] = P[g(X) \in C] = P[X \in B] = P[\zeta \in A]$$

useful events:

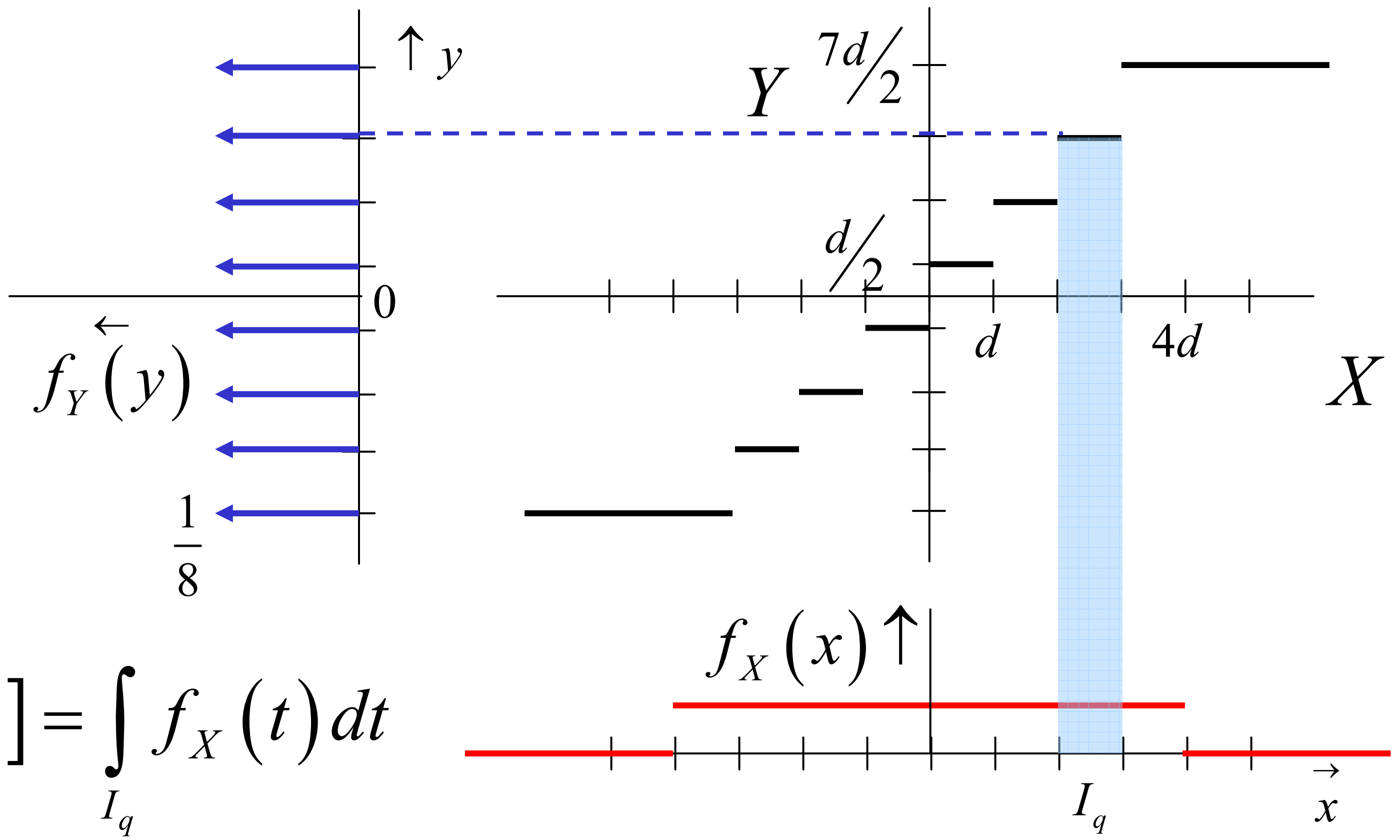
$\{g(X) = y_k\}$ is used to find the magnitude of a jump in the CDF

$\{g(X) \le y\}$ is used to directly find the CDF

$\{y < g(X) \le y + h\}$ is useful in finding the PDF

Ex 3.22 8-level uniform quantizer

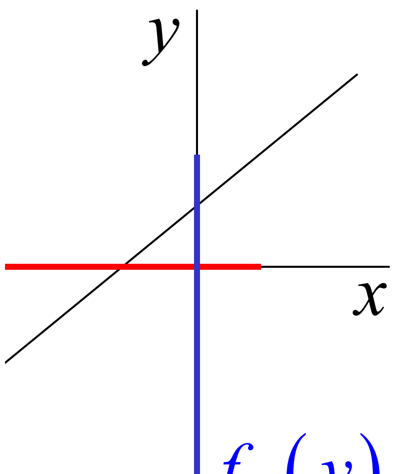
- Let *X* be a sample voltage of a speech waveform; assume *X* is uniform over [-4d,4d]



$$P[Y=q]=\int_{I_q} f_X(t)\,dt$$

Ex 3.23 a linear function $Y = aX + b \quad a \neq 0$

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = P[aX \le y - b]$$



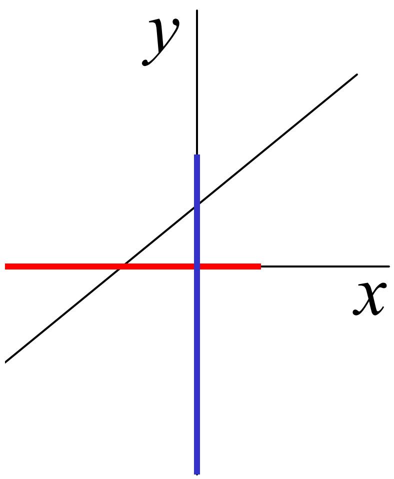
$$= \begin{cases} P\left[X \le \dfrac{y-b}{a}\right] & a > 0 \\ P\left[X \ge \dfrac{y-b}{a}\right] & a < 0 \end{cases} = \begin{cases} F_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ -\dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Ex 3.24 linear function of Gaussian r.v.

$$Y = aX + b \quad a \neq 0$$



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

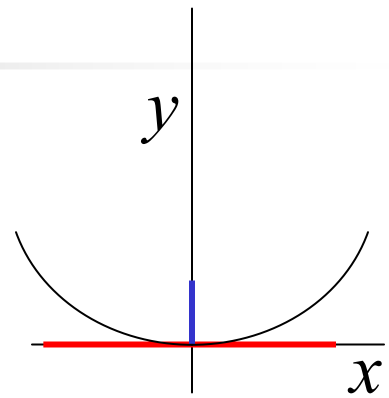
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{|a\sigma|\sqrt{2\pi}} e^{-(y-b-am)^2 / 2(a\sigma)^2}$$

linear function of a Gaussian r.v. is also a Gaussian r.v.

Ex 3.25 square law device

$$Y = X^2$$



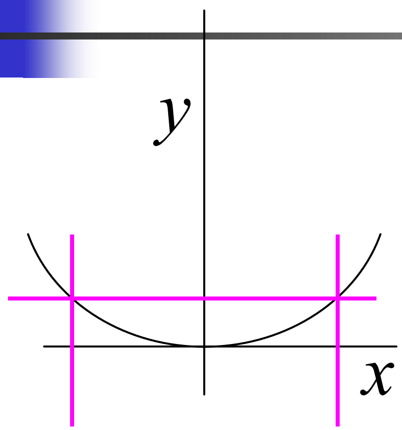
$$F_Y(y) = P[Y \le y] = P\left[X^2 \le y\right]$$

$$= P\left[-\sqrt{y} \le X \le \sqrt{y}\right] = \left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right] u(y)$$

$$f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right] u(y)$$

$$= \left[\frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}}\right] u(y)$$

from Ex 3.26



$$y_0 = g(x) \leftarrow x_0, x_1$$

produces 2 terms in PDF

$$f_Y(y) = \frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}}$$

Redo Ex 3.27

for $y < 0$: $\quad y = x^2$ has no solutions $\Rightarrow f_Y(y) = 0$

for $y \geq 0$: $\quad y = x^2$ has two solutions: $x_0 = \sqrt{y}\,;\, x_1 = -\sqrt{y}$

$dy/dx = 2x$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| dy/dx \right|} \right]_{x=x_k} = \left[\frac{f_X(x)}{|2x|} \right]_{x=x_0} + \left[\frac{f_X(x)}{|2x|} \right]_{x=x_1}$$

$$= \left[\frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}} \right] u(y)$$

Nonlinear function *Y*=g(*X*)

$$P\left[C_y\right] = P\left[B_y\right]$$

equivalent events ↓ induce equal probabilities

$$f_Y(y)\left|dy\right| = f_X(x_1)\left|dx_1\right| + f_X(x_2)\left|dx_2\right| + f_X(x_3)\left|dx_3\right|$$

↓

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left|dy/dx\right|}\right]_{x=x_k} = \sum_k \left[f_X(x)\left|\frac{dx}{dy}\right|\right]_{x=x_k}$$

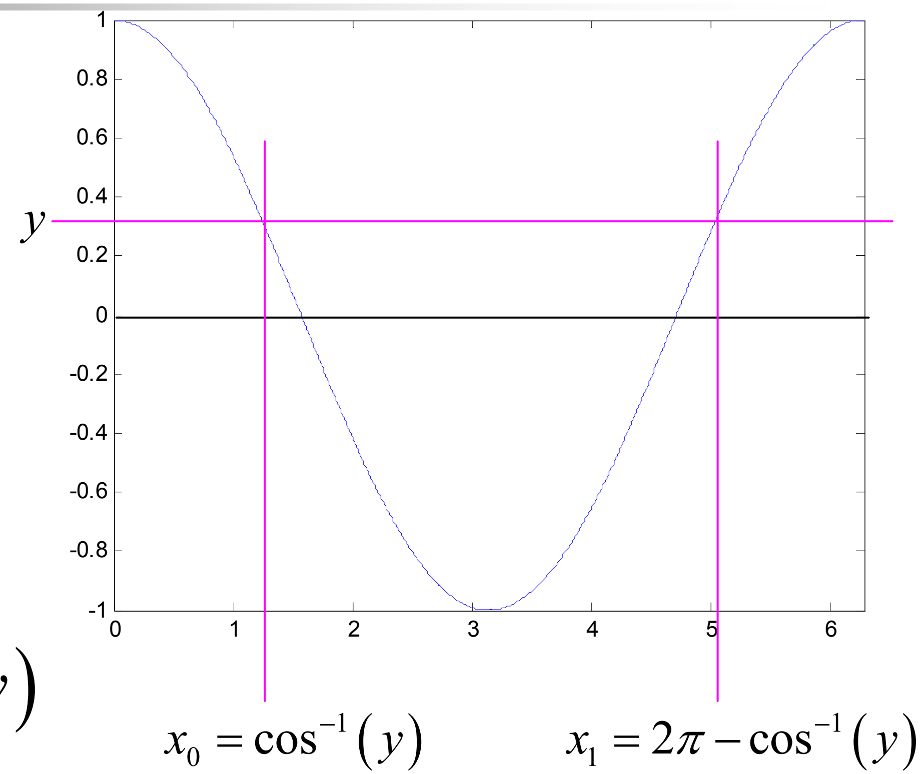
function of *y*

Ex 3.28

$$Y = \cos(X)$$

$$X \sim U(0, 2\pi]$$

$$Y = \cos(X)$$



for $y < -1$ or $y > 1$: no sol^s

$$f_Y(y) = 0$$

for $-1 \le y \le 1$:

$$x_0 = \cos^{-1}(y);\ x_1 = 2\pi - \cos^{-1}(y)$$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = -\sin(x_0) = -\sin\{\cos^{-1}(y)\} = -\sqrt{1-y^2}$$

$$Y = \cos(X)$$

$$X \sim U(0, 2\pi] \rightarrow f_X(x) = \frac{1}{2\pi}\left[u(x) - u(x - 2\pi)\right]$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| dy/dx \right|} \right]_{x = x_k} = \frac{1}{2\pi\left|-\sqrt{1-y^2}\right|} + \frac{1}{2\pi\left|\sqrt{1-y^2}\right|} = \frac{1}{\pi\sqrt{1-y^2}}$$

$$-1 \le y \le 1$$

$$\int_{-\infty}^{y}$$

$$F_Y(y) = \frac{1}{2} + \frac{\sin^{-1}(y)}{\pi} \text{ for } -1 \le y \le 1$$

Y has the arcsine distribution

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$