Diversity techniques
for
flat fading channels

- BER vs. S/N in a flat fading channel
- Different kinds of diversity techniques
- Selection diversity
- Maximum ratio combining (MRC)

Baseline: AWGN Channel

$$y = x + w$$

$$x = \pm a$$

BPSK $p_e = Q\left(\frac{a}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2\mathsf{SNR}}\right)$

$$\mathsf{SNR} := \frac{a^2}{N_0}$$

BER vs. S/N in a flat fading channel

In a flat fading channel (or narrowband system), the CIR (channel impulse response) reduces to a single impulse scaled by a time-varying complex coefficient.

The received (equivalent lowpass) signal is of the form

$$r(t) = a(t)\, e^{j\phi(t)}\, s(t) + n(t)$$

We assume that the phase changes "slowly" and can be perfectly tracked

=> important for coherent detection

BER vs. S/N (cont.)

We assume:

- the time-variant complex channel coefficient changes slowly (=> constant during a symbol interval)
- the channel coefficient magnitude (= attenuation factor) is Rayleigh distributed
- coherent detection of a binary PSK signal (assuming ideal phase synchronization)

Let us define instantaneous S/N and average S/N:

$$\gamma = a^2 E_b / N_0 \qquad \gamma_0 = E\{a^2\} \cdot E_b / N_0$$

BER vs. S/N (cont.)

Since

$$p(a) = \frac{2a}{E\{a^2\}} e^{-a^2 / E\{a^2\}} \qquad a \geq 0,$$

Rayleigh distribution

using

$$p_\gamma(\gamma) = \frac{p_a(a)}{|d\gamma / da|}$$

Exponential distribution

we get

$$p(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \qquad \gamma \geq 0.$$

BER vs. S/N (cont.)

The average bit error probability is

$$P_e = \int_0^\infty P_e(\gamma)\, p(\gamma)\, d\gamma$$

Important formula for obtaining statistical average

where the bit error probability for a certain value of a is

$$P_e(\gamma) = Q\left(\sqrt{2a^2 E_b / N_0}\right) = Q\left(\sqrt{2\gamma}\right).$$

2-PSK

We thus get

$$P_e = \int_0^\infty Q\left(\sqrt{2\gamma}\right) \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} d\gamma = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right).$$

BER vs. S/N (cont.)

Approximation for large values of average S/N is obtained in the following way. First, we write

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) = \frac{1}{2}\left(1 - \sqrt{1 + \frac{-1}{1+\gamma_0}}\right)$$

Then, we use

$$\sqrt{1+x} = 1 + x/2 + \ldots$$

which leads to

$$P_e \approx 1/4\gamma_0$$ for large γ_0 .

Rayleigh Flat Fading Channel

$$y = hx + w$$

$$h \sim \mathcal{CN}(0, 1)$$

BPSK: $x = \pm a$. Coherent detection.

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$$

Averaged over h,

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1 + \mathsf{SNR}}}\right) \approx \frac{1}{4\mathsf{SNR}}$$

at high SNR.

Typical Error Event

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)$$

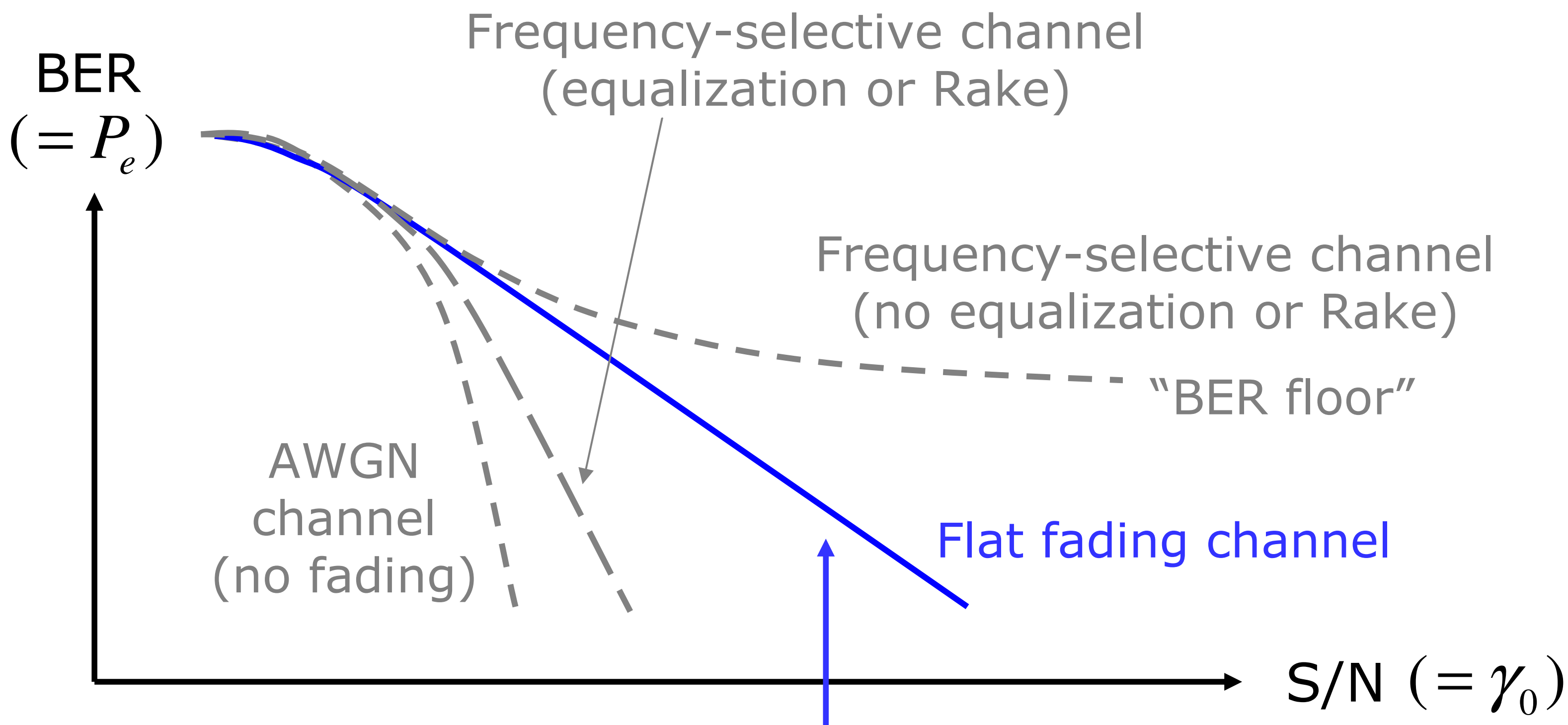
When $|h|^2 >> \frac{1}{\mathsf{SNR}}$, error probability is very small.
When $|h|^2 < \frac{1}{\mathsf{SNR}}$, error probability is large:

$$p_e \approx P\left(|h|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{\mathsf{SNR}}$$

$$|h|^2 \sim \mathsf{exp}(1).$$

Typical error event is due to channel being in deep fade rather than noise being large.

BER vs. S/N (cont.)



$P_e \approx 1/4\gamma_0$ means a straight line in log/log scale

BER vs. S/N for different modulation methods

Modulation	$P_e(\gamma)$	P_e	P_e (for large γ_0)
2-PSK	$Q\left(\sqrt{2\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{1+\gamma_0}}\right)$	$1/4\gamma_0$
DPSK	$e^{-\gamma}/2$	$1/(2\gamma_0+2)$	$1/2\gamma_0$
2-FSK (coh.)	$Q\left(\sqrt{\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)$	$1/2\gamma_0$
2-FSK (non-c.)	$e^{-\gamma/2}/2$	$1/(\gamma_0+2)$	$1/\gamma_0$

Objective of Diversity

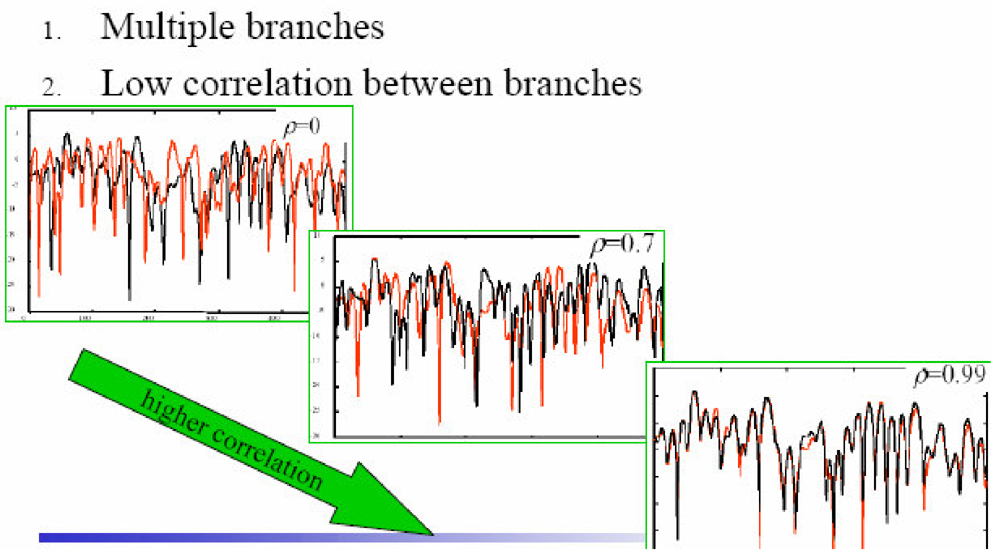
If diversity is not employed, the resulting efficiency would be very low, as it can be deduced from the comparison of AWGN vs. Rayleigh channel BER.

Diversity refers to transmitting and/or receiving the same information via different (preferably independent) ways.

Diversity combats fading and improves the BER performance which

– directly translates to power savings,

– increased system capacity.



Better performance through diversity

Diversity ⇔ the receiver is provided with multiple copies of the transmitted signal. The multiple signal copies should experience *uncorrelated fading* in the channel.

In this case the probability that *all* signal copies fade simultaneously is reduced dramatically with respect to the probability that a *single* copy experiences a fade.

As a rough rule:

P_e is proportional to $\dfrac{1}{\gamma_0^{\,L}}$

Diversity of *L*:th order

BER

Average S/N

Different kinds of diversity methods

Space diversity:

Several receiving antennas spaced sufficiently far apart (separation should be large to reduce correlation between diversity branches).

Time diversity:

Transmission of same signal sequence at different times (time separation should be larger than coherence time).

Frequency diversity:

Transmission of same signal at different frequencies (frequency separation should be larger than coherence bandwidth).

Diversity methods (cont.)

Polarization diversity:

Only two diversity branches are available.

Multipath diversity:

- Equalization (W-TDMA) can be considered a kind of utilization of multipath diversity
- Delay discrimination (RAKE receiver in CDMA)
- Doppler discrimination (questionable in a practical system)
- Angle discrimination (directional antennas).

Selection diversity vs. signal combining

Selection diversity: Signal with best quality is selected.

Equal Gain Combining (EGC)

Signal copies are combined coherently:

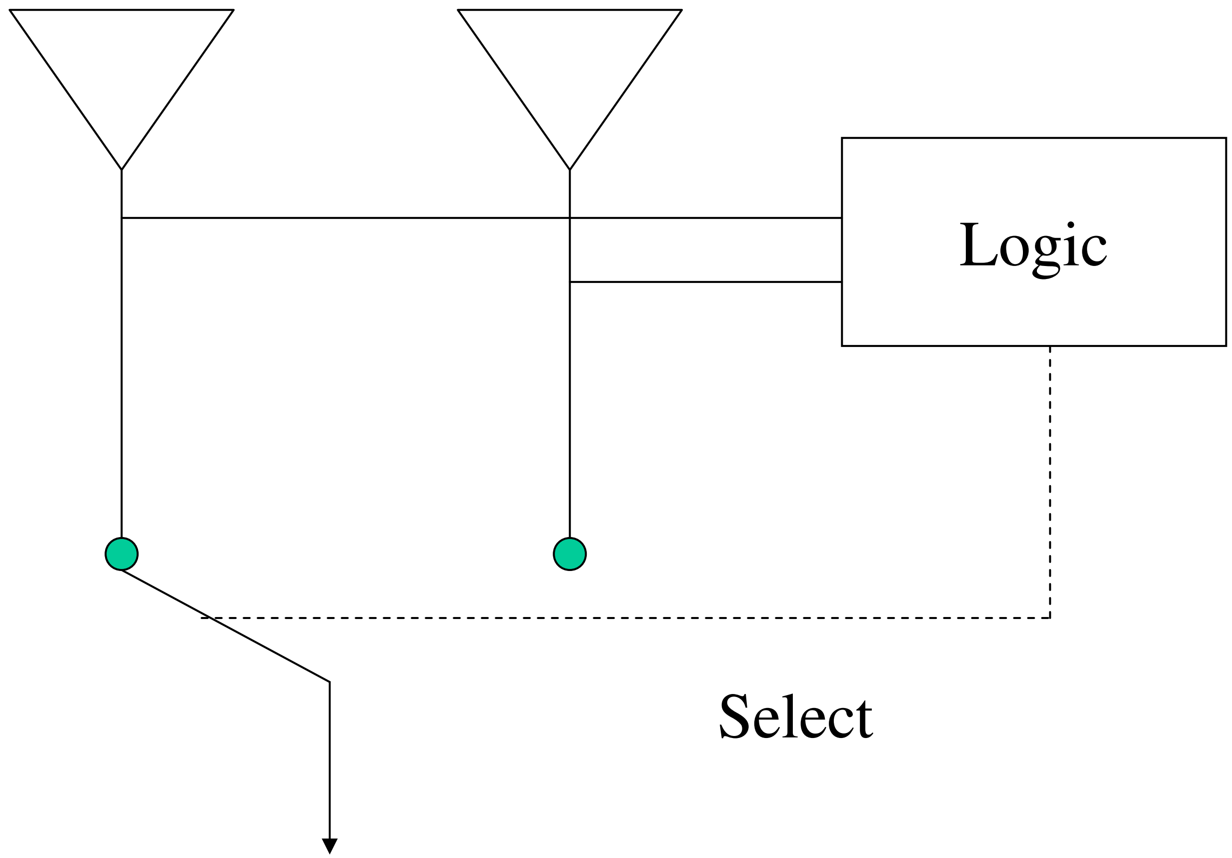
$$Z_{EGC} = \sum_{i=1}^{L} a_i \, e^{j\phi_i} e^{-j\phi_i} = \sum_{i=1}^{L} a_i$$

Maximum Ratio Combining (MRC, best S/N is achieved)

Signal copies are weighted and combined coherently:

$$Z_{MRC} = \sum_{i=1}^{L} a_i \, e^{j\phi_i} \, a_i \, e^{-j\phi_i} = \sum_{i=1}^{L} a_i^{\,2}$$

Selection Combining



Selection diversity

We assume:

(*a*) uncorrelated fading in diversity branches

(*b*) fading in *i*:th branch is Rayleigh distributed

(*c*) => S/N is exponentially distributed:

$$p(\gamma_i) = \frac{1}{\gamma_0} e^{-\gamma_i/\gamma_0}, \quad \gamma_i \geq 0.$$

Probability that S/N in branch *i* is less than threshold y :

$$P(\gamma_i < y) = \int_0^y p(\gamma_i)\, d\gamma_i = 1 - e^{-y/\gamma_0} .$$

Selection diversity (cont.)

Probability that S/N in every branch (i.e. all *L* branches) is less than threshold y :

$$P\left(\gamma_1, \gamma_2, \ldots, \gamma_L < y\right) = \left[\int_0^y p\left(\gamma_i\right) d\gamma_i\right]^L = \left[1 - e^{-y/\gamma_0}\right]^L$$

Note: this is true only if the fading in different branches is independent (and thus uncorrelated) and we can write

$$p\left(\gamma_1, \gamma_2, \ldots, \gamma_L\right) = p\left(\gamma_1\right) p\left(\gamma_2\right) \ldots p\left(\gamma_L\right).$$

Selection diversity (cont.)

Differentiating the cdf (cumulative distribution function) with respect to y gives the pdf

$$p(y) = L\left[1 - e^{-y/\gamma_0}\right]^{L-1} \cdot \frac{e^{-y/\gamma_0}}{\gamma_0}$$

which can be inserted into the expression for average bit error probability

$$P_e = \int_0^\infty P_e(y)\, p(y)\, dy\,.$$

The mathematics is unfortunately quite tedious ...

Selection diversity (cont.)

… but as a general rule, for large γ_0 it can be shown that

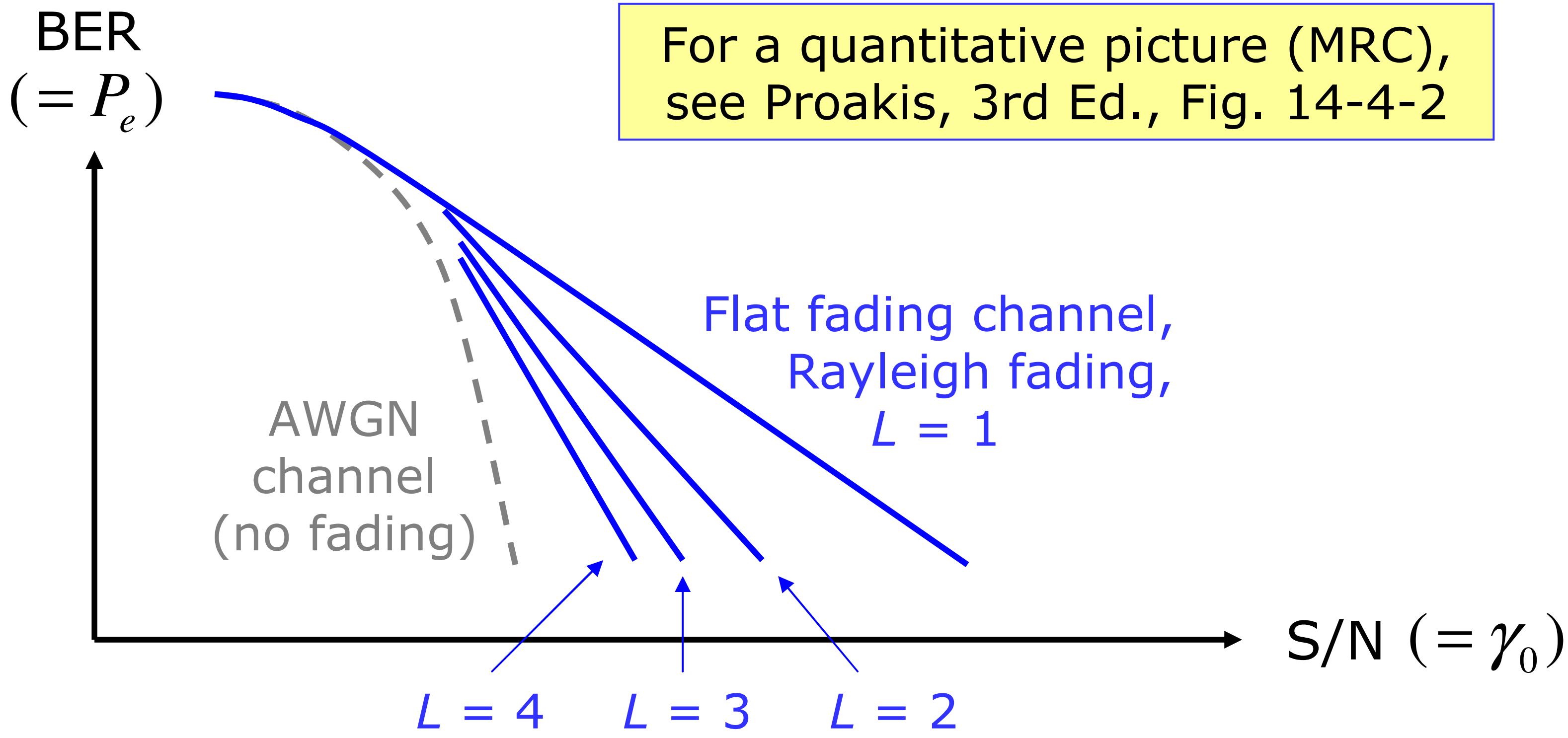
$$P_e \text{ is proportional to } \frac{1}{\gamma_0^{\ L}}$$

regardless of modulation scheme (2-PSK, DPSK, 2-FSK).

The largest diversity gain is obtained when moving from *L* = 1 to *L* = 2. Successively smaller gains are obtained when *L* is further increased.

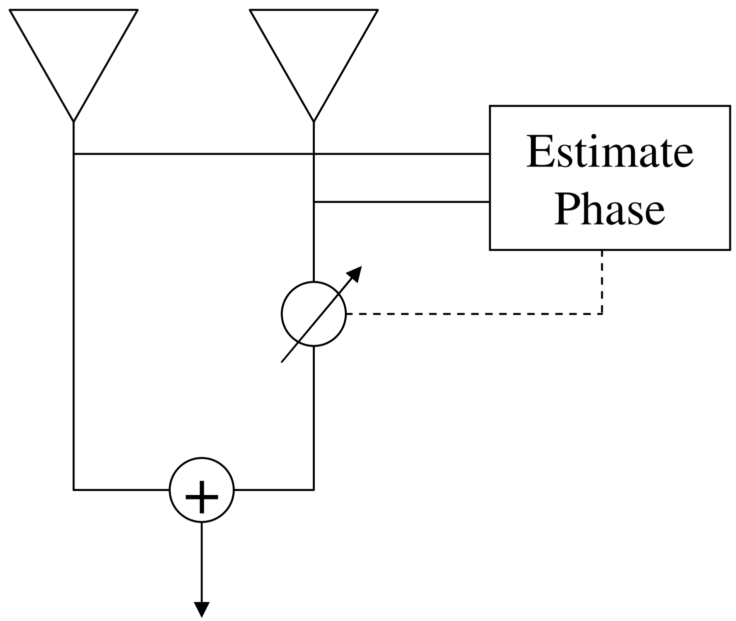
This behaviour is typical for all diversity techniques.

BER vs. S/N (diversity effect)

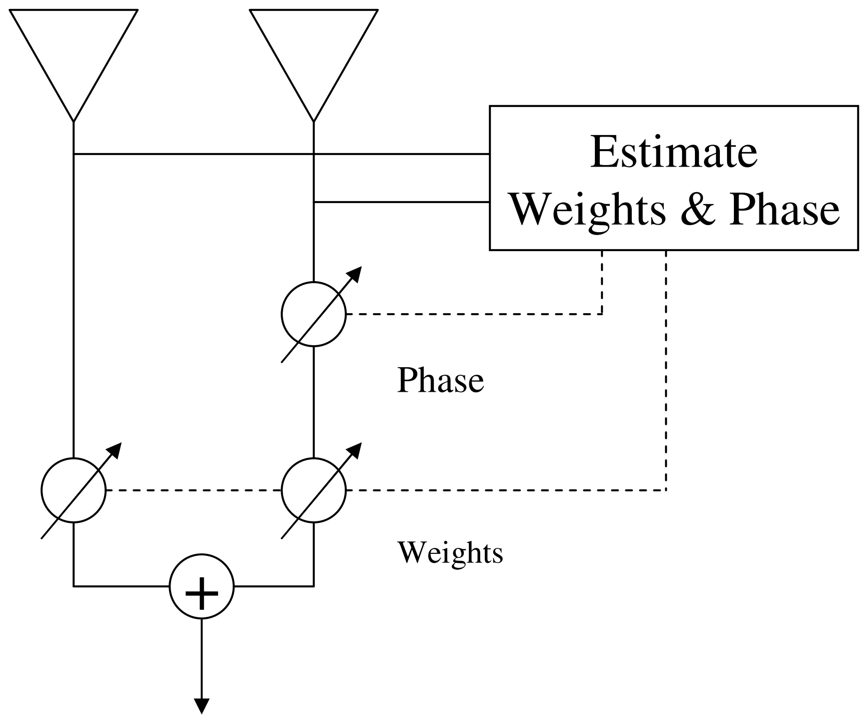


For a quantitative picture (MRC),
see Proakis, 3rd Ed., Fig. 14-4-2

Equal Gain Combining



Maximal Ratio Combining



Why is MRC optimum peformance?

Let us investigate the performance of a signal combining method in general using arbitrary weighting coefficients g_i.

Signal magnitude and noise energy/bit at the output of the combining circuit:

$$Z = \sum_{i=1}^{L} g_i \cdot a_i \qquad\qquad N_t = N_0 \sum_{i=1}^{L} g_i^{\,2}$$

S/N after combining:

$$\gamma = \frac{Z^2 E_b}{N_t} = \frac{E_b \left(\sum g_i a_i \right)^2}{N_0 \sum g_i^{\,2}}$$

Why is MRC optimum peformance? (cont.)

Applying the Schwarz inequality

$$\left(\sum g_i a_i\right)^2 \leq \sum g_i{}^2 \sum a_i{}^2$$

it can be easily shown that in case of equality we must have $g_i = a_i$ which in fact is the definition of MRC.

Thus for MRC the following important rule applies (the rule also applies to SIR = Signal-to-Interference Ratio):

$$\gamma = \sum_{i=1}^{L} \gamma_i$$

Output S/N = sum of branch S/N values

MRC performance

Rayleigh fading => S/N in *i*:th diversity branch is

$$\gamma_i = \frac{E_b}{N_0} a_i^{\,2} = \frac{E_b}{N_0}\left(x_i^{\,2} + y_i^{\,2} \right)$$

Rayleigh distributed magnitude

Gaussian distributed quadrature components

In case of *L* uncorrelated branches with same fading statistics, the MRC output S/N is

$$\gamma = \frac{E_b}{N_0}\left(a_1^{\,2} + a_2^{\,2} \ldots + a_L^{\,2} \right) = \frac{E_b}{N_0}\left(x_1^{\,2} + y_1^{\,2} \ldots + x_L^{\,2} + y_L^{\,2} \right)$$

MRC performance (cont.)

The pdf of γ follows the *chi-square distribution* with 2*L* degrees of freedom

Reduces to exponential pdf when *L* = 1

$$p(\gamma) = \frac{\gamma^{L-1}}{{\gamma_0}^L \Gamma(L)} e^{-\gamma/\gamma_o} = \frac{\gamma^{L-1}}{{\gamma_0}^L (L-1)!} e^{-\gamma/\gamma_o}$$

Gamma function — Factorial

For 2-PSK, the average BER is

$$P_e = \int_0^\infty P_e(\gamma)\, p(\gamma)\, d\gamma$$

$$P_e = \left(\frac{1-\mu}{2}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k$$

$$P_e(\gamma) = Q\left(\sqrt{2\gamma}\right)$$

$$\mu = \sqrt{\gamma_0 / (1+\gamma_0)}$$

MRC performance (cont.)

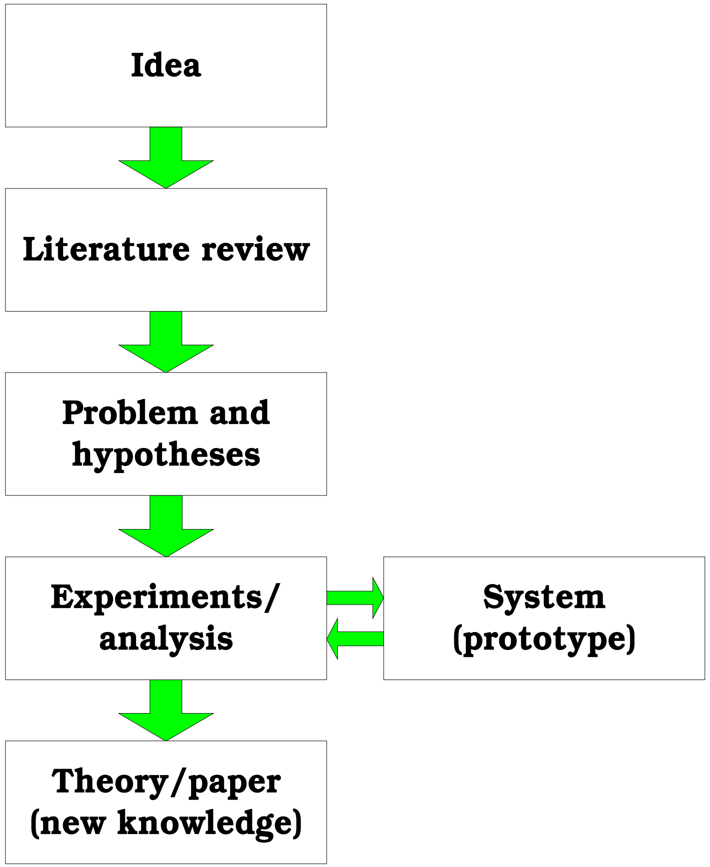
For large values of average S/N this expression can be approximated by

$$p(\gamma) = \left(\frac{1}{4\gamma_0}\right)^L \binom{2L-1}{L}$$

which again is according to the general rule

P_e is proportional to $\frac{1}{\gamma_0{}^L}$.

RESEARCH METHODS: From Problem and Hypothesis to Experiments



SOME DEFINITIONS

- *Research:* Careful study or investigation to discover new knowledge
    - *basic research* (no specific application in mind)
    - *applied research* (ideas into operational form)
- *Development:* Systematic use of the existing knowledge

- Note. Research and development are closely related. In research a prototype is often developed.

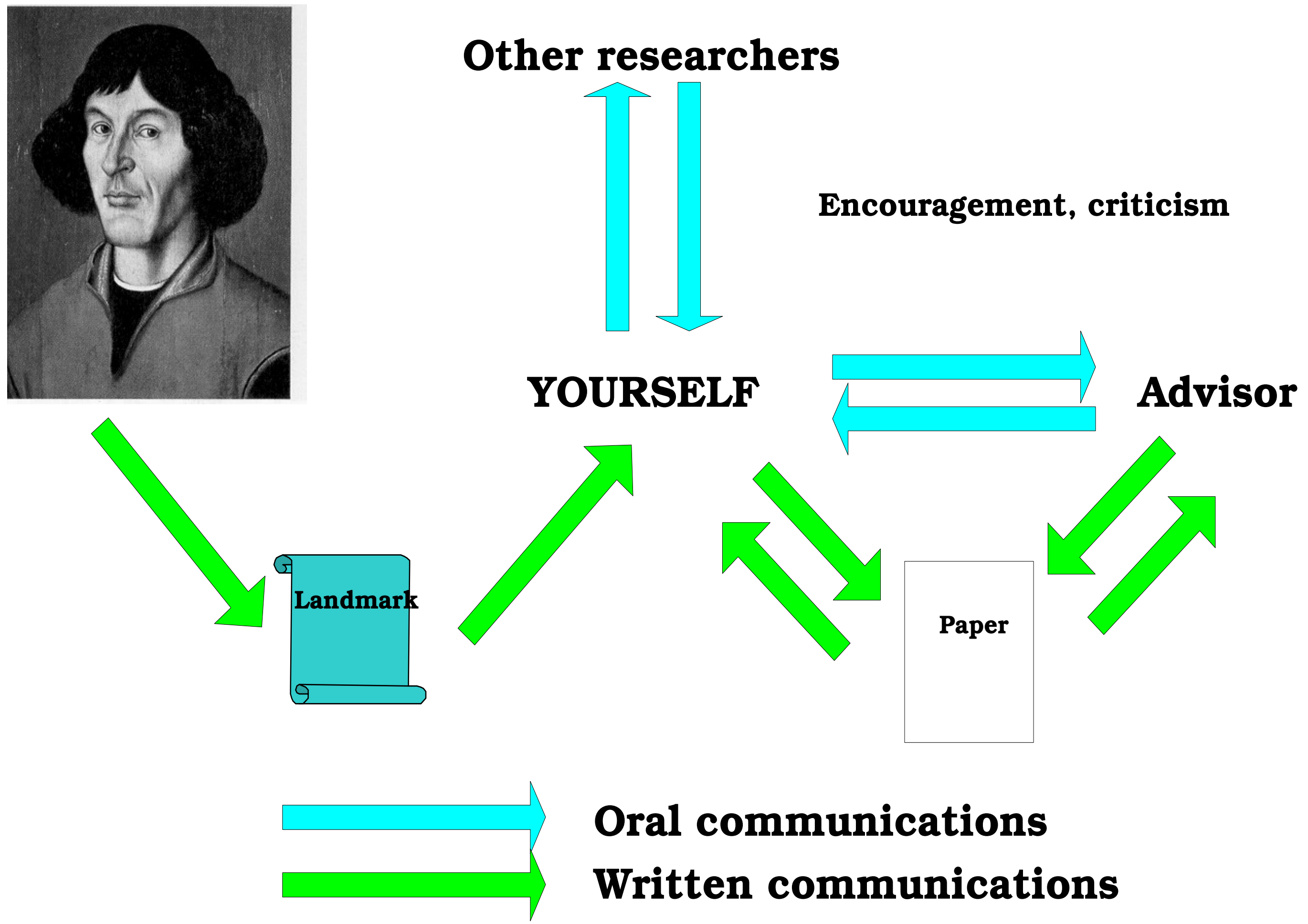
HOW DOES A RESEARCHER WORK?

1. Make always notes in a notebook (day book)

2. Make plans for the future all the time (outlines, roadmaps)

3. Discuss, ask questions and argue (criticism)

COMMUNICATIONS IMPROVE CREATIVITY



RESEARCH IDEAS

To find research ideas, use your own *intuition/expertise* and..

- know the *literature*, especially original landmark papers (write *brief* well-organized summaries)

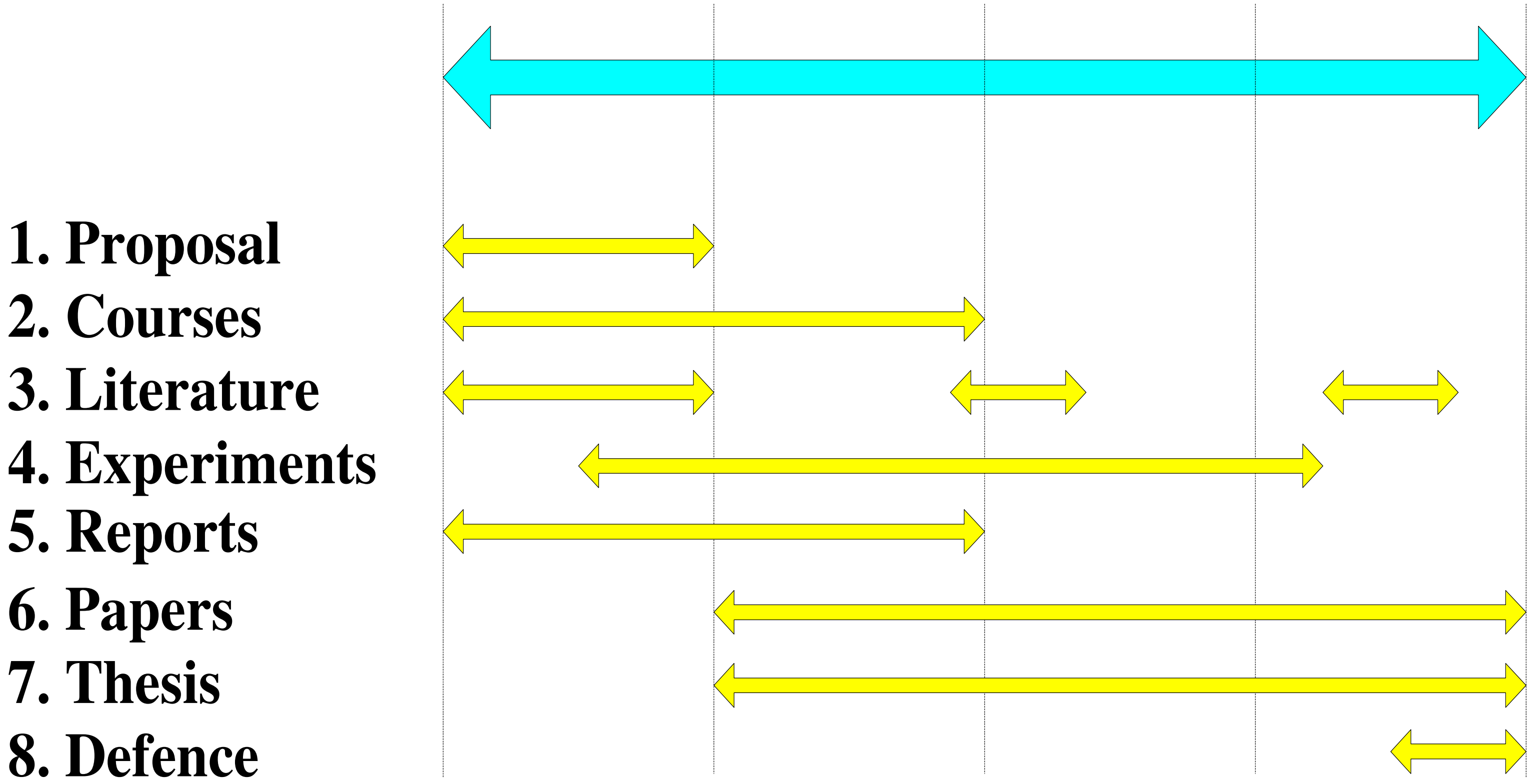
- do *experiments* early in your studies, use your colleagues' experience

- *discuss* with colleagues and students and teach them (seminars)

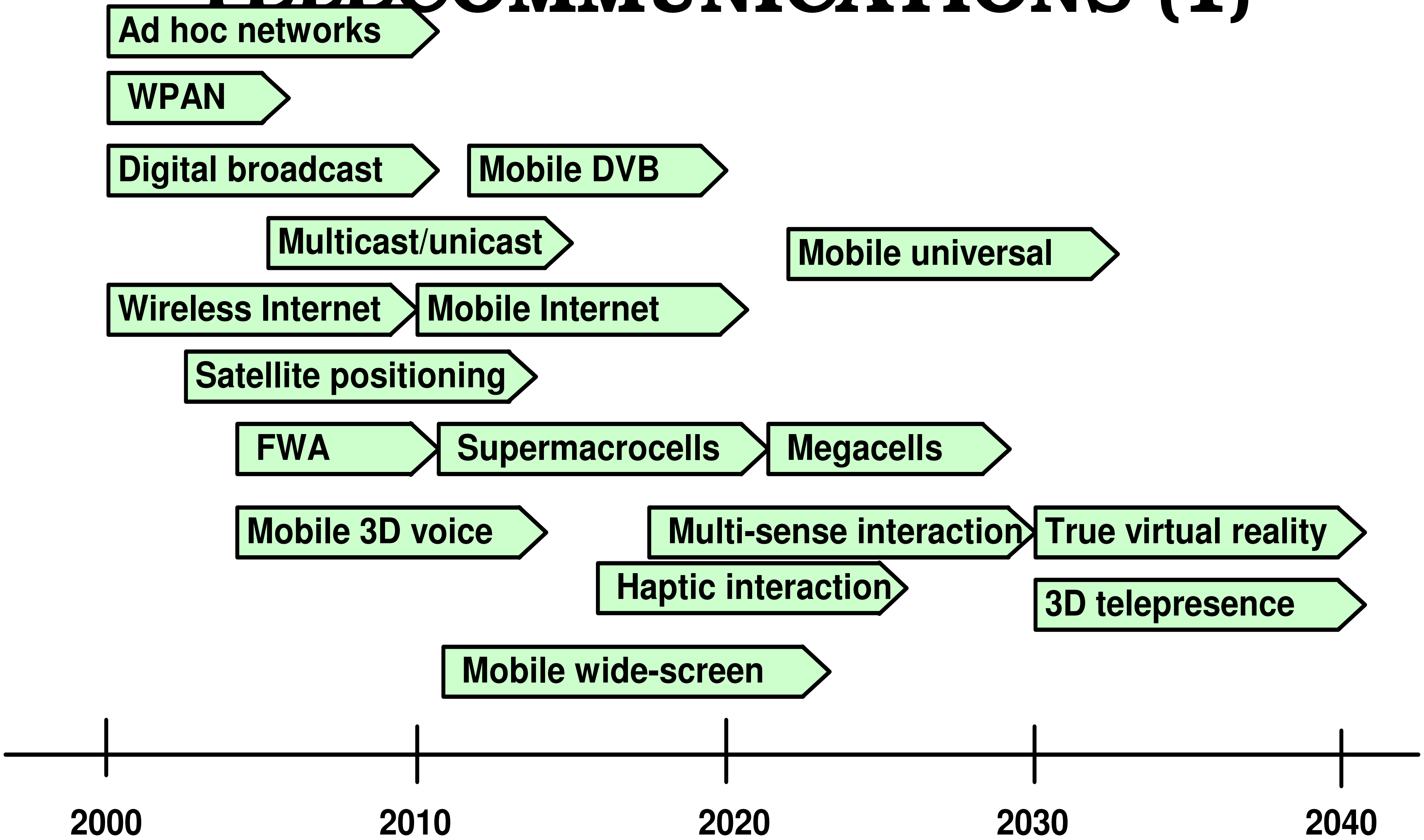
RESEARCH PROPOSAL

- Abstract
- Introduction
  - problem and hypothesis
- Review of the literature
  - good organization, concept analysis
- Materials and methods
  - system requirements, system specifications
  - plan for operation, experimental procedures
  - analytical and other tools
- Results
  - results (for example experimental data) to be expected
  - *publication* and other dissemination of research results
- Discussion and conclusions
  - originality, open questions, limitations
  - validation, significance, applications
- Time frame, budget
  - *intermediate objectives*
- Bibliography
  - list of references

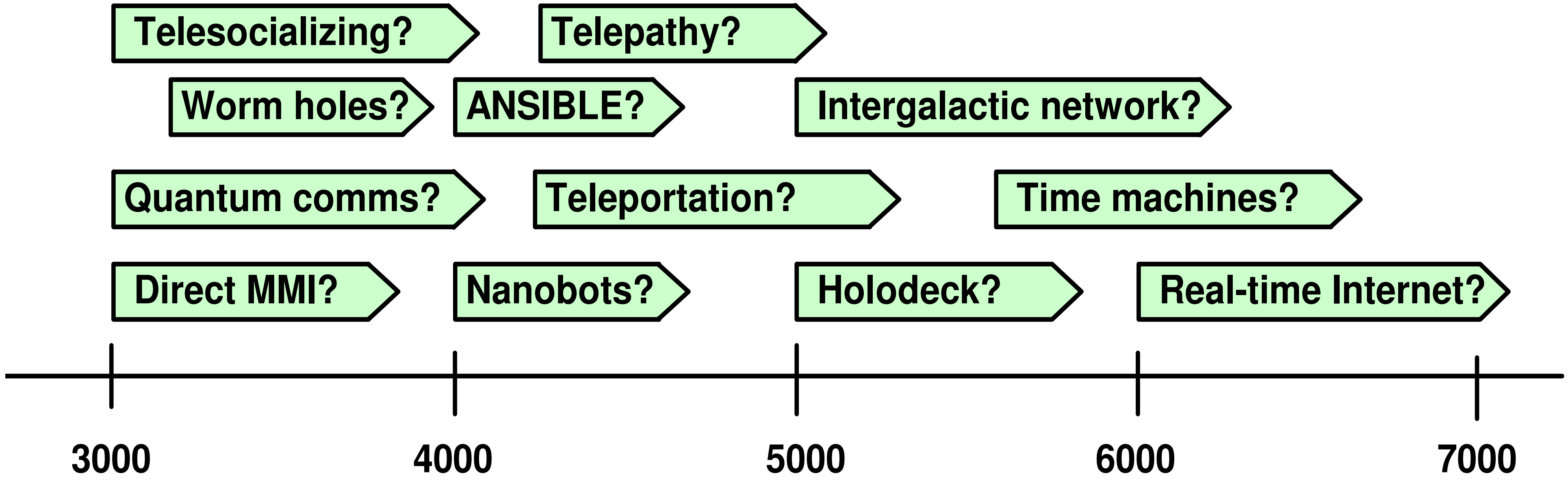
TIMING OF DOCTORAL THESIS (4 years)



ROADMAP AND VISION OF TELECOMMUNICATIONS (1)



ROADMAP AND VISION OF TELECOMMUNICATIONS (2)



HOW TO START

1. Find a suitable *advisor* and a good group.

2. Look for a good *idea*, study literature & discuss, do not reinvent the wheel.

3. Define the *problem*, limit the scope, find the right approach and *hypotheses* (= possible solutions), write a *research proposal*.

4. Analyze the system, make *experiments* (simulations, prototypes) and discuss the results, use right tools.

5. Write a *paper* or thesis and listen carefully to comments and be prepared to argue and defend your claims (opponents try to find weak points in your reasoning!).

Advisor is your best friend

Look for a good advisor

- Be there for the length of your project
- Experience on research in the same area (a doctor)
- Pedagogical skills, know the big picture, know literature
- Respected by colleagues, critical, tough methodologist
- Interested in your topic, gives comments, you respect him/her

How to keep your advisor?

- Orient your advisor (system model, block diagrams, table of contents)
- Follow instructions (make notes), but also discuss and argue
- Make concise progress reports (organize the material, limit the scope)
- Do not expect ready-made solutions, but ways of thinking
- Advisor needs also credit for his/her work in the form of publications
- Get into the driver’s seat!