Sample Space

- The sample space S of a random experiment is defined as the set of all possible outcomes.
- An outcome or sample point ζ of a random experiment is a result that cannot be decomposed into other results. $\zeta \in S$
- One and only one outcome occurs when a random experiment is performed.
  - *Outcomes are mutually exclusive – they cannot occur simultaneously*

Sample Spaces – example experiments

- E_1: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

$$S_1 = \{1, 2, \ldots, 50\}$$

- E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

$$S_2 = \{(1,b),(2,b),(3,w),(4,w)\}$$

- E_3: Toss a coin three times and note the sequence of heads and tails.

$$S_3 = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

- E_4: Toss a coin three times and note the number of heads.

$$S_4 = \{0,1,2,3\}$$

sample spaces are sets

Sample Spaces

- E_5: Count the number of voice packets containing only silence produced from a group of *N* speakers in a 10ms period.

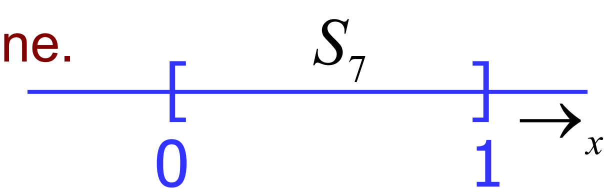
$$S_5 = \{0, 1, 2, \ldots, N\}$$

- E_6: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

$$S_6 = \{1, 2, 3, \ldots\}$$

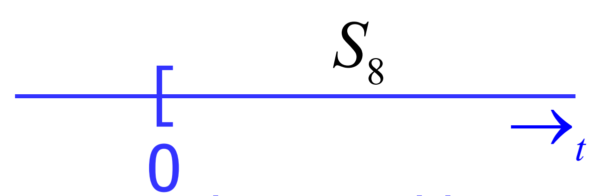
- E_7: Pick a number at random between zero and one.

$$S_7 = \{x : 0 \le x \le 1\} = [0, 1]$$



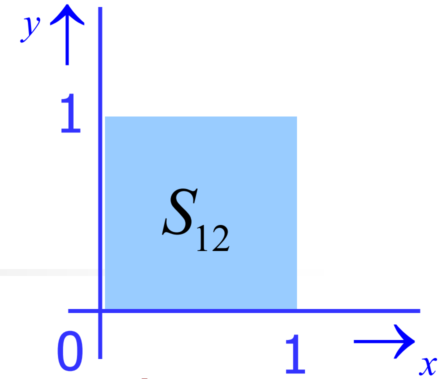
- E_8: Measure the time between two message arrivals at a message center.

$$S_8 = \{t : t \ge 0\} = [0, \infty)$$



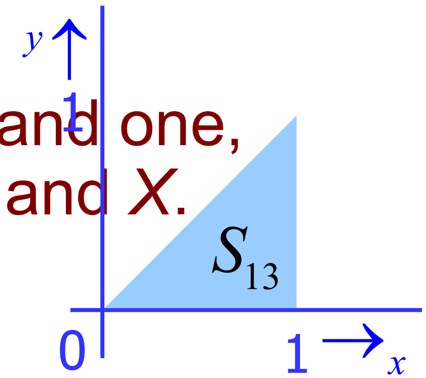
sample spaces visualized as intervals on real line

Sample Spaces



- E_{12}: Pick two numbers at random between zero and one.

$$S_{12} = \{(x, y) : 0 \le x \le 1 \ and \ 0 \le y \le 1\}$$



- E_{13}: Pick a number *X* at random between zero and one, then pick a number *Y* at random between zero and *X*.

$$S_{13} = \{(x, y) : 0 \le y \le x \le 1\}$$

- E_{14}: A system component is installed at time t=0. For t≥0 let $X(t)$=1 as long as the component is functioning, and let $X(t)$=0 after the component fails.

$$S_{14} = \left\{ X(t) : X(t) = \begin{cases} 1 & 0 \le t < t_0 \\ 0 & t \ge t_0 \end{cases}, \ time \ of \ component \ failure \ t_0 \right\}$$

sample spaces visualized as regions of the plane

Sample spaces

- Finite, countably infinite, uncountably infinite
- Discrete sample space
  - *If S is countable*
    - Outcomes can be put in 1-to-1 correspondence with the positive integers
- Continuous sample space
  - *If S is not countable*
- Can be multi-dimensional
  - *Can sometimes be written as Cartesian product of other sets*

Events

- A subset of *S*
  - *Does the outcome satisfy certain conditions?*
    - E_{10}: Determine the value of a voltage waveform at time t_1.

$$S_{10} = \{v : -\infty < v < \infty\} = (-\infty, \infty)$$

  - *Is the voltage negative?*
- Event *A* occurs iff the outcome of the experiment ζ is in the subset

iff = if and only if

$$A = \{\zeta : -\infty < \zeta < 0\}$$

Special events

- Certain event S
  - *Consists of all outcomes, hence occurs always*
- Impossible or null event $\varnothing$
  - *Contains no outcomes, hence never occurs*

Events – example experiments

- E_1: select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.

  "An even-numbered ball is selected" $A_1 = \{2, 4, \ldots, 48, 50\}$

- E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.

  "The ball is white and even-numbered" $A_2 = \{(4, w)\}$

- E_3: Toss a coin three times and note the sequence of heads and tails.

  "The three tosses give the same outcome" $A_3 = \{HHH, TTT\}$

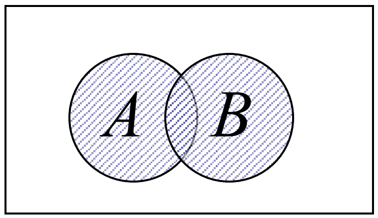
- E_4: Toss a coin three times and note the number of heads.

  "The number of heads equals the number of tails" $A_4 = \varnothing$

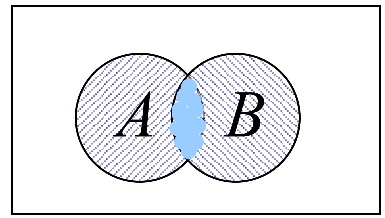
Set (event) operations

- Union $A \bigcup B$
- Intersection $A \bigcap B$
  - *Mutually exclusive* $A \bigcap B = \varnothing$
- Complement $A^c \text{ such that } A^c \bigcup A = S \text{ and } A^c \bigcap A = \varnothing$
- Implies
  - *all outcomes in A are also outcomes in B* $A \subset B$
- Equal
  - *A and B contain the same outcomes* $A = B$

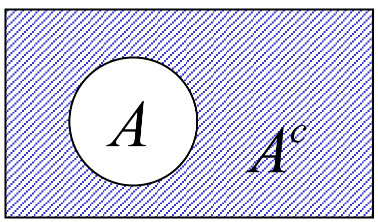
Venn diagrams

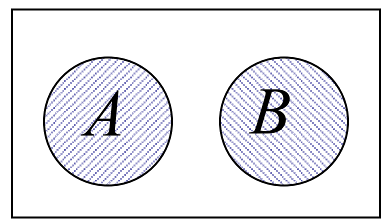


$A \bigcup B$

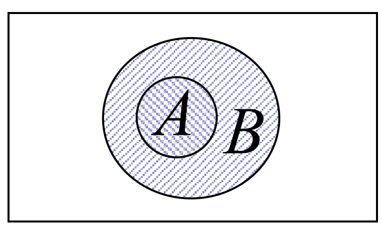


$A \bigcap B$





$A \bigcap B = \varnothing$



$A \subset B$

properties

- Commutative $A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$
- Associative $A \cup (B \cup C) = (A \cup B) \cup C$

  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- deMorgan's rules $(A \cap B)^c = A^c \cup B^c$

  $(A \cup B)^c = A^c \cap B^c$

Repeated union/intersection operations

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n$$

occurs if one or more of the events A_k occur

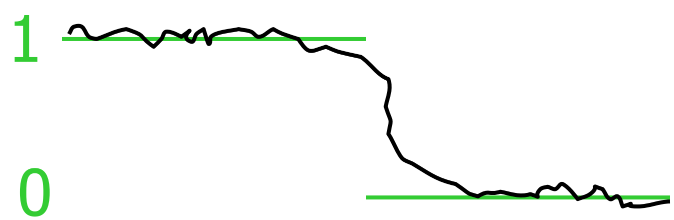
$$\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n$$

occurs if each of the events A_k occur

$$\bigcup_{k=1}^{\infty} A_k \qquad \bigcap_{k=1}^{\infty} A_k$$

Conditional probability

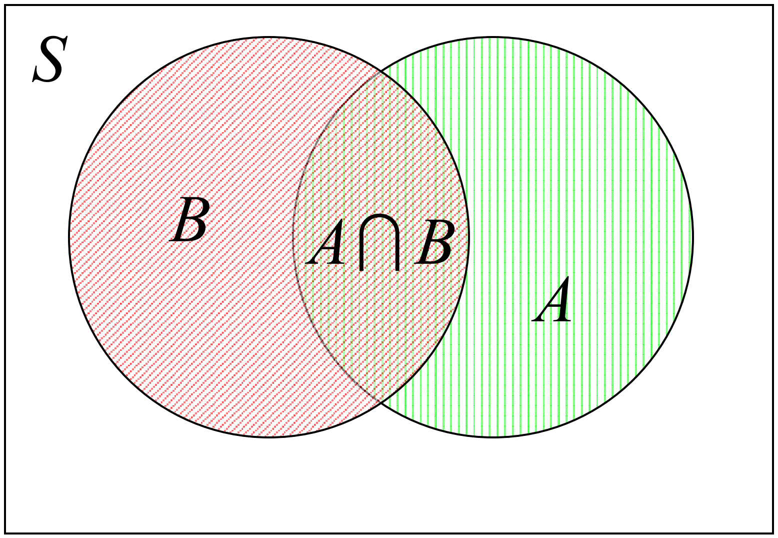
- Are two events *A* and *B* related, in the sense that one tells us something about the other?
  - *We observe/measure something to learn about something else that's not directly measurable*



- Conditional probability of event *A* given that event *B* has occurred

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad for\ P[B] > 0$$

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad \textit{for } P[B] > 0$$



event B has occurred $\zeta \in B$

$$S_{\cdot|B} = B$$

$P[A \mid B]$ deals with $\zeta \in A \cap B$

a renormalization of probability for the reduced sample space

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]} \quad \textit{for } P[B] > 0$$

Ex 2.49 satisfies the Axioms of probability

$$0 \le P[A \cap B], \quad 0 < P[B] \quad \Rightarrow \quad 0 \le P[A \mid B] \qquad \text{Axiom I}$$

$$A \cap B \subset B \quad \Rightarrow \quad P[A \cap B] \le P[B] \quad \Rightarrow \quad P[A \mid B] \le 1$$

$$B \subset S \quad \Rightarrow \quad P[S \mid B] = \frac{P[S \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1 \qquad \text{Axiom II}$$

$$\textit{If } A \cap C = \varnothing \quad \Rightarrow \quad (A \cap B) \cap (C \cap B) = \varnothing$$

$$\textit{then } P[A \cup C \mid B] = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}$$

Axiom III

$$= \frac{P[(A \cap B)] + P[(C \cap B)]}{P[B]} = P[A \mid B] + P[C \mid B]$$

Ex Conditional probability

- Select a ball from an urn containing 2 black balls, labeled 1 and 2, and 2 white balls, labeled 3 and 4

$$S = \{(1,b),(2,b),(3,w),(4,w)\}$$

- Assuming equi-probable outcomes, find *P*[*A*|*B*] and *P*[*A*|*C*], where

  A={(1,b),(2,b)} “black ball selected”

  B={(2,b),(4,w)} “even-numbered ball selected”

  C={(3,w),(4,w)} “number of ball is >2”

$$\left.\begin{aligned} P[A\cap B] &= P[(2,b)] \\ P[A\cap C] &= P[\varnothing] = 0 \end{aligned}\right\}$$

$$P[A\mid B] = \frac{P[A\cap B]}{P[B]} = \frac{.25}{.5} = .5 = P[A]$$

$$P[A\mid C] = \frac{P[A\cap C]}{P[C]} = \frac{0}{.5} = 0 \neq P[A]$$

knowing B doesn’t help, knowing C does

$$P[A \mid B] \triangleq \frac{P[A \cap B]}{P[B]}$$

$$\Downarrow$$

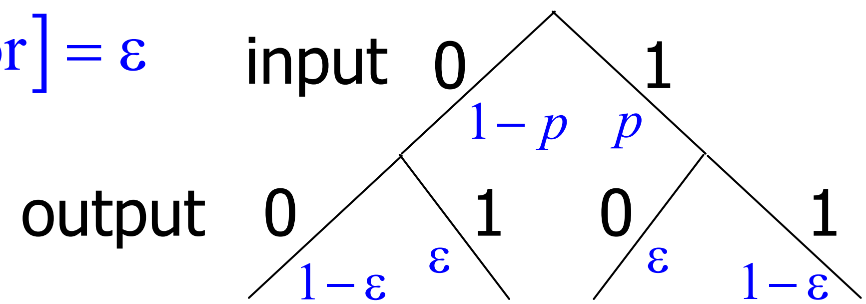
$$P[A \cap B] = P[A \mid B]P[B]$$
$$= P[B \mid A]P[A]$$

Ex Conditional probability

binary communications channel

$$P[1\,\text{sent}] = p$$

$$P[\text{random decision error}] = \varepsilon$$



$$P[T_0 \cap R_0] = P[R_0 \mid T_0]P[T_0] = (1-\varepsilon)(1-p)$$

$$P[T_0 \cap R_1] = P[R_1 \mid T_0]P[T_0] = \varepsilon(1-p)$$

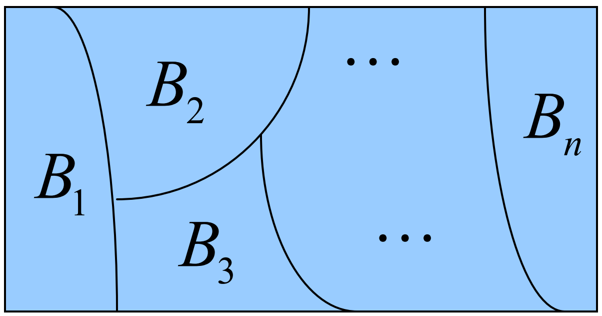
$$P[T_1 \cap R_0] = P[R_0 \mid T_1]P[T_1] = \varepsilon p$$

$$P[T_1 \cap R_1] = P[R_1 \mid T_1]P[T_1] = (1-\varepsilon)p$$

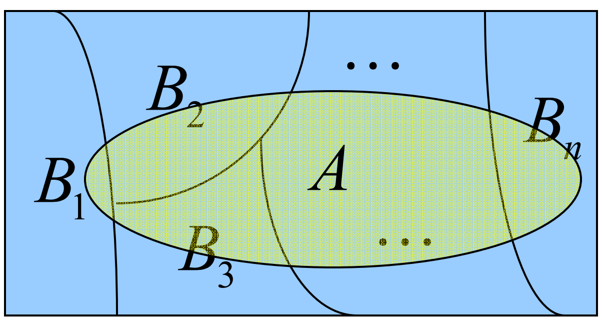
Total probability theorem

a partition of *S:* $\{B_1, B_2, \cdots, B_n\} \ni \bigcup_{i=1}^{n} B_i = S \text{ and } B_i \cap B_j = \varnothing \, \forall i \neq j$

mutually exclusive



$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

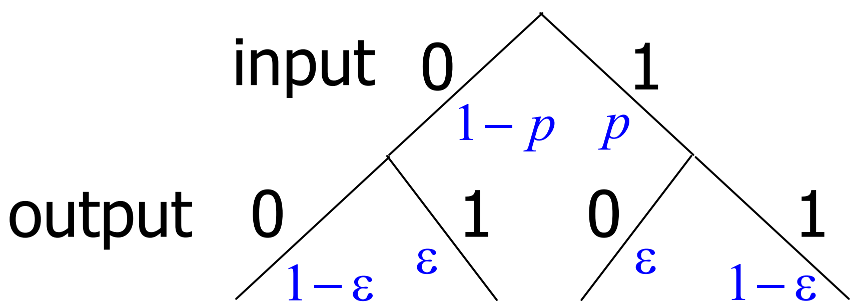


$$P[A] = \sum_{i=1}^{n} P[A \cap B_i] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i]$$

Ex

$$P[A] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i]$$

Very useful when experiment consists of a sequence of 2 subexperiments



T_0 and T_1 partition S

$$P[R_1] = P[R_1 \mid T_1] P[T_1] + P[R_1 \mid T_0] P[T_0]$$
$$= (1-\varepsilon) p + \varepsilon (1-p)$$
$$P[R_0] = P[R_0 \mid T_1] P[T_1] + P[R_0 \mid T_0] P[T_0]$$
$$= \varepsilon p + (1-\varepsilon)(1-p)$$

Bayes' Rule

Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of S then

$$P\left[B_j \mid A\right] = \frac{P\left[B_j \cap A\right]}{P\left[A\right]} = \frac{P\left[A \mid B_j\right] P\left[B_j\right]}{\sum_{k=1}^{n} P\left[A \mid B_k\right] P\left[B_k\right]}$$

a posteriori probability

the partition corresponds to a priori events (of interest)
A corresponds to a measurement/observation

Ex

- A 1 is received; what is the probability that a 1 was transmitted?

$$P[T_1 \mid R_1] = \frac{P[R_1 \mid T_1]P[T_1]}{P[R_1]} = \frac{(1-\varepsilon)p}{(1-\varepsilon)p + \varepsilon(1-p)} \overset{p=0.5}{=} \frac{(1-\varepsilon)/2}{1/2} = (1-\varepsilon)$$

Bayes' Rule

total probability $P[R_1] = P[R_1 \mid T_1]P[T_1] + P[R_1 \mid T_0]P[T_0]$

$$= (1-\varepsilon)p + \varepsilon(1-p)$$

$$P[T_0 \mid R_1] = \frac{P[R_1 \mid T_0]P[T_0]}{P[R_1]} = \frac{\varepsilon(1-p)}{(1-\varepsilon)p + \varepsilon(1-p)} \overset{p=0.5}{=} \frac{\varepsilon/2}{1/2} = \varepsilon$$

Independence of events

- Knowledge of the occurrence of event *B* does not alter the probability of some other event *A*
  - *A does not depend on B*

$$P[A] = P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

events A and B are independent if

$$P[A \cap B] = P[A]P[B]$$

← problematic if $P[B] = 0$

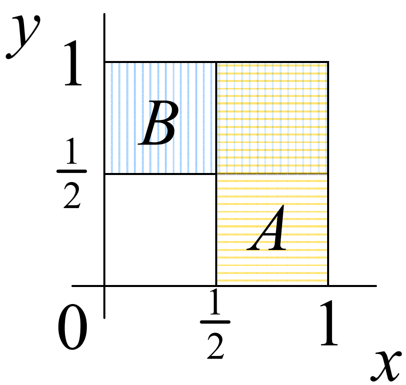
$\Downarrow\Uparrow$ $P[B] \neq 0$ $\qquad$ $\Downarrow\Uparrow$ $P[A] \neq 0$

$$P[A \mid B] = P[A] \qquad P[B \mid A] = P[B]$$

Ex 2-D continuous

two numbers x and y are selected at random between 0 and 1

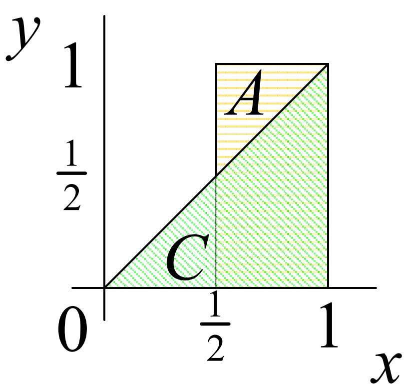
events $A = \{x > 0.5\}$, $B = \{y > 0.5\}$, $C = \{x > y\}$



$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{1/4}{1/2} = \frac{1}{2} = P[A]$$

A and *B* are independent

ratio of proportions has remained the same



$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{3/8}{1/2} = \frac{3}{4} \neq \frac{1}{2} = P[A]$$

ratio of proportions has increased

i.e. we gained "knowledge" from measuring *C*

Independence of *A*, *B*, and *C*

1. pairwise independence:
$$\begin{cases} P[A\cap B]=P[A]P[B] \\ P[A\cap C]=P[A]P[C] \\ P[B\cap C]=P[B]P[C] \end{cases}$$

2. knowledge of joint occurrence, of any two, does not affect the third: $P[C\mid A\cap B]=P[C]$

$$\frac{P[A\cap B\cap C]}{P[A\cap B]}=P[C]$$

$$P[A\cap B\cap C]=P[A\cap B]P[C]$$
$$=P[A]P[B]P[C]$$

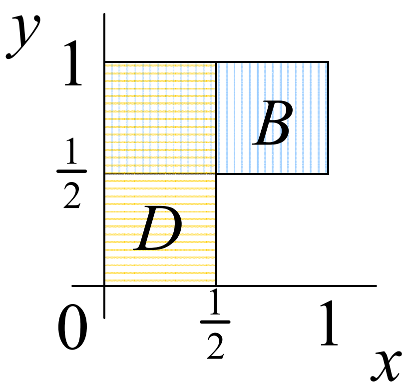
A,B,C are independent
if the probability of the intersection
of any pair or triplet of events
equals the product of the probabilities
of the individual events

Ex pairwise independence is not enough

two numbers x and y are selected at random between 0 and 1

events: $B = \{y > 0.5\}$, $D = \{x < 0.5\}$

$$F = \{x < 0.5; y < 0.5\} \cup \{x > 0.5; y > 0.5\}$$

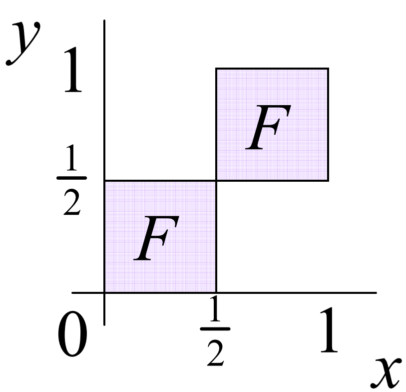


$$P[B \cap D] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[D]$$

$$P[B \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[B]P[F]$$

$$P[D \cap F] = \frac{1}{4} = \frac{1}{2}\frac{1}{2} = P[D]P[F]$$

pairwise independent



$$P[B \cap D \cap F] = P[\varnothing] = 0$$

$$P[B]P[D]P[F] = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$$

violates 2^{nd} condition

Independence of *n* events

- Probability of an event is not affected by the joint occurrence of any subset of the other events

events $B_1, B_2, \cdots, B_n$

are said to be independent if for $k=2,\ldots,n$

$$P\left[B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}\right] = P\left[B_{i_1}\right] P\left[B_{i_2}\right] \cdots P\left[B_{i_k}\right]$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n$

$2^n - n - 1$ possible intersections to evaluate! Δ

- Assuming independence of the events of separate experiments is more common
  - *E.g. one coin toss is independent of any before/after*

independent experiments

Sequential experiments

- Many random experiments can be viewed as sequential experiments consisting of a sequence of simpler subexperiments
- The subexperiments may, or may not, be independent

Sequences of independent experiments

- Random experiment consisting of performing experiments E_1, $E_2,\ldots,E_n$
- Outcome is an *n*-tuple $s=(s_1,s_2,\ldots,s_n)$ where s_k is the outcome of the *k*-th subexperiment
- Sample space of sequential experiment contains *n*-tuples; it is denoted by the Cartesian product of the individual sample spaces $S_1 x S_2 x \ldots x S_n$
- Physical considerations will often indicate that the outcome of any given subexperiment cannot affect the outcomes of the other subexperiments; it is then reasonable to assume that the events $A_1,A_2,\ldots,A_n$ – where A_k concerns only the *k*-th subexperiment – are independent:

$$P\left[A_1 \cap A_2 \cap \cdots \cap A_n\right] = P\left[A_1\right]P\left[A_2\right]\cdots P\left[A_n\right]$$

facilitates computing all probabilities of events of the sequential experiment

Ex

- Select 10 numbers at random from [0,1]
- Find P[first 5 #'s <1/4, last 5 #'s >1/2]

$$A_k = \left\{ x < \frac{1}{4} \right\} \text{ for } k = 1, \ldots, 5 \qquad A_k = \left\{ x > \frac{1}{2} \right\} \text{ for } k = 6, \ldots, 10$$

- Assuming selection of a number is independent of other selections

$$P\left[A_1 \cap A_2 \cap \cdots \cap A_n\right] = P\left[A_1\right] P\left[A_2\right] \cdots P\left[A_n\right] = \left(\frac{1}{4}\right)^5 \left(\frac{1}{2}\right)^5$$

Bernoulli trial

- Perform an experiment once; note whether event *A* occurs (if so, it's called a "success," if not, it's called a "failure")
- Find probability of *k* successes in *n* independent repetitions of a Bernoulli trial
  - *It's like coin tosses, with an unfair coin*

$$P[A] = p$$

Binomial probability law

- Probability of *k* successes in *n* independent Bernoulli trials

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, \cdots, n$$

"success in *k* particular positions & failure elsewhere"

# of distinct ways, $N_n(k)$, that such sequences can be chosen

"*n* choose *k*" $\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$ binomial coefficient