Bayes' Rule

Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of S then

$$P\left[B_j \mid A\right] = \frac{P\left[B_j \cap A\right]}{P[A]} = \frac{P\left[A \mid B_j\right] P\left[B_j\right]}{\sum_{k=1}^{n} P\left[A \mid B_k\right] P\left[B_k\right]}$$

↑ a posteriori probability

the partition corresponds to a priori events (of interest)
A corresponds to a measurement/observation

Ex

- A 1 is received; what is the probability that a 1 was transmitted?

$$P[T_1 \mid R_1] = \frac{P[R_1 \mid T_1]P[T_1]}{P[R_1]} = \frac{(1-\varepsilon)p}{(1-\varepsilon)p + \varepsilon(1-p)} \overset{p=0.5}{=} \frac{(1-\varepsilon)/2}{1/2} = (1-\varepsilon)$$

Bayes' Rule

total probability $P[R_1] = P[R_1 \mid T_1]P[T_1] + P[R_1 \mid T_0]P[T_0]$

$$= (1-\varepsilon)p + \varepsilon(1-p)$$

$$P[T_0 \mid R_1] = \frac{P[R_1 \mid T_0]P[T_0]}{P[R_1]} = \frac{\varepsilon(1-p)}{(1-\varepsilon)p + \varepsilon(1-p)} \overset{p=0.5}{=} \frac{\varepsilon/2}{1/2} = \varepsilon$$

Independence of events

- Knowledge of the occurrence of event *B* does not alter the probability of some other event *A*
  - *A does not depend on B*

$$P[A] = P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

events *A* and *B* are independent if

$$P[A \cap B] = P[A]P[B]$$

problematic if $P[B] = 0$

$\Downarrow\Uparrow$ $P[B] \neq 0$

$$P[A \mid B] = P[A]$$

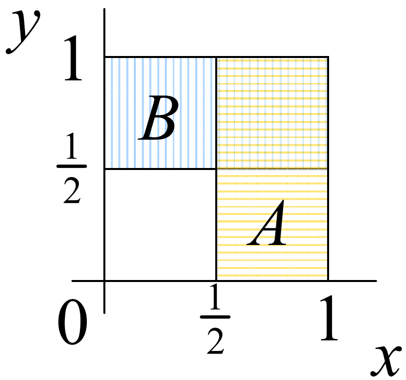
$\Downarrow\Uparrow$ $P[A] \neq 0$

$$P[B \mid A] = P[B]$$

Ex 2-D continuous

two numbers x and y are selected at random between 0 and 1

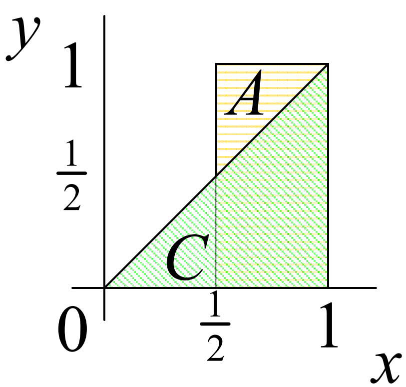
events $A = \{x > 0.5\}$, $B = \{y > 0.5\}$, $C = \{x > y\}$



$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{1/4}{1/2} = \frac{1}{2} = P[A]$$

A and *B* are independent

ratio of proportions has remained the same



$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{3/8}{1/2} = \frac{3}{4} \neq \frac{1}{2} = P[A]$$

ratio of proportions has increased

i.e. we gained "knowledge" from measuring *C*

Independence of *A*, *B*, and *C*

1. pairwise independence:
$$\begin{cases} P[A\cap B]=P[A]P[B] \\ P[A\cap C]=P[A]P[C] \\ P[B\cap C]=P[B]P[C] \end{cases}$$

2. knowledge of joint occurrence, of any two, does not affect the third: $P[C\mid A\cap B]=P[C]$

$$\frac{P[A\cap B\cap C]}{P[A\cap B]}=P[C]$$

$$P[A\cap B\cap C]=P[A\cap B]P[C]$$
$$=P[A]P[B]P[C]$$

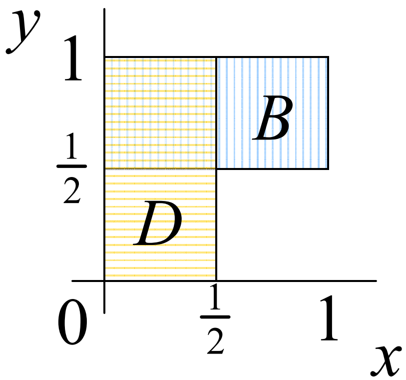
A,B,C are independent
if the probability of the intersection
of any pair or triplet of events
equals the product of the probabilities
of the individual events

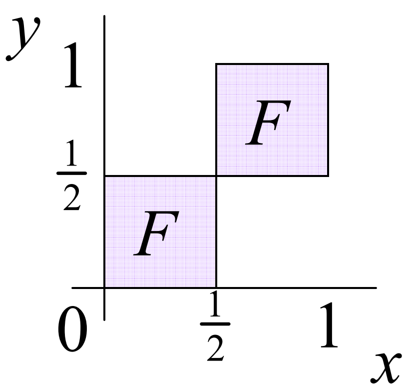
Ex pairwise independence is not enough

two numbers x and y are selected at random between 0 and 1

events: $B = \{y > 0.5\}$, $D = \{x < 0.5\}$

$$F = \{x < 0.5; y < 0.5\} \cup \{x > 0.5; y > 0.5\}$$





$$P[B \cap D] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[D]$$

$$P[B \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[F]$$

$$P[D \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[D]P[F]$$

pairwise independent

$$P[B \cap D \cap F] = P[\varnothing] = 0$$

$$P[B]P[D]P[F] = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$$

violates 2^{nd} condition

Independence of *n* events

- Probability of an event is not affected by the joint occurrence of any subset of the other events

events $B_1, B_2, \cdots, B_n$

are said to be independent if for *k*=2,…,*n*

$$P\left[B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}\right] = P\left[B_{i_1}\right] P\left[B_{i_2}\right] \cdots P\left[B_{i_k}\right]$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n$

$2^n - n - 1$ possible intersections to evaluate! Δ

- Assuming independence of the events of separate experiments is more common
  - *E.g. one coin toss is independent of any before/after*

independent experiments

Sequential experiments

- Many random experiments can be viewed as sequential experiments consisting of a sequence of simpler subexperiments
- The subexperiments may, or may not, be independent

Sequences of independent experiments

- Random experiment consisting of performing experiments E_1, $E_2,\ldots,E_n$
- Outcome is an n-tuple $s=(s_1,s_2,\ldots,s_n)$ where s_k is the outcome of the k-th subexperiment
- Sample space of sequential experiment contains n-tuples; it is denoted by the Cartesian product of the individual sample spaces $S_1 \text{x} S_2 \text{x} \ldots \text{x} S_n$
- Physical considerations will often indicate that the outcome of any given subexperiment cannot affect the outcomes of the other subexperiments; it is then reasonable to assume that the events $A_1,A_2,\ldots,A_n$ – where A_k concerns only the k-th subexperiment – are independent:

$$P\left[A_1 \cap A_2 \cap \cdots \cap A_n\right] = P\left[A_1\right]P\left[A_2\right]\cdots P\left[A_n\right]$$

facilitates computing all probabilities of events of the sequential experiment

Ex

- Select 10 numbers at random from [0,1]
- Find P[first 5 #'s <1/4, last 5 #'s >1/2]

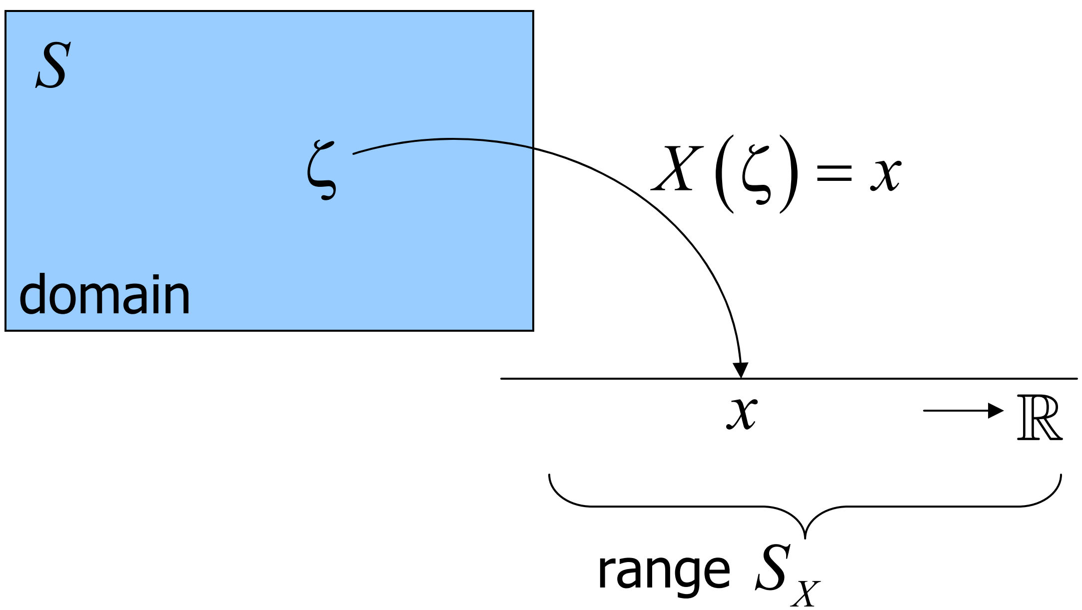
$$A_k = \left\{ x < \frac{1}{4} \right\} \text{ for } k = 1,\ldots,5 \qquad A_k = \left\{ x > \frac{1}{2} \right\} \text{ for } k = 6,\ldots,10$$

- Assuming selection of a number is independent of other selections

$$P\left[A_1 \cap A_2 \cap \cdots \cap A_n\right] = P\left[A_1\right]P\left[A_2\right]\cdots P\left[A_n\right] = \left(\frac{1}{4}\right)^5 \left(\frac{1}{2}\right)^5$$

Random variable

- A random variable X is a function that assigns a real number, $X(\zeta)$, to each outcome ζ in the sample space of a random experiment



Ex

- Toss a coin three times. Note sequence of Heads and Tails.

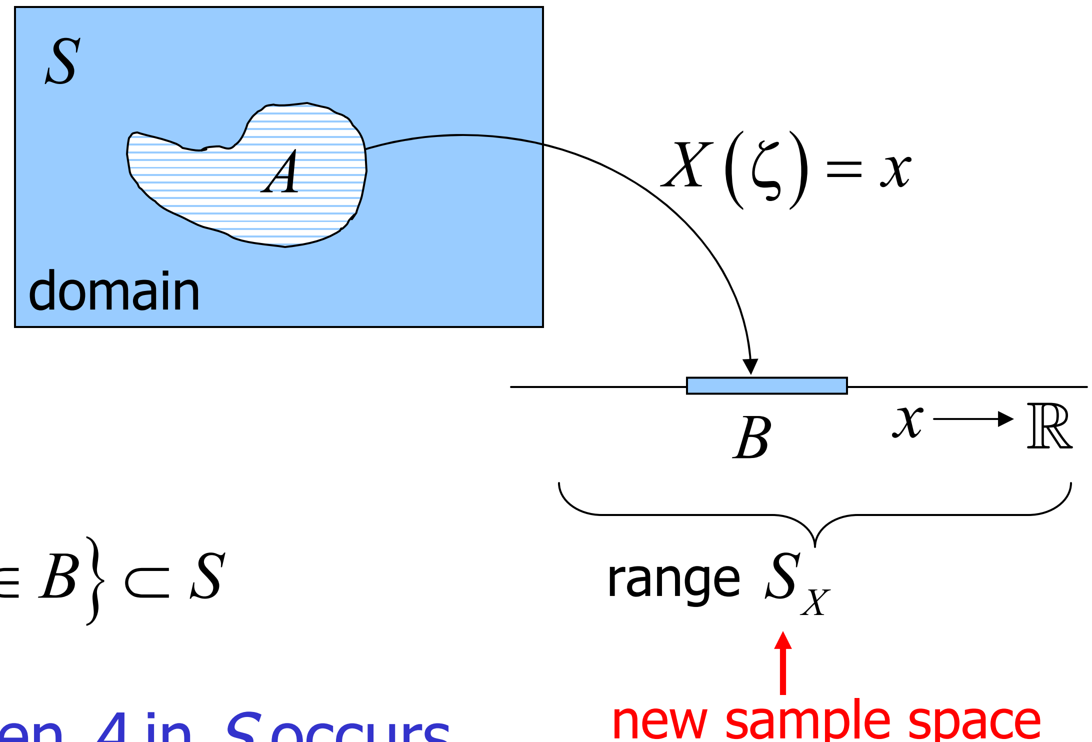
$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- Let X be the # of Heads in three coin tosses

$$\begin{array}{lcccccccc} \zeta: & HHH & HHT & HTH & THH & HTT & THT & TTH & TTT \\ X(\zeta): & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{array}$$

- *X* is a r.v. taking on values in the set $S_X = \{0, 1, 2, 3\}$

Finding probabilities involving r.v. *X*



$$B \subset S_X$$

$$A = \{\zeta : X(\zeta) \in B\} \subset S$$

B in S_X occurs when *A* in *S* occurs

$$P[B] = P[A] = P\left[\{\zeta : X(\zeta) \in B\}\right] = P\left[X^{-1}(B)\right]$$

A and *B* are equivalent events

Ex

- Event $\{X=k\}=\{k$ Heads in three coin tosses$\}$ occurs when the outcome of the coin tossing experiment contains three Heads. The probability of the event $\{X=k\}$ is given by the sum of the probabilities of the corresponding outcomes or elementary events.

$$p_0 = P[X=0] = P\{TTT\} = (1-p)^3$$

$$p_1 = P[X=1] = P\{HTT\} + P\{THT\} + P\{TTH\} = 3(1-p)^2\, p$$

$$p_2 = P[X=2] = P\{HHT\} + P\{HTH\} + P\{THH\} = 3(1-p)\, p^2$$

$$p_3 = P[X=3] = P\{HHH\} = p^3$$

- The p_k's can be used to find the probabilities of all events involving X, i.e. we can deal with sample space S_X and p_k's, instead of dealing with sample space S and the probabilities of ζ

Cumulative distribution function (cdf)

- Defined as: $F_X(x) \triangleq P[X \le x]$ for $-\infty < x < \infty$

$$F_X(x) = P\left[X \in (-\infty, x]\right] = P\left[\zeta : X(\zeta) \le x\right]$$

function of variable *x*

convenient way of specifying the probability of
all semi-infinite intervals of the real line

events of interest – when dealing with numbers – are intervals of the real line, and their complements, unions, and intersections
and probabilities of all these can be expressed in terms of the cdf!

CDF properties

i. $0 \le F_X(x) \le 1$

ii. $\lim_{x \to \infty} F_X(x) = 1$

iii. $\lim_{x \to -\infty} F_X(x) = 0$

iv. $F_X(a) \le F_X(b)$ for $a < b$

non-decreasing

v. $F_X(b) = \lim_{h \to 0} F_X(b+h) = F_X(b^+)$ for $h > 0$ Δ

continuous from the right

More CDF properties

$$\{X \le a\} \cup \{a < X \le b\} = \{X \le b\}$$

$$\Downarrow$$

$$F_X(a) + P\left[\{a < X \le b\}\right] = F_X(b)$$

$$\Downarrow$$

vi. $$P\left[\{a < X \le b\}\right] = F_X(b) - F_X(a)$$

$$\Downarrow$$

$$P\left[\{b - \varepsilon < X \le b\}\right] = F_X(b) - F_X(b - \varepsilon)$$

$$\Downarrow \; \varepsilon \to 0$$

vii. $$P[X = b] = F_X(b) - F_X(b^-)$$

magnitude of jump in CDF at *b*

if CDF is continuous at *b*, then $\{X = b\}$ is a zero-probability event

Other types of interval

$$\{a \le X \le b\} = \{X = a\} \cup \{a < X \le b\}$$

$$\Downarrow$$

$$P\left[\{a \le X \le b\}\right] = P\left[X = a\right] + F_X(b) - F_X(a)$$

$$= F_X(a) - F_X\left(a^-\right) + F_X(b) - F_X(a)$$

$$= F_X(b) - F_X\left(a^-\right)$$

for CDF that is continuous at *a* and *b*

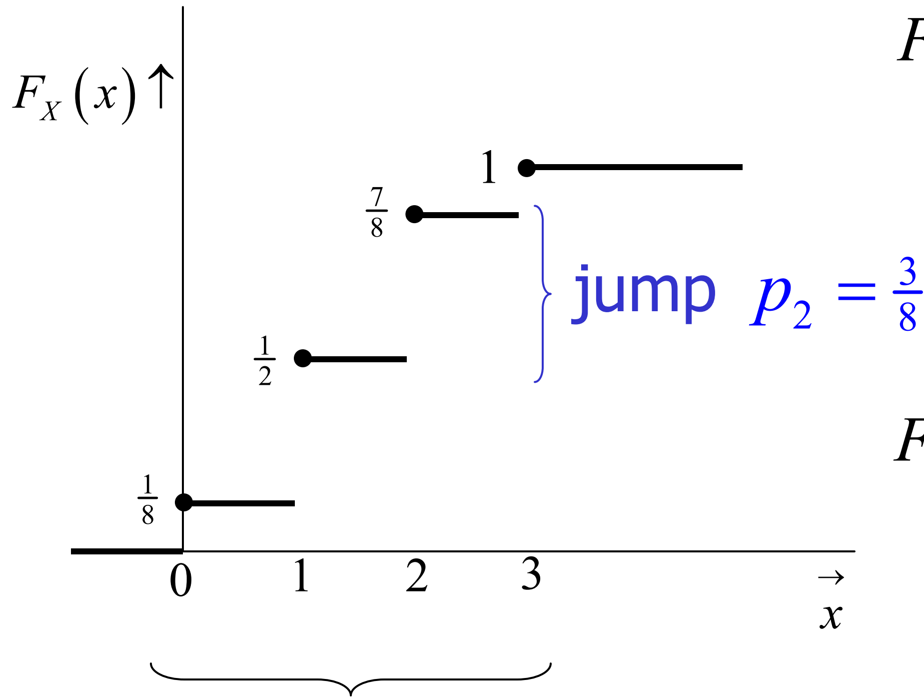
$$P[a < X < b] = P[a \le X < b] = P[a < X \le b] = P[a \le X \le b]$$

$$\{X \le x\} \cup \{X > x\} = \{-\infty < X < \infty\}$$

$$\Downarrow$$

viii. $P[X > x] = 1 - F_X(x)$

Ex CDF # heads in three coin tosses



$$F_X(2-\delta) = P[X \le 2-\delta]$$
$$= P[\{0 \; or \; 1 \; heads\}]$$
$$= p_0 + p_1$$

Δ

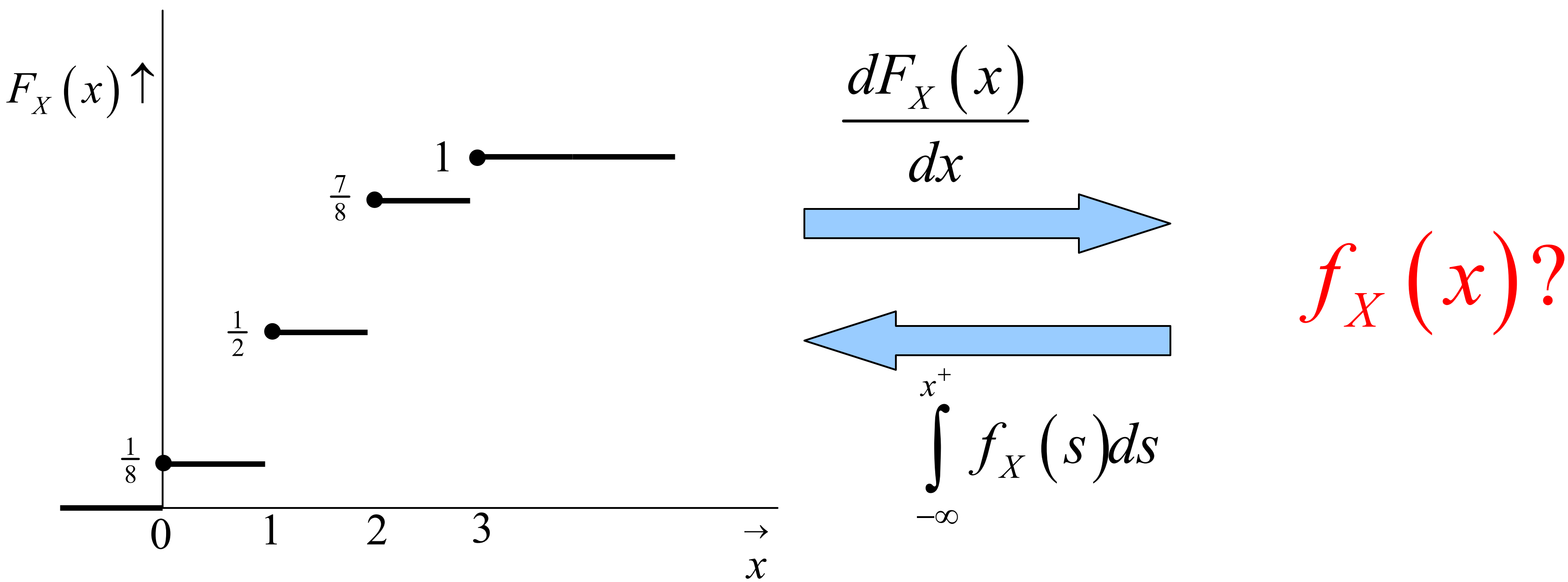
$$F_X(2+\delta) = P[X \le 2+\delta]$$
$$= P[\{0 \; or \; 1 \; or \; 2 \; heads\}]$$
$$= p_0 + p_1 + p_2$$

$$S_X = \{0,1,2,3\}$$

$$p_i = \left\{\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\right\} \Longrightarrow F_X(x) = \sum_{i:x_i \le x} p_i = \sum_{i=0}^{3} p_i u(x - x_i)$$

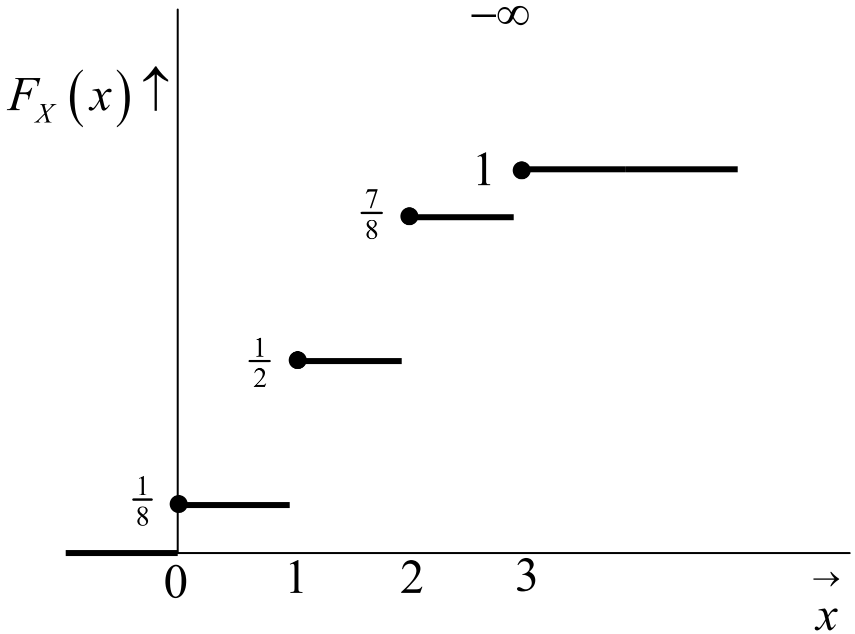
Derivative of CDF?

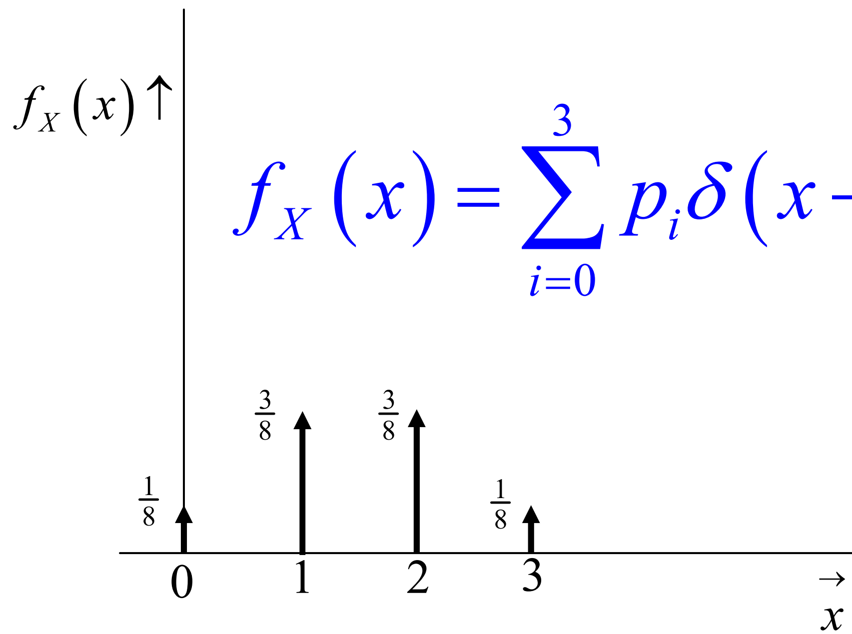
$$\text{Find } f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X\left(x^+\right) - F_X(-\infty) = F_X(x)$$



Derivative of CDF?

$$f_X(x) \ni \int_{-\infty}^{x^+} f_X(s)\,ds = F_X(x^+) - F_X(-\infty) = F_X(x)$$





$$f_X(x) = \sum_{i=0}^{3} p_i \delta(x - x_i)$$

$$\int_{-\infty}^{x^+} \delta(s-a)\,ds = u(x-a)$$

Ex 3.5 message transmission time

- The transmission time *X* of messages in a communication system obeys the exponential probability law with parameter λ

$$P[X > x] = e^{-\lambda x} \quad x > 0$$

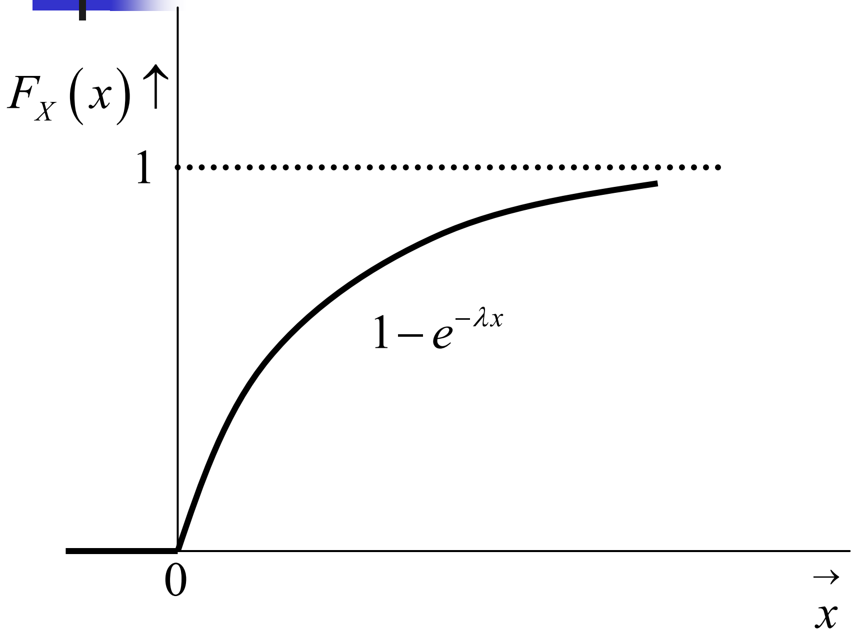
Find cdf $F_X(\mathrm{x})$. Find $P[T < X < 2T]$, where $T = \lambda^{-1}$.

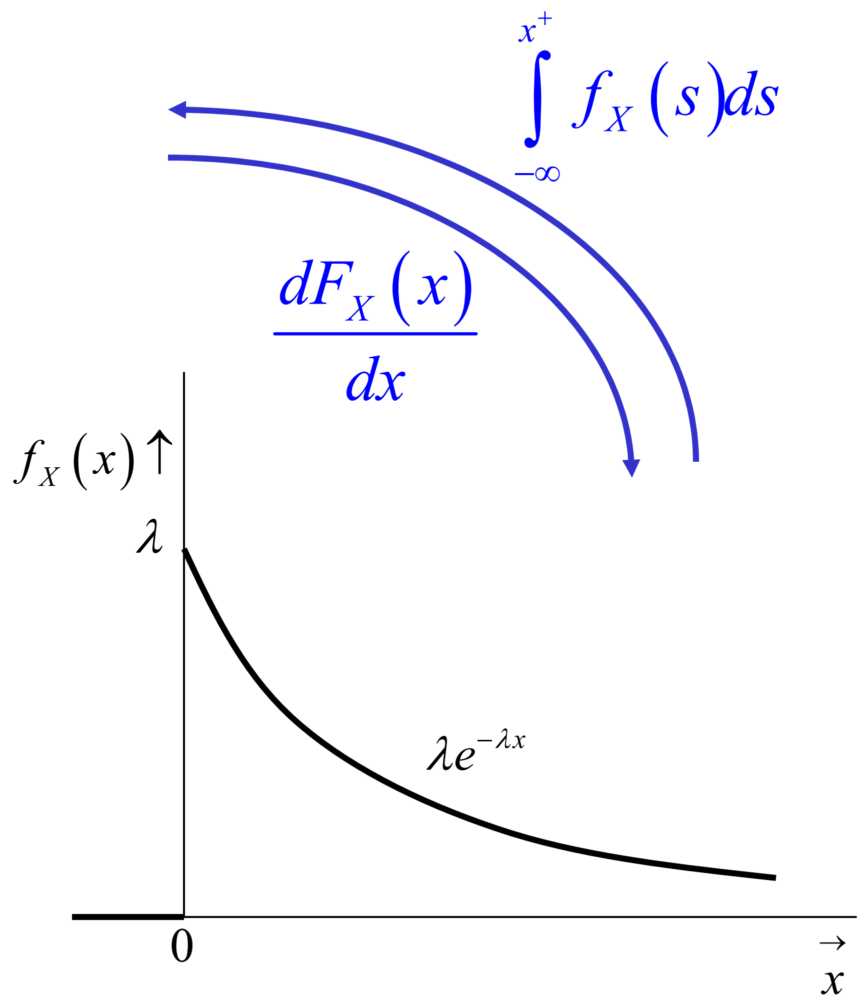
$$F_X(x) = P[X \le x] = 1 - P[X > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$P[T < X \le 2T] = F_X(2T) - F_X(T)$$

$$= 1 - e^{-\lambda 2T} - \left(1 - e^{-\lambda T}\right) = -e^{-2} + e^{-1}$$

Ex 3.5 message transmission time





continuous for all *x*

⇩

derivative exists everywhere except for *x*=0

Ex 3.6 waiting time in queue

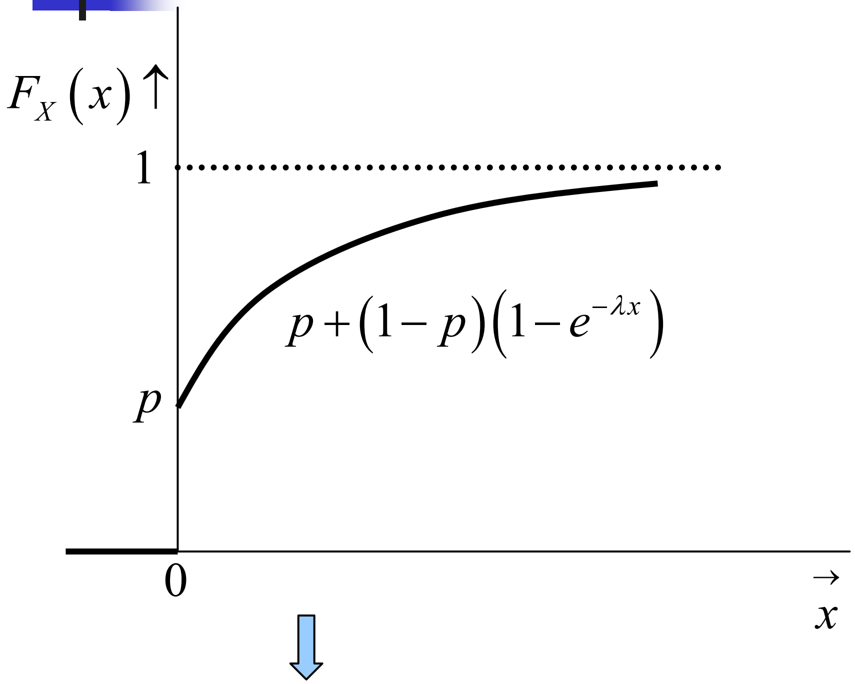
- Customer waiting time *X* is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy; the probability for finding the system idle is *p.*

$$F_X(x) = P[X \le x]$$

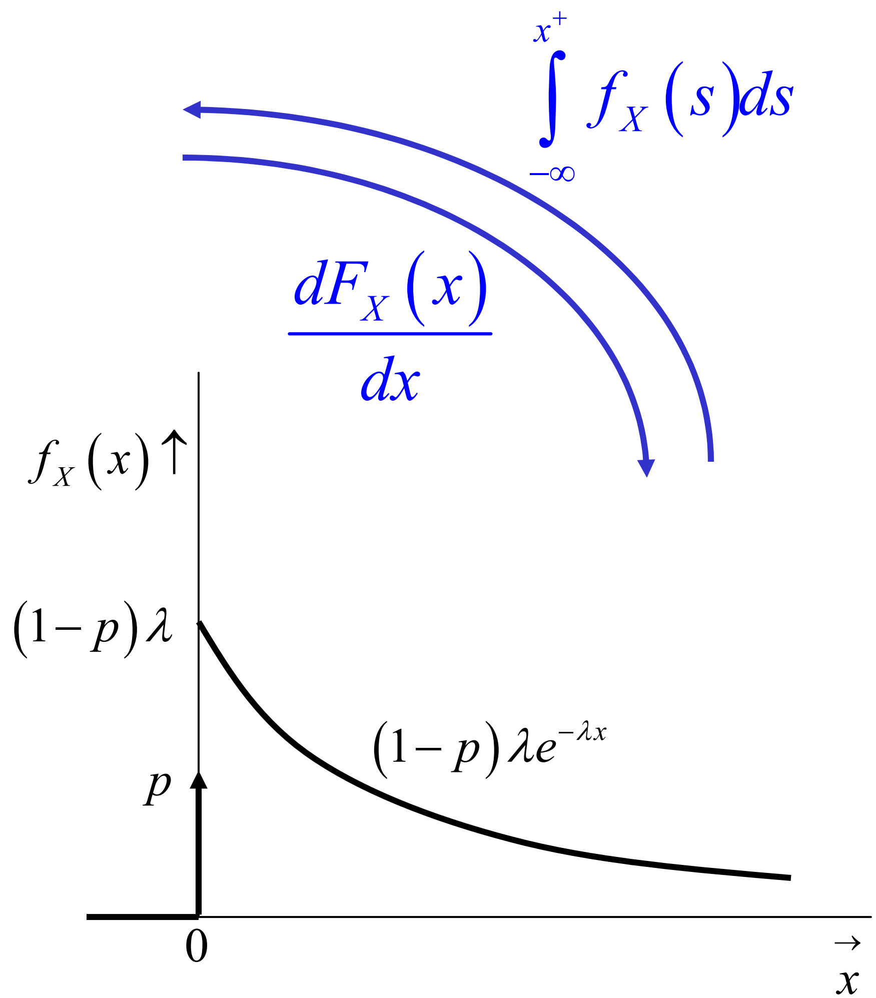
$$= P[X \le x \mid idle]\,p + P[X \le x \mid busy](1-p)$$

$$= u(x)\,p + \left(1-e^{-\lambda x}\right)(1-p)\,u(x)$$

Ex 3.6 waiting time in queue



sum of step function
and continuous function



Types of random variables

- Discrete
  - *CDF is right-continuous, staircase function of x, with jumps at a countable set of points* x_k
  - *Probability mass function* $p_X(x_k)$
  - *CDF can be written as weighted sum of unit step functions*
- Continuous
  - *CDF is continuous everywhere*
  - *CDF can be written as the integral of a non-negative function (its derivative)*
  - $P[X=x]=0$ *for all* x

Types of random variables

- Mixed
  - *CDF that has jumps on a countable set of points x_0, x_1, x_2,… and also increases continuously over at least one interval of values of x.*
  - $F_X(x)=pF_D(x)+(1-p)F_C(x)$ *with* $0<p<1$
  - *Can be viewed as being produced by a two-step process*
    - An unfair coin is tossed: if Heads, a discrete r.v. is generated according to $F_D(x)$; if Tails, a continuous r.v. is generated according to $F_C(x)$

More PDF properties

ii. $P[a \le X \le b] = \int_{a^-}^{b^+} f_X(x)\,dx$

iii. $F_X(x) = \int_{-\infty}^{x^+} f_X(t)\,dt$

the PDF completely specifies the behavior of continuous r.v.'s

iv. $\int_{-\infty}^{\infty} f_X(t)\,dt = F_X(\infty) = 1$ unit probability mass

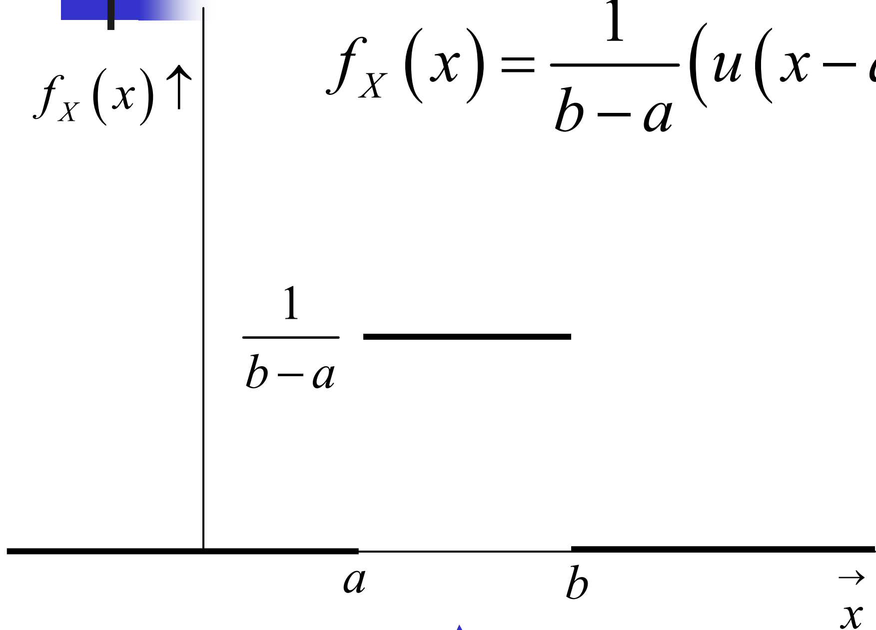
PDF is like "mass" density

a valid PDF can be formed from any nonnegative, piecewise continuous, integrable function

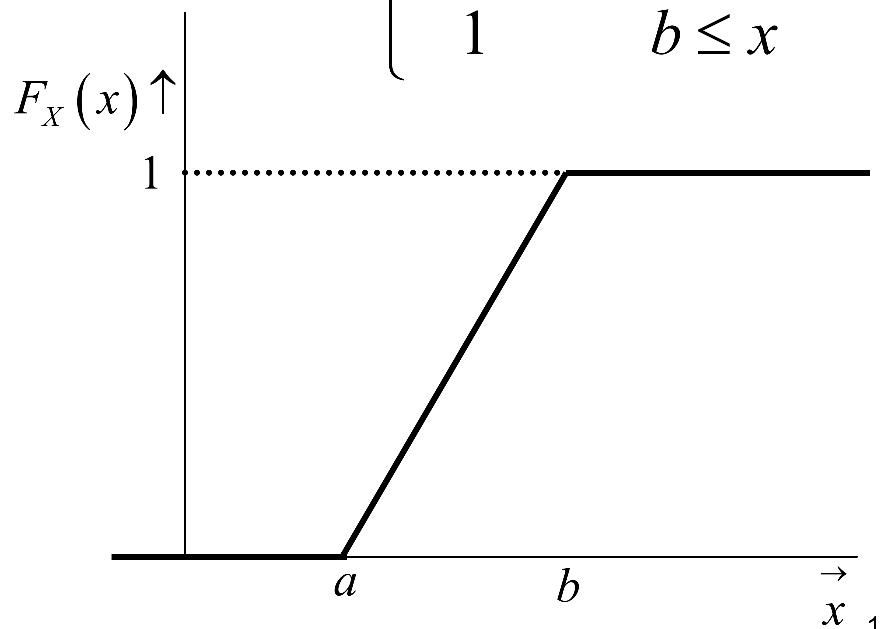
PDF – like CDF – is defined over $(-\infty,\infty)$

Ex 3.7 Uniform random variable

$$f_X(x) = \frac{1}{b-a}\left(u(x-a) - u(x-b)\right)$$



$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases}$$



$$\frac{dF_X(x)}{dx}$$

$$\int_{-\infty}^{x^+} f_X(s)\,ds$$

not magnitude

Ex 3.8 normalization

- PDF of sample values of speech waveforms is found to decay exponentially - at a rate α - for positive and negative values

$$f_X(x) = ce^{-\alpha|x|} \qquad -\infty < x < \infty$$

normalization:

$$\int_{-\infty}^{\infty} ce^{-\alpha|x|}dx = 2\int_{0}^{\infty} ce^{-\alpha|x|}dx = 2c\frac{e^{-\alpha\infty} - e^{-\alpha 0}}{-\alpha} = 1$$

⇩

$$c = \frac{\alpha}{2}$$

$$P\left[|X| < \upsilon\right] = \frac{\alpha}{2}\int_{-\upsilon}^{\upsilon} e^{-\alpha|x|}dx = \alpha\int_{0}^{\upsilon} e^{-\alpha x}dx = \alpha\frac{e^{-\alpha\upsilon} - e^{-\alpha 0}}{-\alpha} = 1 - e^{-\alpha\upsilon}$$

Conditional CDF's and PDF's

$$F_X(x \mid A) \triangleq \frac{P\left[\{X \le x\} \cap A\right]}{P[A]} \quad \text{if } P[A] > 0$$

$$f_X(x \mid A) \triangleq \frac{d}{dx} F_X(x \mid A)$$

$$f_X(x \mid A) \ni \int_{-\infty}^{x^+} f_X(s \mid A)\, ds = F_X(x \mid A)$$

generalized concept

Ex 3.10 conditional CDF & PDF

- The lifetime *X* of a machine has a continuous CDF. Find the conditional CDF & PDF given the event *A*={*X*>*t*} (machine still working at *t*)

$$F_X(x \mid X>t) = P[X \le x \mid X>t] = \frac{P[\{X \le x\} \cap \{X>t\}]}{P[X>t]}$$

$$= \begin{cases} 0 & x \le t \quad \leftarrow P[\phi] \\ \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & t < x \end{cases}$$

$\frac{d}{dx}$

$$f_X(x \mid X>t) = \begin{cases} 0 & x \le t \\ \dfrac{f_X(x)}{1 - F_X(t)} & t < x \end{cases}$$

Bernoulli r.v.

- *A* is an event related to the outcomes of a random experiment

indicator function for *A*: $I_A(\zeta) \triangleq \begin{cases} 0 & \zeta \notin A \\ 1 & \zeta \in A \end{cases}$

$$S_X = \{0,1\}$$

assigns # to outcome

$I_A(\zeta)$ is a r.v.

pmf: $p_I(0) = 1 - p\,; \quad p_I(1) = p = P[A]$

$I_A(\zeta) = 1 \sim$ "success" $\rightarrow$ Bernoulli r.v.

"tossing of a biased coin": Bernoulli r.v. is a model for this fundamental mechanism for generating randomness"

Binomial r.v.

- Random experiment repeated *n* independent times. Let *X* be the number of times event *A* occurs in these *n* trials.

$$X = I_1 + I_2 + \cdots + I_n$$

sum of Bernoulli r.v.'s (indicator functions for *A* in trial *j*)

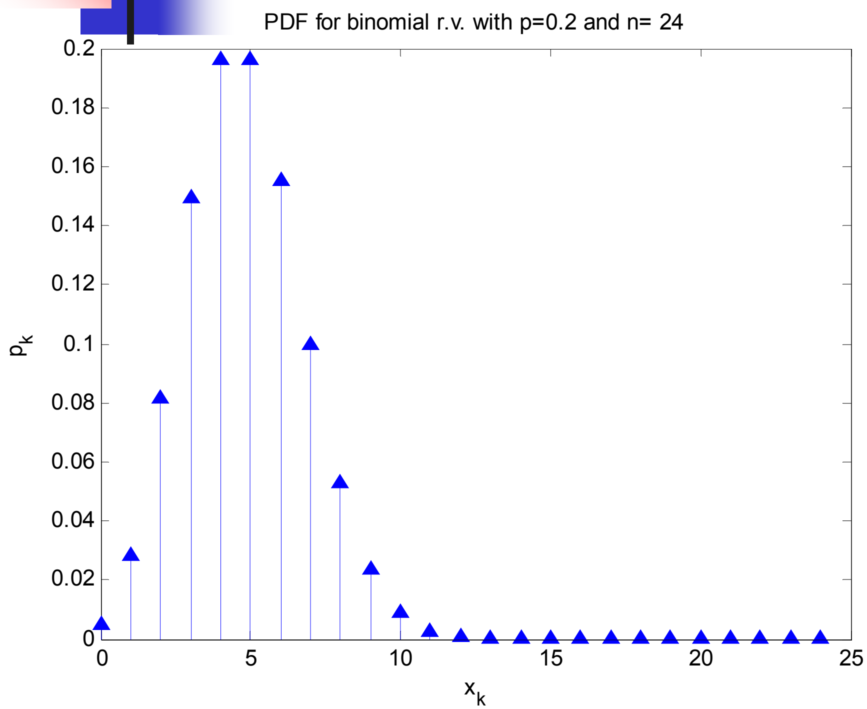
$$S_X = \{0, 1, \cdots, n\}$$

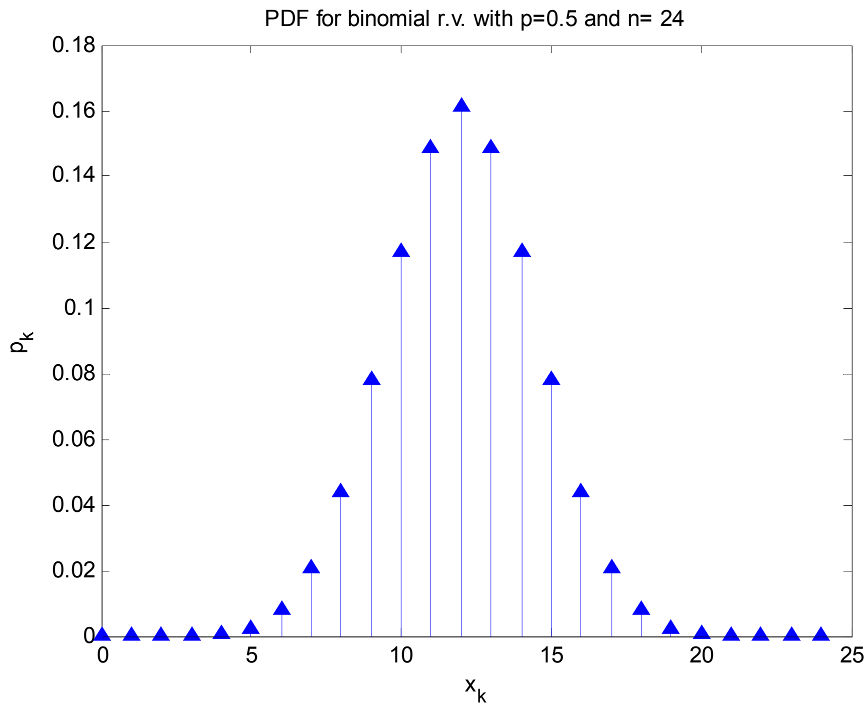
$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \cdots, n$$

Binomial PDF

$$k_{\max} = \arg\max_k P\left[X = k\right] = \left\lfloor (n+1)p \right\rfloor$$

$$\text{if } \left\lfloor (n+1)p \right\rfloor == \left\lceil (n+1)p \right\rceil \text{ then also max at } k_{\max -1}$$





Arises in applications where there are two types of object (heads/tails, good/defective, correct/in-error, active/silent), and we're interested in the number of type 1 objects in a randomly selected batch of size *n*, and the type of each object is independent of the types of the other objects in the batch

Geometric r.v.

- # of independent Bernoulli trials until first occurrence of “success”

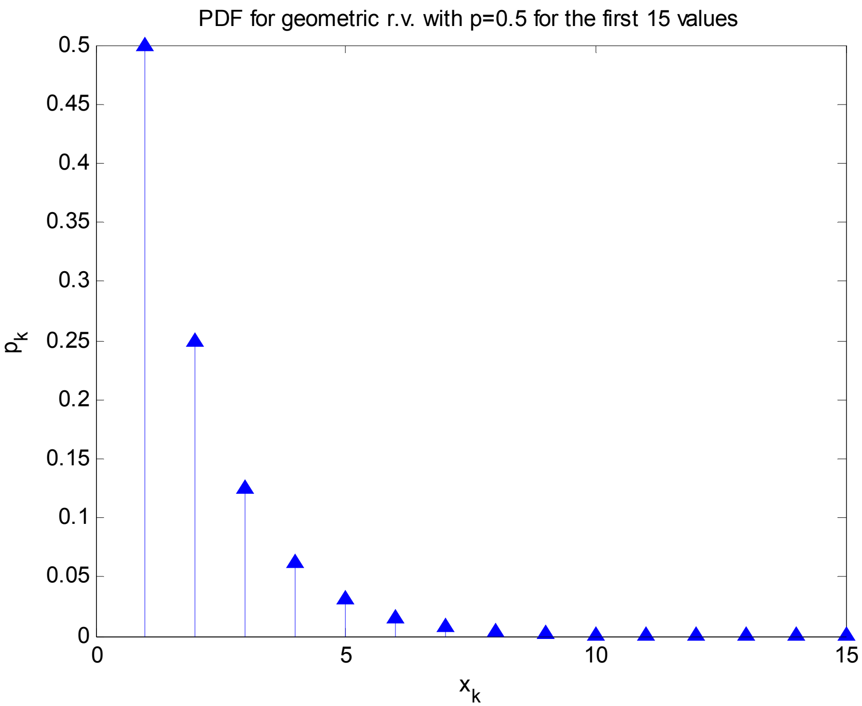
$$S_X = \{1, 2, \cdots\}$$

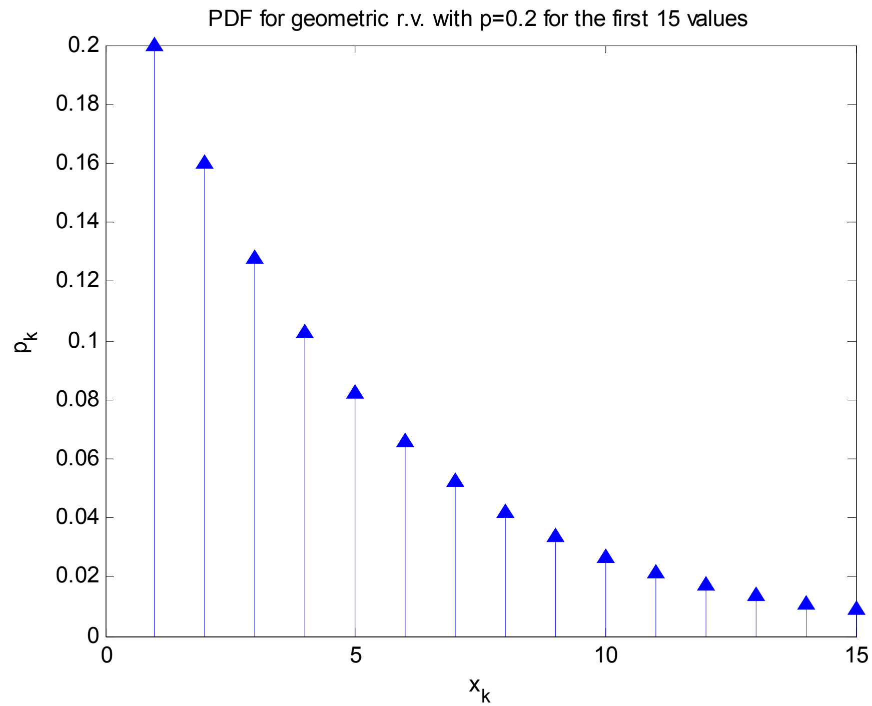
$$p = P[A] = P["success"] \text{ in each Bernoulli trial}$$

$$P[M = k] = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, \cdots$$

geometric decay

Geometric PDF

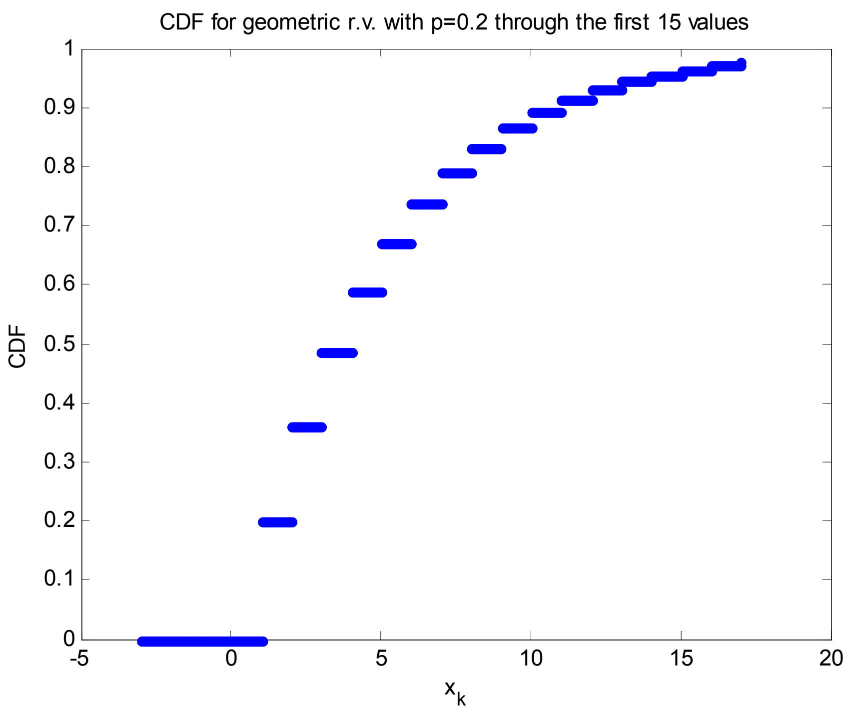




decay like 0.5^k and 0.8^k respectively

Geometric CDF

$$P[M \le k] = \sum_{j=1}^{k} (1-p)^{j-1} p = \frac{1-(1-p)^k}{1-(1-p)} p = 1-(1-p)^k = 1-q^k$$

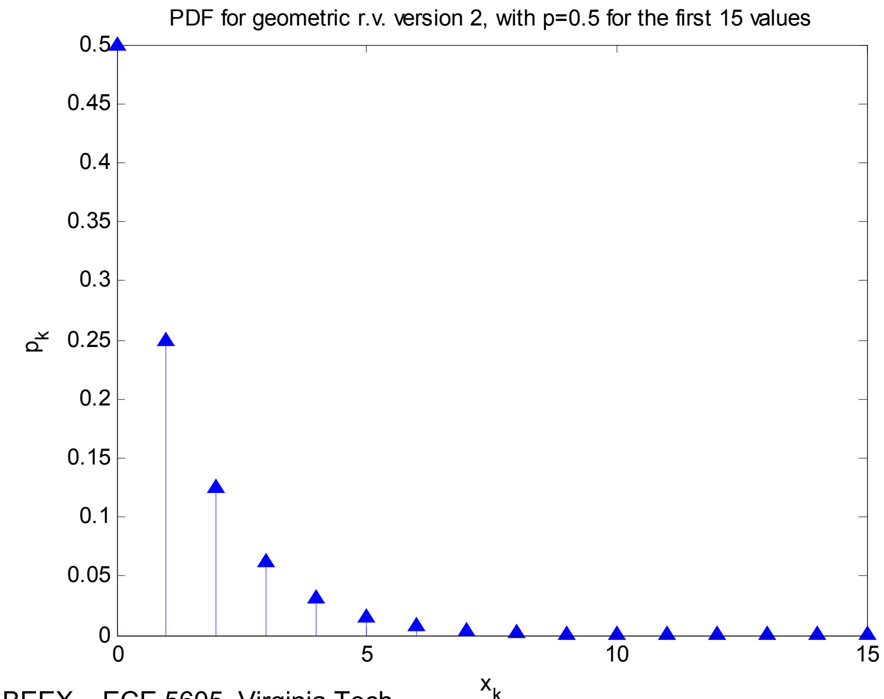


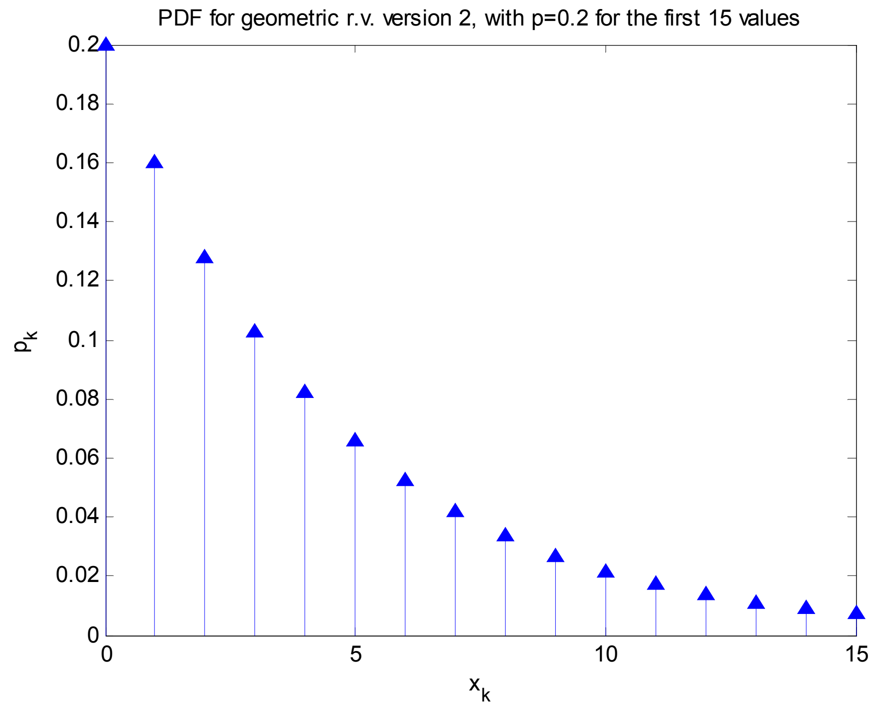
Geometric r.v. – version 2

If our interest is in the # of failures before a success occurs:

$$P[M'=k]=P[M=k+1]=(1-p)^k\,p \quad \text{for } k=0,1,2,\cdots$$

M' is also a geometric r.v.





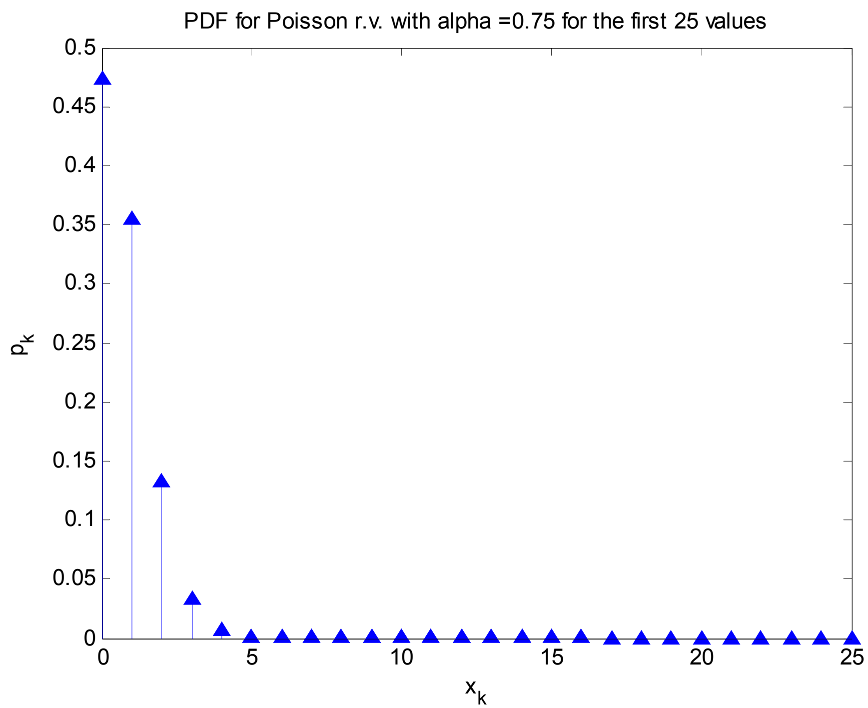
Poisson r.v.

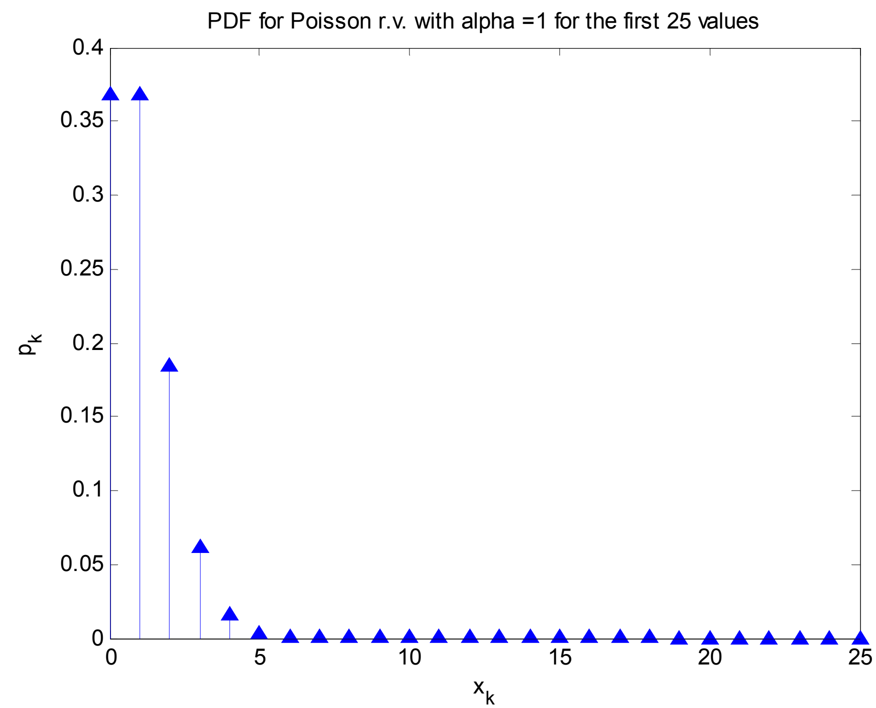
- Interested in counting the # occurrences of an event in a certain time period or in a certain region in space
  - *Events occur completely "at random"*
    - Emissions from radioactive substances
    - Counts of demands for telephone connections
    - Counts of defects in a semiconductor chip

$$P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha} \text{ for } k=0,1,2,\cdots$$

α is the average # of event occurrences in a specified interval or region in space

Poisson PDF



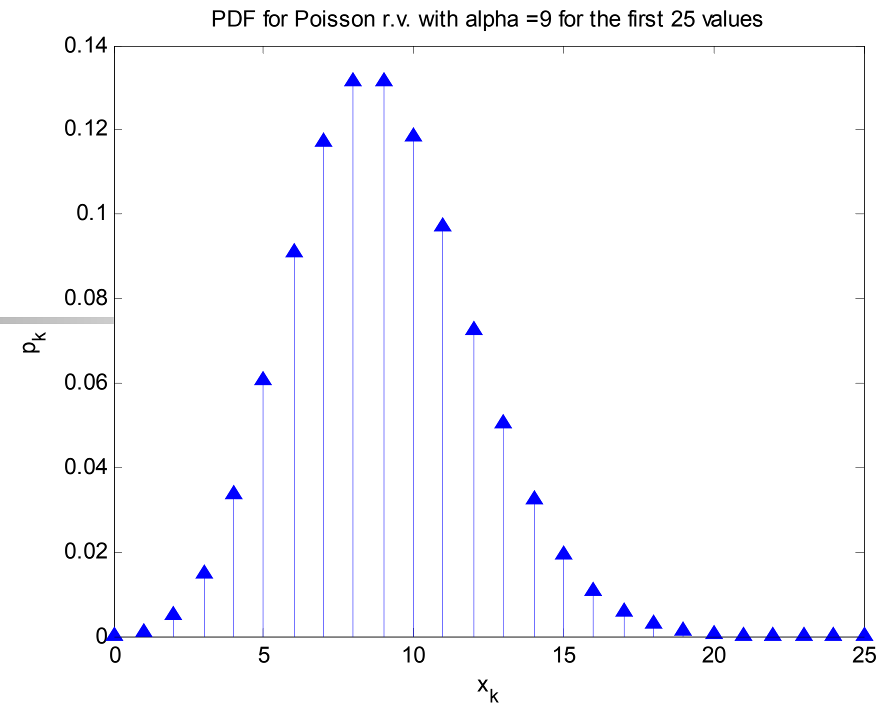


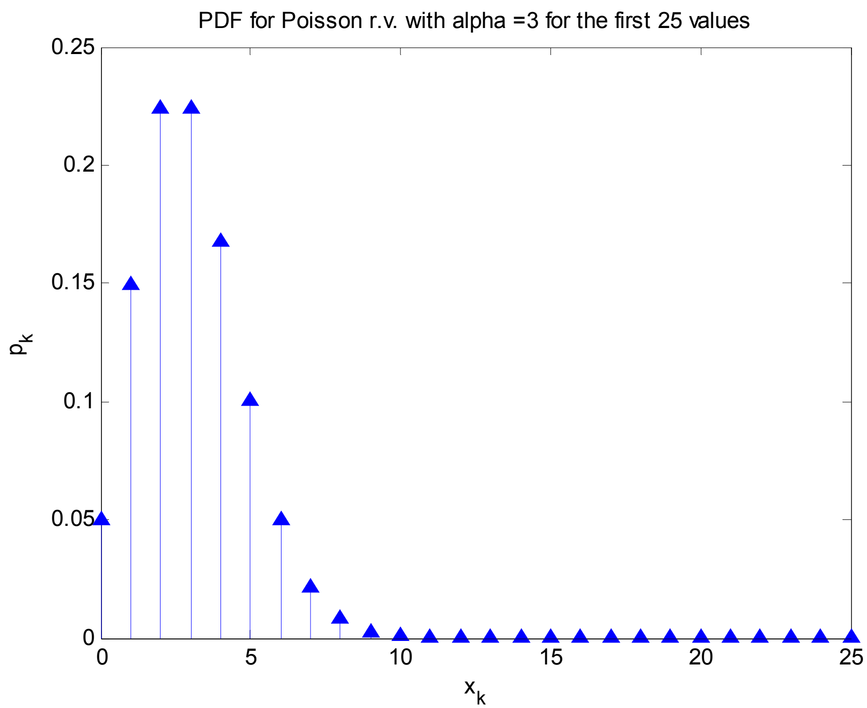
$\max P[N=k]$ is at 0 for $\alpha < 1$

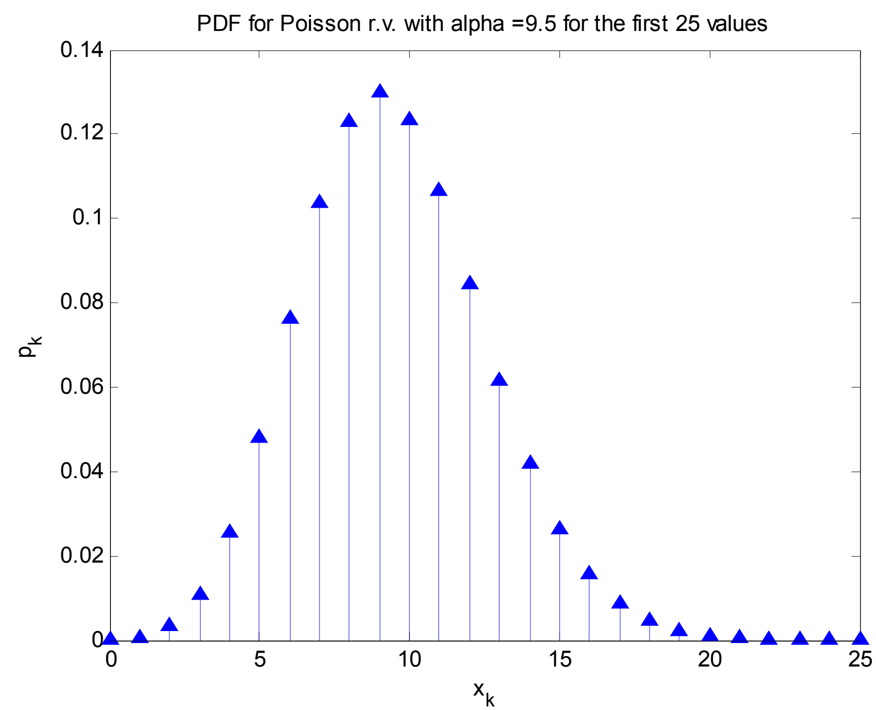
$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$

Poisson PDF

$\max P[N=k]$ is at $k=\alpha$ and at $k=\alpha-1$ for integer $\alpha \geq 1$







$\max P[N=k]$ is at $\lfloor \alpha \rfloor$ for $\alpha > 1$

Poisson PDF

$$P[N=k]=\frac{\alpha^k}{k!}e^{-\alpha} \text{ for } k=0,1,2,\cdots$$

$$\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}e^{-\alpha}=e^{-\alpha}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}=e^{-\alpha}e^{\alpha}=1$$

$P[S]=1$

if n is large and p is small, then for $\alpha=np$

$$p_k=\binom{n}{k}p^k(1-p)^{n-k}\simeq\frac{\alpha^k}{k!}e^{-\alpha} \quad k=0,1,\cdots$$

Poisson PMF is the limiting form of the binomial PMF when the number of Bernoulli trials is made very large and the probability of success is kept small, so that $\alpha=np$

recall: numerical problems in calculating binomial coefficients

Ex probability of bit error in comms

- *P*[bit error]=10^{-3}. *P*[≥5 bit errors in block of 10^3 bits]

Bernoulli trials with "success" corresponding to bit error

$$p_k = \binom{1000}{k} 10^{-3k} \left(1 - 10^{-3}\right)^{1000-k} \simeq \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \cdots$$

$$\alpha = np = 10^3 10^{-3} = 1$$

$$P[N \geq 5] = 1 - P[N < 5] \simeq 1 - \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= 1 - e^{-1}\left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right\} = 0.00366$$

Ex

- Requests for telephone connections arrive at a switching office at the rate of λ calls per second. It is known that the number of requests follows a Poisson r.v. What is *P*[no call requests in *t* sec]? What is *P*[≥*n* call requests in *t* sec]?

average # requests in a *t*-sec period is $\alpha=\lambda t$

⇩

N(*t*), the # requests in *t* sec, is Poisson with $\alpha=\lambda t$

$$P\left[N(t)=0\right]=\frac{(\lambda t)^0}{0!}e^{-\lambda t}=e^{-\lambda t}$$

$$P\left[N(t)\geq n\right]=1-P\left[N(t)<n\right]=1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

Exponential r.v.

- Arises in modeling of the time between occurrence of events, and in modeling lifetime of devices and systems; λ is the rate at which events occur

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$$F_X(x) = \left(1 - e^{-\lambda x}\right) u(x)$$

shown earlier

- For a Poisson r.v., the time between events is an exponentially distributed r.v. with parameter $\lambda = \frac{\alpha}{T}$ events per second

- Binomial $\rightarrow$ Poisson
- Geometric $\rightarrow$ exponential

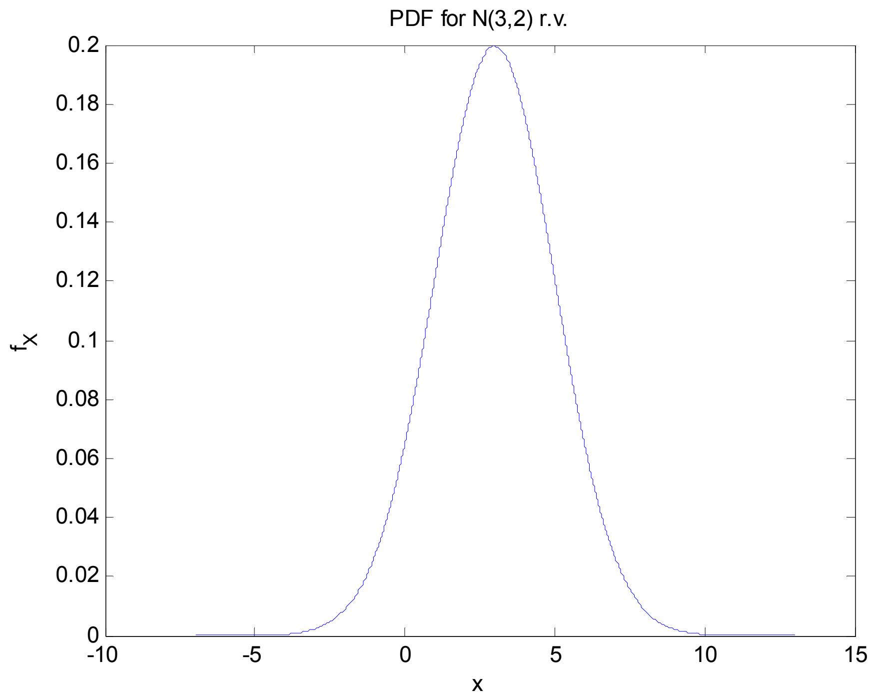
Gaussian (normal) r.v.

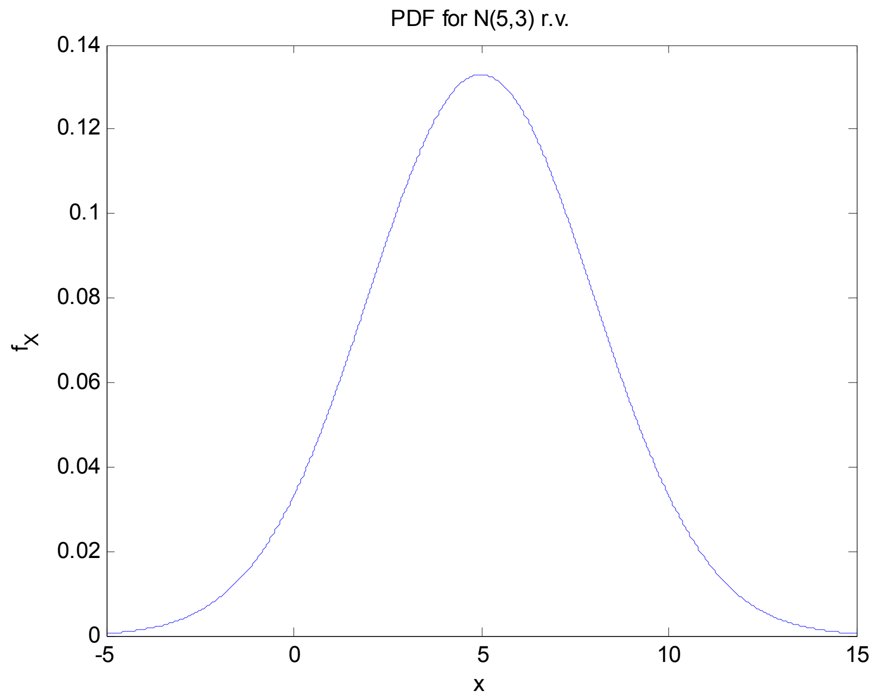
- In many situations in man-made and natural phenomena one deals with a r.v. *X* that consists of a large sum of "small" r.v.'s
  - *Exact PDF becomes complex and unwieldy*
- Under fairly general conditions, as the number of components becomes large (CLT), the CDF approaches that of the normal r.v.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \qquad -\infty < x < \infty$$

$$= N(m, \sigma)$$

Gaussian (normal) r.v. - PDF





the "bell-shaped" curve

Ex normal PDF integrates to 1

$$\left[\int_{-\infty}^{\infty} f_X(x)dx\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}dx\int_{-\infty}^{\infty} e^{-y^2/2}dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dxdy$$

Cartesian

$COV: x = r\cos\theta$

$COV: y = r\sin\theta$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2/2}\,rdrd\theta$$

polar

$$= \int_{0}^{\infty} e^{-r^2/2}\,rdr = e^{-r^2/2}\Big|_{0}^{\infty} = 1$$

Gaussian (normal) r.v. - CDF

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(s-m)^2 / 2\sigma^2} ds$$

$\Downarrow$ COV : $t = \frac{s-m}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right) \text{ where } \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

$\Uparrow$

CDF for $N(0,1)$ r.v.

"standard normal"

Q-function

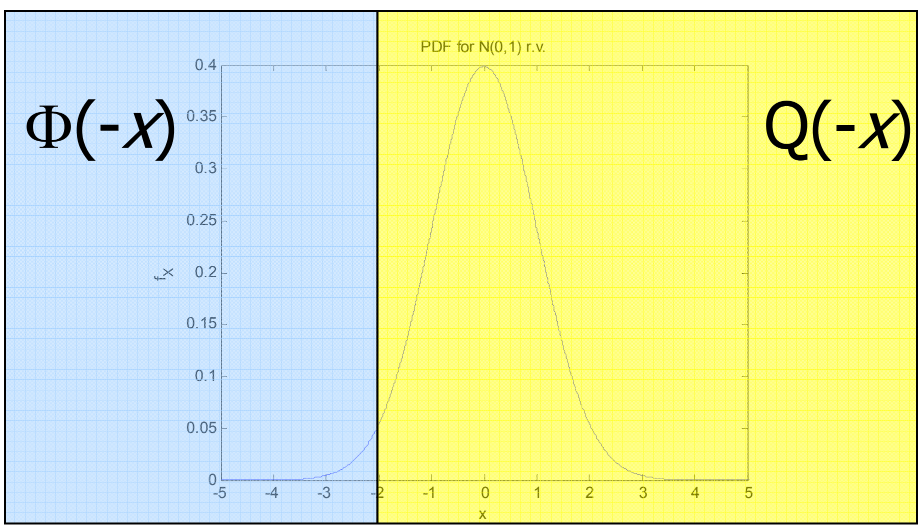
used by EE as error probability

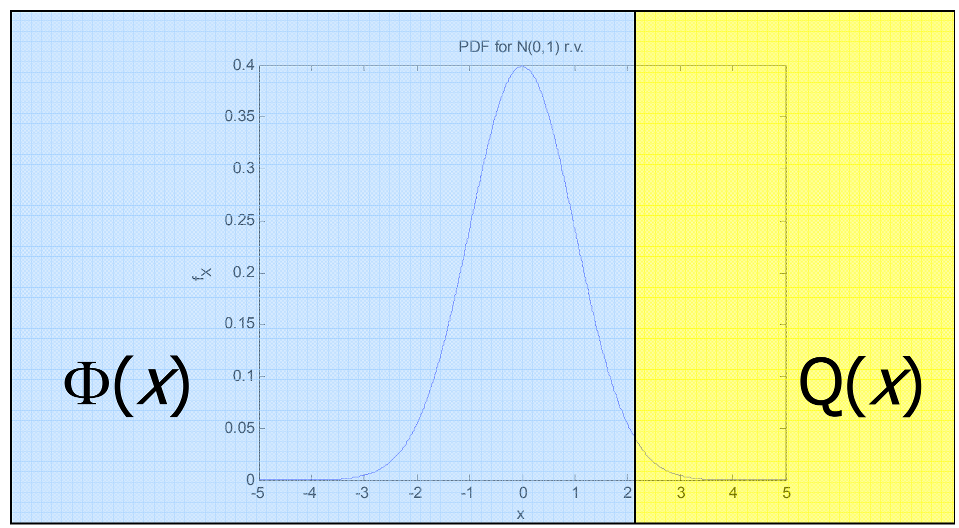
$$Q(x) \triangleq 1-\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

probability of the "tail"

$$Q(0) = 0.5$$

$$Q(-x) = 1-\Phi(-x) = 1-Q(x)$$





Gaussian (normal) r.v.

- Plays an important role in communication systems, where transmission of signals is subject to noise
  - *Noise resulting from the thermal motion of electrons, can – from physical principles – be shown to have a Gaussian PDF*

Ex

- A communication system accepts a positive voltage *V* as input and outputs a voltage $Y=\alpha V+N$, where $\alpha=10^{-2}$ and *N* is ~$N(0,2)$. Find $V \ni P[Y<0]=10^{-6}$

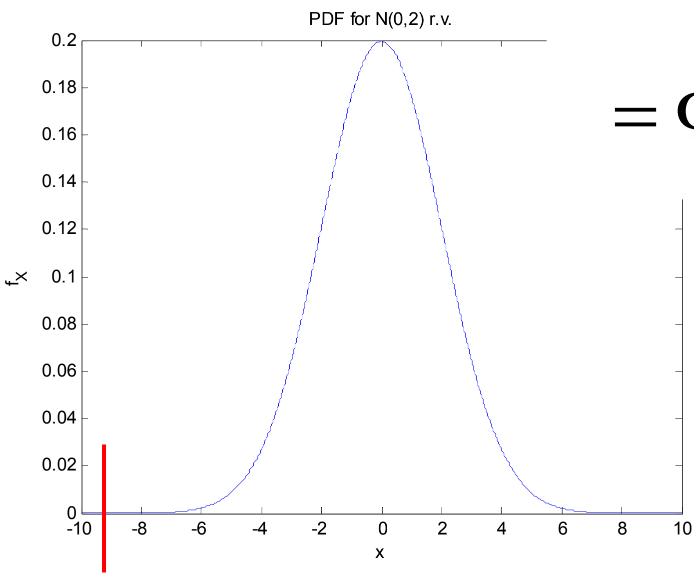
$$P\left[Y<0\right]=P\left[\alpha V+N<0\right]=P\left[N<-\alpha V\right]$$

$$=\Phi\left(\frac{-\alpha V}{\sigma}\right)=Q\left(\frac{\alpha V}{\sigma}\right)=10^{-6}$$

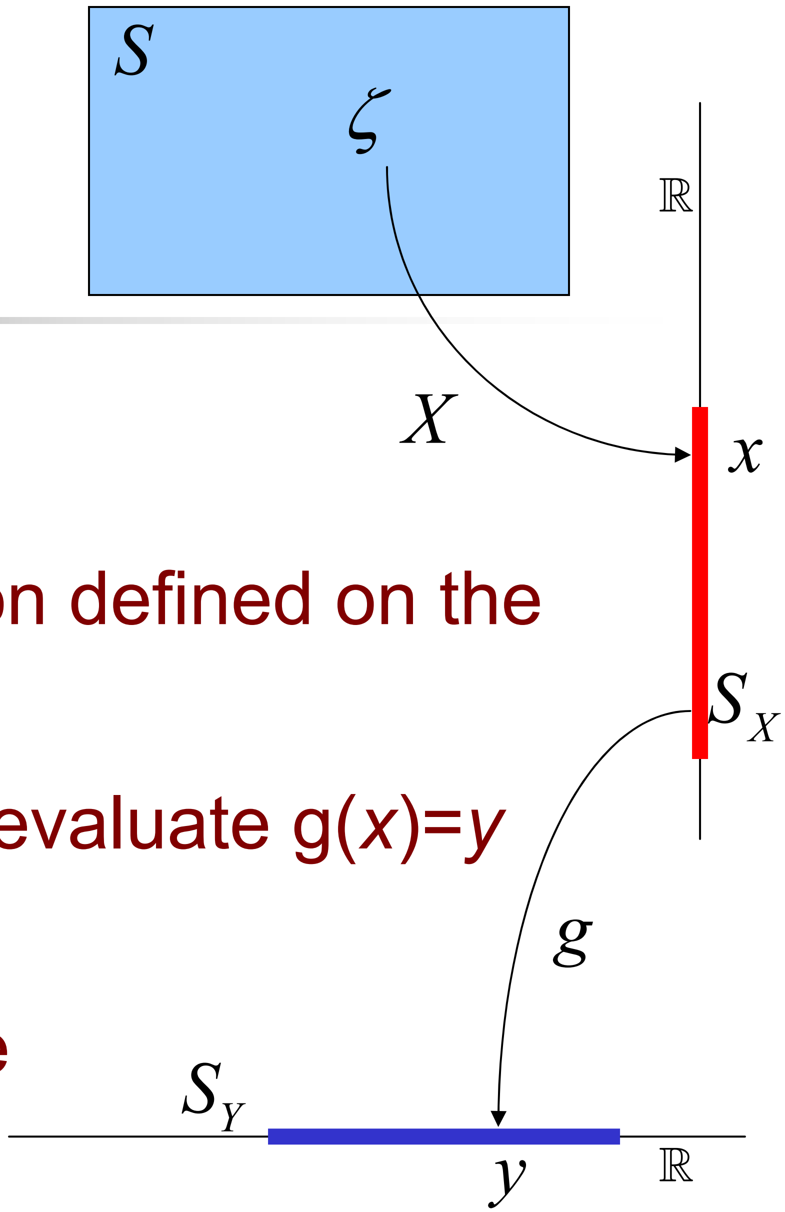
Table

$$\frac{\alpha V}{\sigma}=4.7535$$

$$V=4.7535\frac{\sigma}{\alpha}=950.6$$



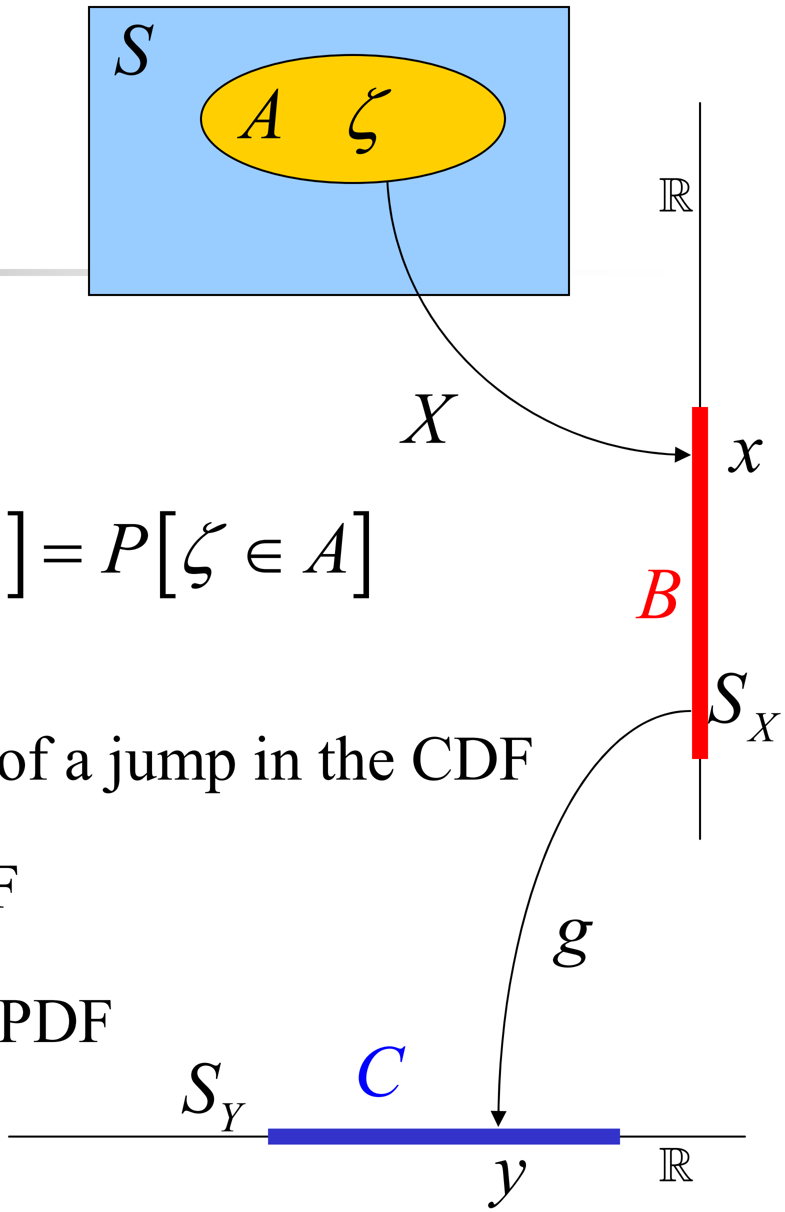
Functions of a r.v.



- *X* is a random variable
- g(*x*) is a real-valued function defined on the real line
- *Y*=g(*X*), i.e. for every *X*=*x*, evaluate g(*x*)=*y* and assign it to *Y*
- *Y* is also a random variable
- Find CDF and PDF of *Y*

ultimately probabilities are induced by the underlying experiment

Induced probability



equivalent events

$$P[Y \in C] = P[g(X) \in C] = P[X \in B] = P[\zeta \in A]$$

useful events:

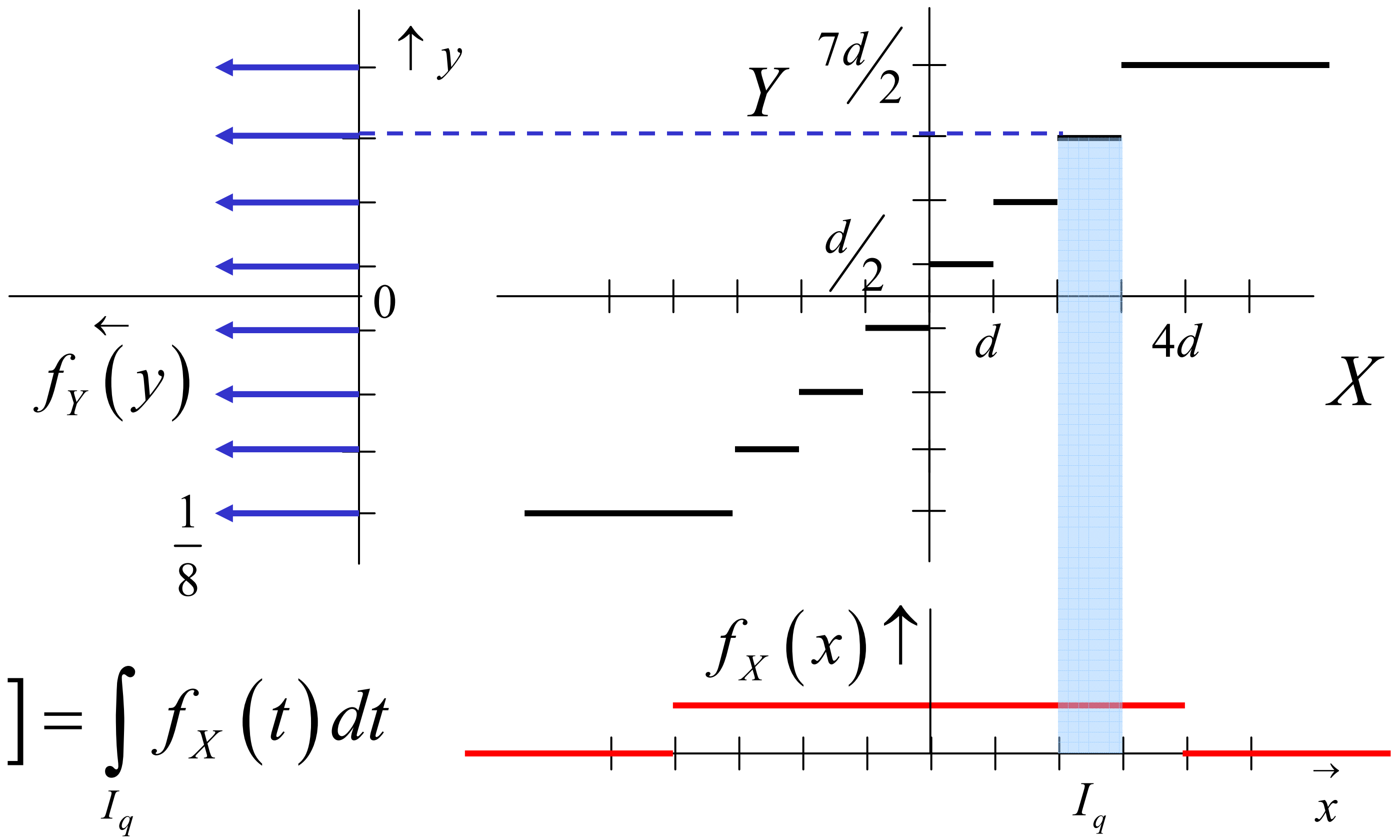
$\{g(X) = y_k\}$ is used to find the magnitude of a jump in the CDF

$\{g(X) \le y\}$ is used to directly find the CDF

$\{y < g(X) \le y + h\}$ is useful in finding the PDF

Ex 3.22 8-level uniform quantizer

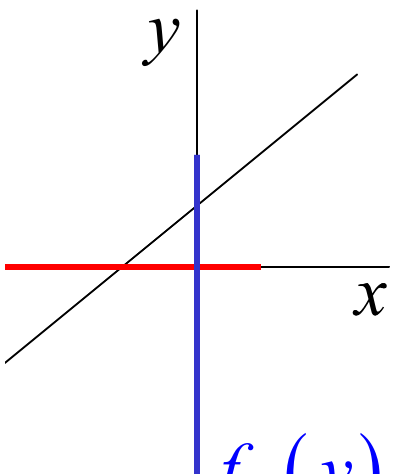
- Let *X* be a sample voltage of a speech waveform; assume *X* is uniform over [-4d,4d]



$$P[Y=q]=\int_{I_q} f_X(t)\,dt$$

Ex 3.23 a linear function $Y = aX + b \quad a \neq 0$

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = P[aX \le y - b]$$



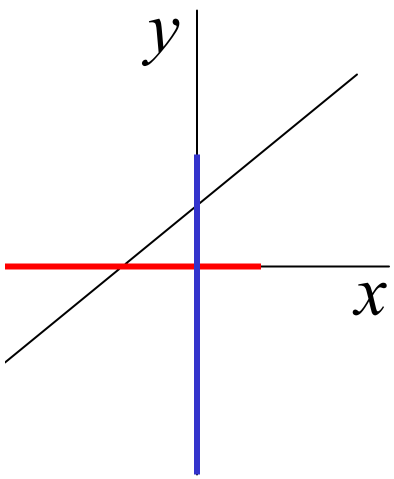
$$= \begin{cases} P\left[X \le \dfrac{y-b}{a}\right] & a > 0 \\ P\left[X \ge \dfrac{y-b}{a}\right] & a < 0 \end{cases} = \begin{cases} F_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a > 0 \\ -\dfrac{1}{a} f_X\left(\dfrac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Ex 3.24 linear function of Gaussian r.v.

$$Y = aX + b \quad a \neq 0$$



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2 / 2\sigma^2} \quad -\infty < x < \infty$$

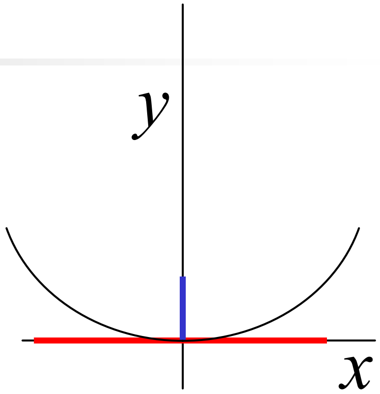
$$f_Y(y) = \frac{1}{|a\sigma|\sqrt{2\pi}} e^{-(y-b-am)^2 / 2(a\sigma)^2}$$

linear function of a Gaussian r.v. is also a Gaussian r.v.

Ex 3.25 square law device

$$Y = X^2$$

$$\Downarrow$$



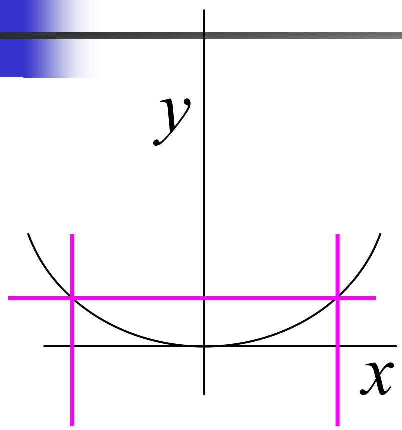
$$F_Y(y) = P[Y \le y] = P\left[X^2 \le y\right]$$

$$= P\left[-\sqrt{y} \le X \le \sqrt{y}\right] = \left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right]u(y)$$

$$f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right]u(y)$$

$$= \left[\frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}}\right]u(y)$$

from Ex 3.26



$$y_0 = g(x) \leftarrow x_0, x_1$$

produces 2 terms in PDF

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

Redo Ex 3.27

for $y < 0$: $\quad y = x^2$ has no solutions $\Rightarrow f_Y(y) = 0$

for $y \geq 0$: $\quad y = x^2$ has two solutions: $x_0 = \sqrt{y}\ ;\ x_1 = -\sqrt{y}$

$dy/dx = 2x$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left| dy/dx \right|} \right]_{x=x_k} = \left[\frac{f_X(x)}{|2x|} \right]_{x=x_0} + \left[\frac{f_X(x)}{|2x|} \right]_{x=x_1}$$

$$= \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \right] u(y)$$

Nonlinear function $Y = g(X)$

$$P\left[C_y\right] = P\left[B_y\right]$$

equivalent events ⇓ induce equal probabilities

$$f_Y(y)\left|dy\right| = f_X(x_1)\left|dx_1\right| + f_X(x_2)\left|dx_2\right| + f_X(x_3)\left|dx_3\right|$$

⇓

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left|dy/dx\right|}\right]_{x=x_k} = \sum_k \left[f_X(x)\left|\frac{dx}{dy}\right|\right]_{x=x_k}$$

function of y

Ex 3.28

$$Y = \cos(X)$$

$$X \sim U(0, 2\pi]$$

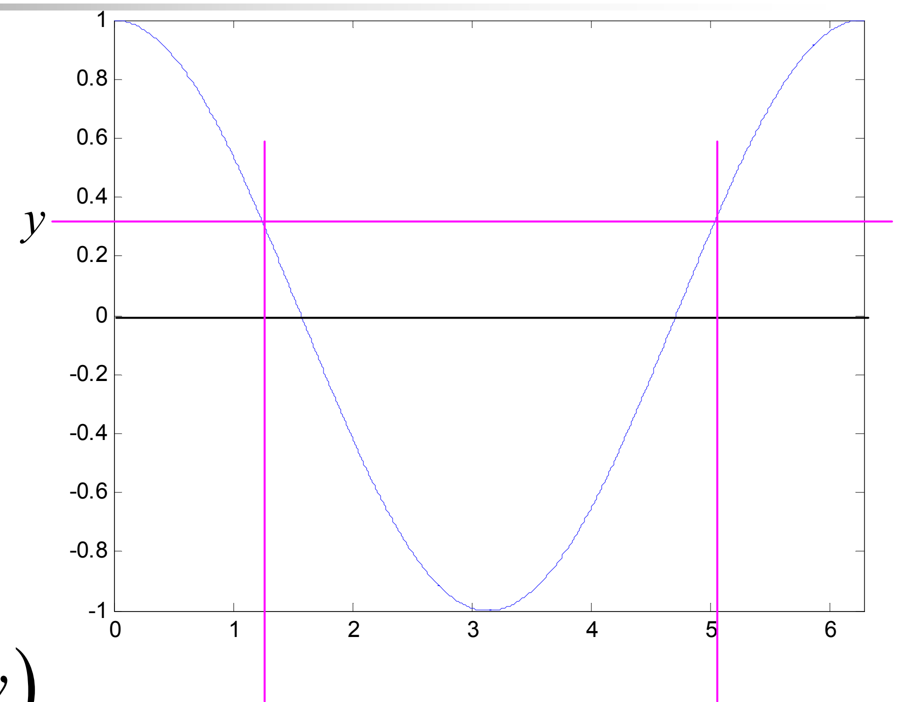
$$\Downarrow \; Y = \cos(X)$$

for $y < -1$ or $y > 1$: no sol^s

$$f_Y(y) = 0$$

for $-1 \le y \le 1$:

$$x_0 = \cos^{-1}(y)\,;\; x_1 = 2\pi - \cos^{-1}(y)$$



$$x_0 = \cos^{-1}(y) \qquad x_1 = 2\pi - \cos^{-1}(y)$$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = -\sin(x_0) = -\sin\left\{\cos^{-1}(y)\right\} = -\sqrt{1-y^2}$$

$$Y = \cos(X)$$

$$X \sim U(0, 2\pi] \rightarrow f_X(x) = \frac{1}{2\pi}\left[u(x) - u(x - 2\pi)\right]$$

$$f_Y(y) = \sum_k \left[\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right]_{x=x_k} = \frac{1}{2\pi\left|-\sqrt{1-y^2}\right|} + \frac{1}{2\pi\left|\sqrt{1-y^2}\right|} = \frac{1}{\pi\sqrt{1-y^2}}$$

$$-1 \le y \le 1$$

$$\int_{-\infty}^{y}$$

$$F_Y(y) = \frac{1}{2} + \frac{\sin^{-1}(y)}{\pi} \text{ for } -1 \le y \le 1$$

Y has the arcsine distribution

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$