EC 7xx Wireless Communications
Spring 2007

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Overview, Probabilities, Random variables, Random process

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Syllabus

- Tentatively

Week 1	Overview, Probabilities, Random variables, Random process
Week 2	Wireless channels, Statistical Channel modelling, Path loss models
Week 3	Cellular concept and system design fundamentals
Week 4	Modulation techniques, single and multi-carrier
Week 5	Diversity techniques
Week 6	Equalization techniques
Week 7	Mid Term exam
Week 8	802.11 and Mac evaluation
Week 9	Energy models in 802.11
Week 10	Wimax and Mac layer
Week 11	Presentations
Week 12	Presentations
Week 13	Presentations
Week 14	Presentations
Week 15	Final Exam

Grades

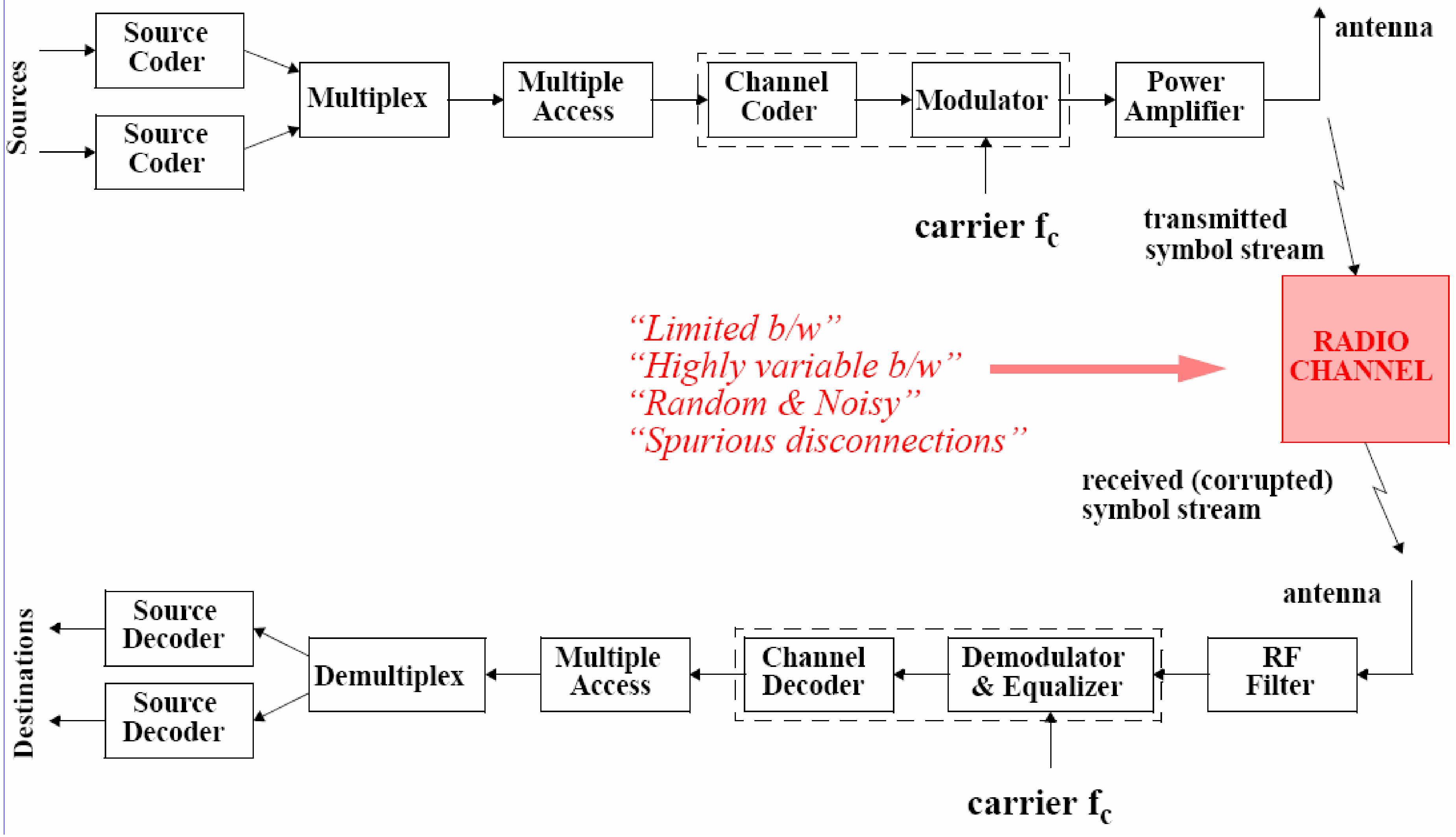
Load	Percentage	Date
7th Week Exam	25%	24 April 2007
Final Exam	40%	
Participation	10%	
Presentation and Report	25%	10th week and up

Presentation and Report

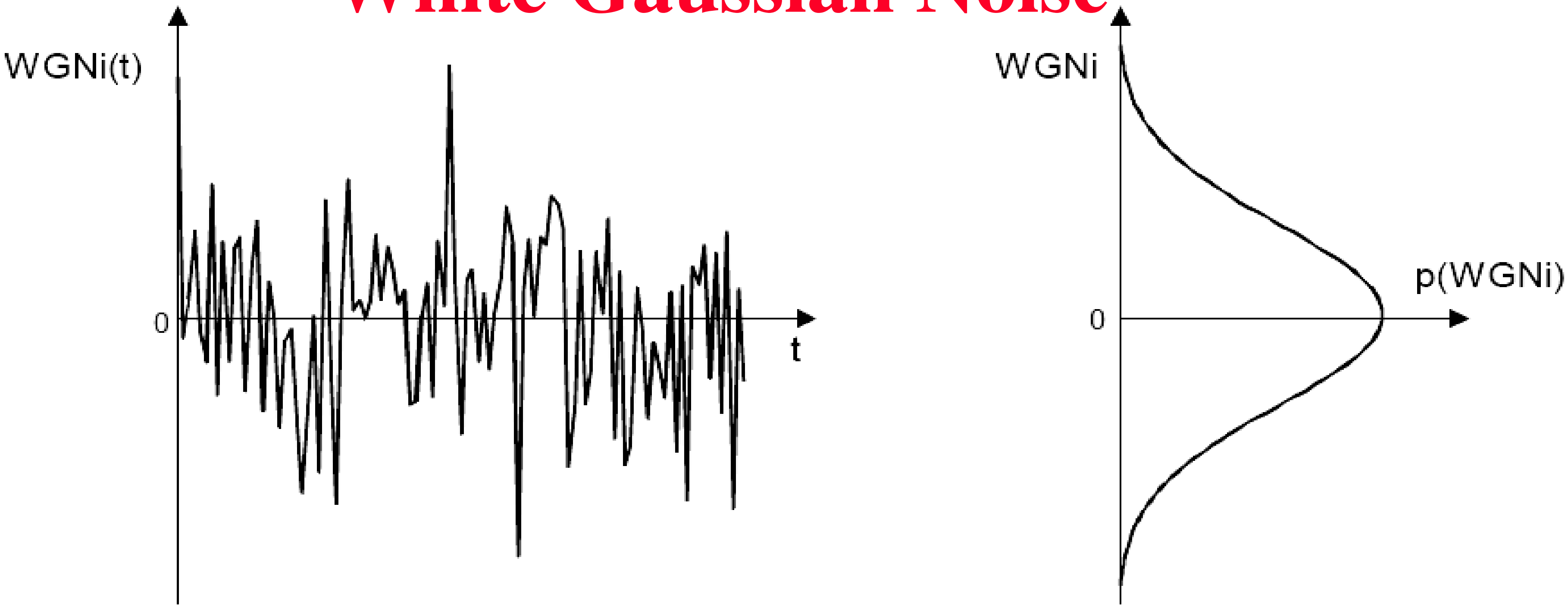
- Roadmap

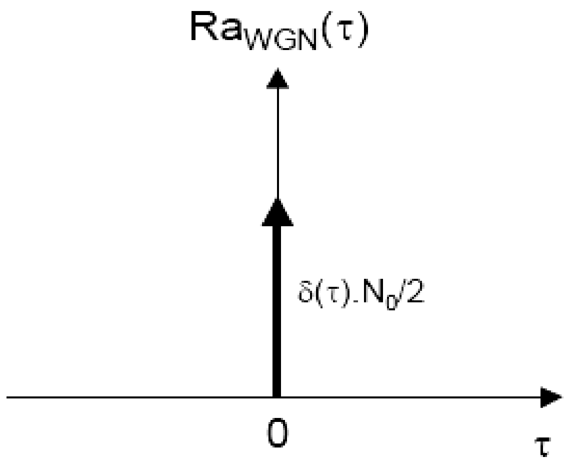
Week 4	Point Distribution
Week 8	Progress report 5%
Week 13	Presentation starts 10%
Week 15	Report 10%

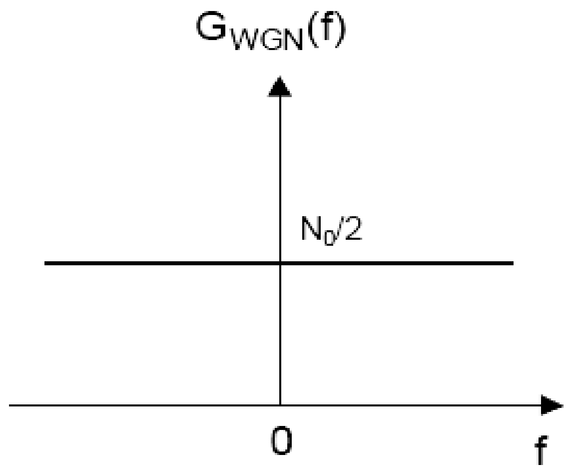
Simplified View of a Digital Radio Link



White Gaussian Noise







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Probability

- Think of probability as modeling an experiment
  - Eg: tossing a coin!
- The set of all possible *outcomes* is the *sample space*: S

- Classic "Experiment":
- Tossing a die: S = {1,2,3,4,5,6}
  - Any subset A of S is an *event*:
  - A = {*the outcome is even*} = {2,4,6}

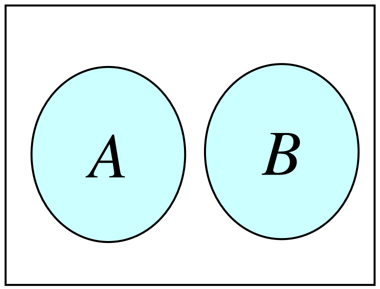
Probability of Events: Axioms

- P is the Probability Mass function if it maps each event A, into a real number ***P(A)***, and:

i.) $P(A) \geq 0$ for every event $A \subseteq S$

ii.) *P(S)* = 1

iii.)If A and B are mutually exclusive events then,



$A \cap B = \phi$

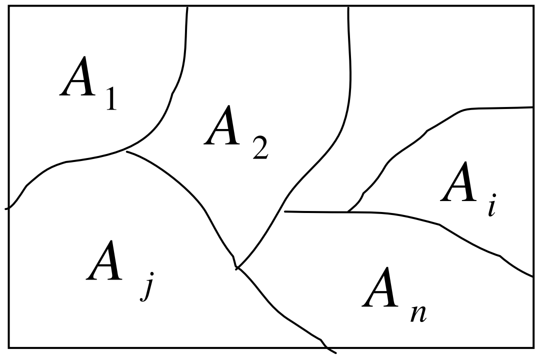
$$P(A \cup B) = P(A) + P(B)$$

Probability of Events

…In fact for any sequence of pair-wise-mutually-exclusive events, we have

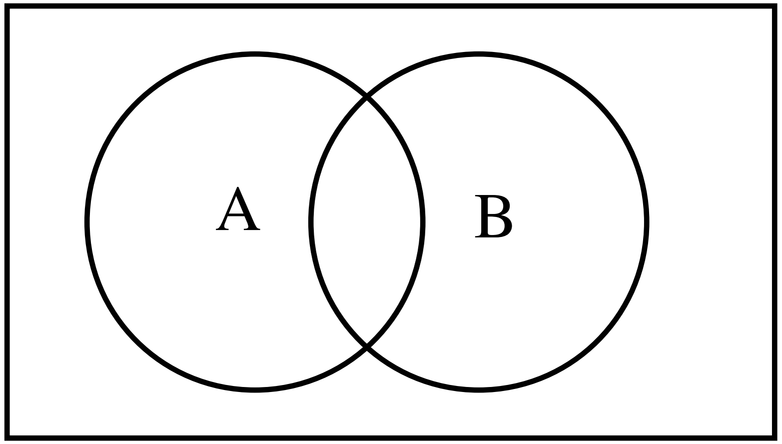
$$A_1, A_2, A_3, \ldots \quad (\text{i.e. } A_i A_j = 0 \text{ for any } i \neq j)$$

$$A_i \cap A_j = \phi, \quad \text{and} \quad \bigcup_{i=1} A_i = S$$



$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Approximations/Bounds: <u>Union Bound</u>



$$P(A \cup B) <= P(A) + P(B)$$

$$\mathbf{P(A_1 \cup A_2 \cup \ldots A_N) <= \Sigma_{i=1..N} P(A_i)}$$

- Applications:
  - Getting bounds on BER (bit-error rates),
  - In general, bounding the tails of prob. distributions

- We will use this in the analysis of error probabilities with various coding schemes

Approximations/Bounds: log(1+x)

- **$\log_2(1+x) \approx x$ for <u>small</u> x**

- Application: Shannon capacity w/ AWGN noise:
  - Bits-per-Hz = C/B = $\log_2(1+\gamma)$
  - If we can increase SNR (γ) linearly when γ is small (i.e. very poor, eg: cell-edge)…
  - … we get a <u>linear</u> increase in capacity.

- When γ is **<u>large</u>**, of course increase in γ gives only a diminishing return in terms of capacity: $\log(1+\gamma)$

Schwartz Inequality & Matched Filter

- Inner Product ($\mathbf{a}^T\mathbf{x}$) <= Product of Norms (i.e. $|\mathbf{a}||\mathbf{x}|$)
  - Projection length <= Product of Individual Lengths
- This is the *Schwartz Inequality*!
  - Equality happens when **a** and **x** are in the same direction (i.e. $\cos\theta = 1$, when $\theta = 0$)

- Application: “matched” filter
  - Received vector $\mathbf{y} = \mathbf{x} + \mathbf{w}$ (zero-mean AWGN)
  - Note: **w** is infinite dimensional
  - Project **y** to the subspace formed by the finite set of transmitted symbols **x**: y’
  - y’ is said to be a “**sufficient statistic**” for detection, i.e. reject the noise dimensions outside the signal space.
  - This operation is called “*matching*” to the signal space (projecting)
  - Now, pick the x which is closest to y’ in distance (ML detection = nearest neighbor)

Conditional Probability

- $P(A \mid B)$ = (conditional) probability that the outcome is in A given that we know the outcome in B

$$P(A \mid B) = \frac{P(AB)}{P(B)} \qquad P(B) \neq 0$$

- **Example: Toss one die.**

$$P(i = 3 \mid \text{i is odd}) =$$

- **Note that:** $P(AB) = P(B)P(A \mid B) = P(A)P(B \mid A)$

Independence

- Events A and B are <u>independent</u> if $P(AB) = P(A)P(B)$.
- Also: $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$
- <u>Example</u>: A card is selected at random from an ordinary deck of cards.
  - A=event that the card is an ace.
  - B=event that the card is a diamond.

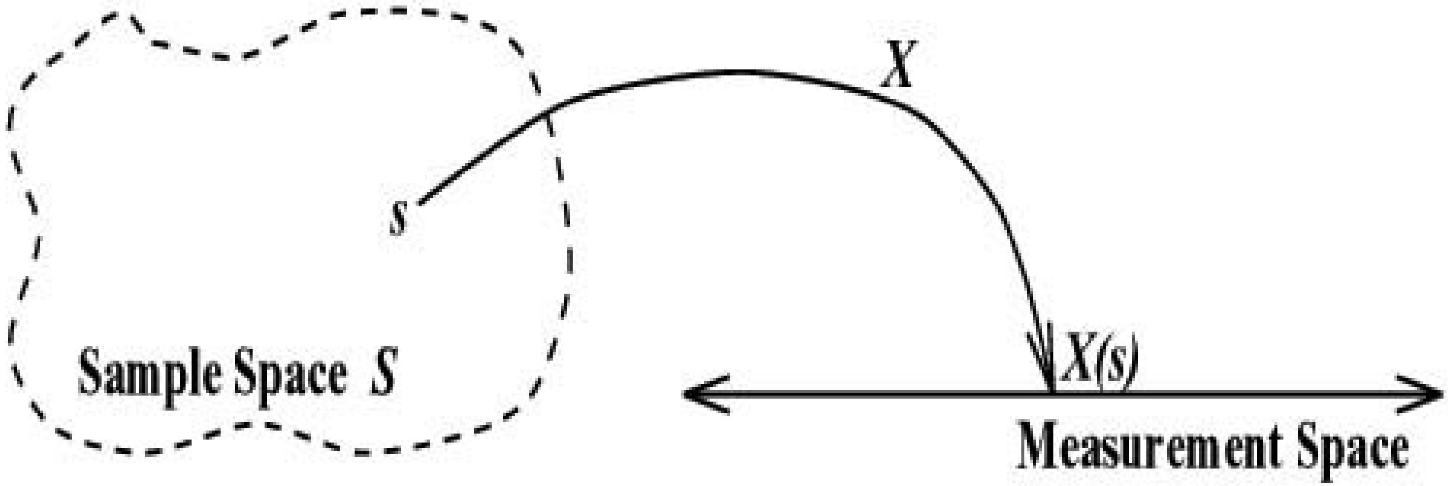
$P(AB) =$

$P(A) =$ $P(B) =$

$P(A)P(B) =$

Random Variable as a Measurement

- Thus a random variable can be thought of as a measurement (yielding a real number) on an experiment
  - Maps "*events*" to "*real numbers*"
  - We can then talk about the pdf, define the mean/variance and other moments



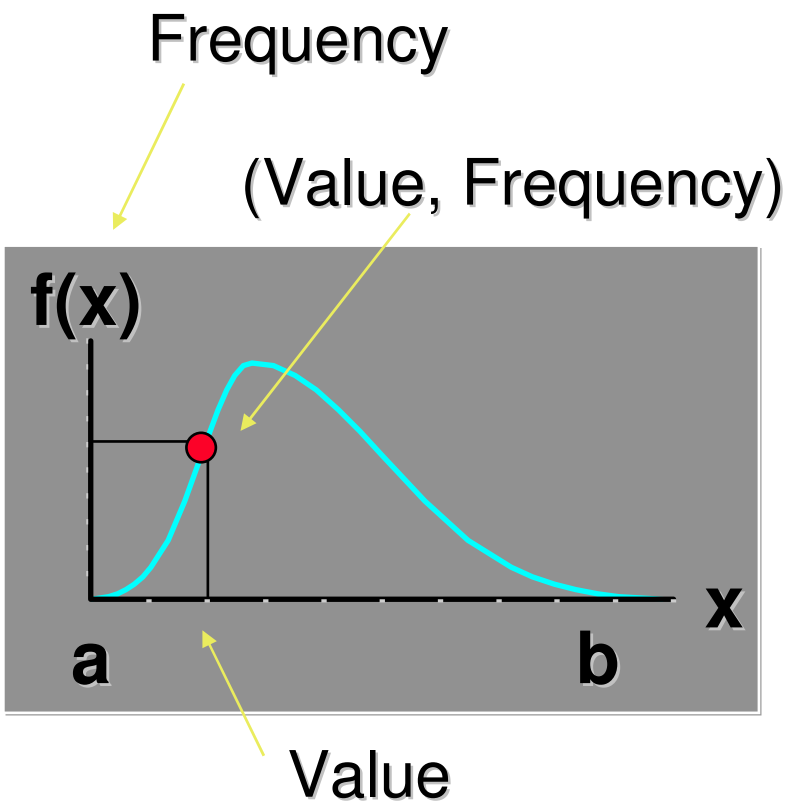
Continuous Probability Density Function

- 1. Mathematical Formula
- 2. Shows All Values, *x*, & Frequencies, f(*x*)
  - f(*X*) Is ***Not*** Probability
- 3. Properties

$$\int_{\text{All } X} f(x)dx = 1$$

(Area Under Curve)

$$f(x) \geq 0, \; a \leq x \leq b$$



Cumulative Distribution Function

- The cumulative distribution function (CDF) for a random variable ***X*** is

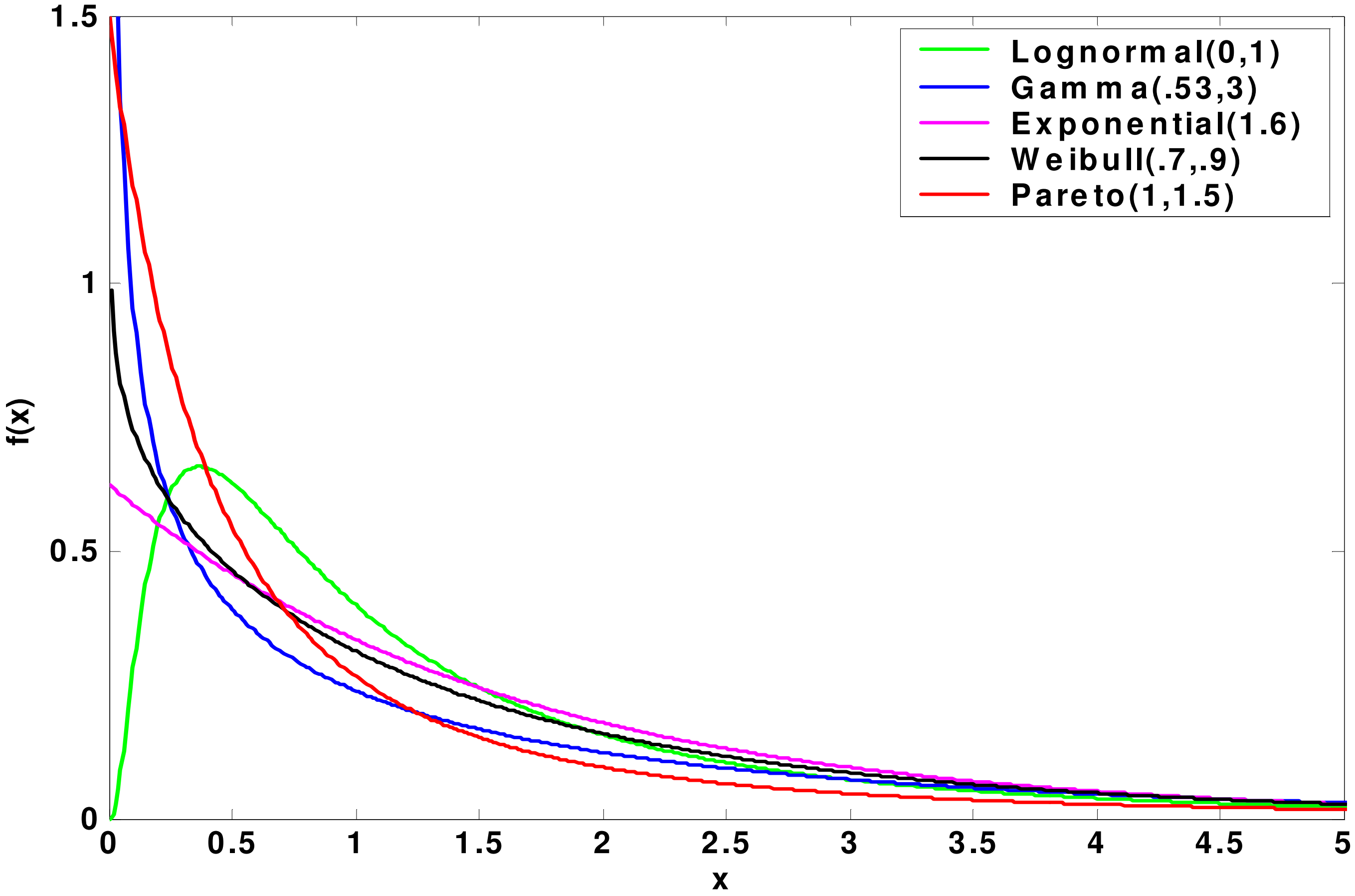
$$F_X(x) = P(X \le x) = P(\{s \in S \mid X(s) \le x\})$$

- Note that $F_X(x)$ is non-decreasing in ***x***, i.e.

$$x_1 \le x_2 \Rightarrow F_x(x_1) \le F_x(x_2)$$

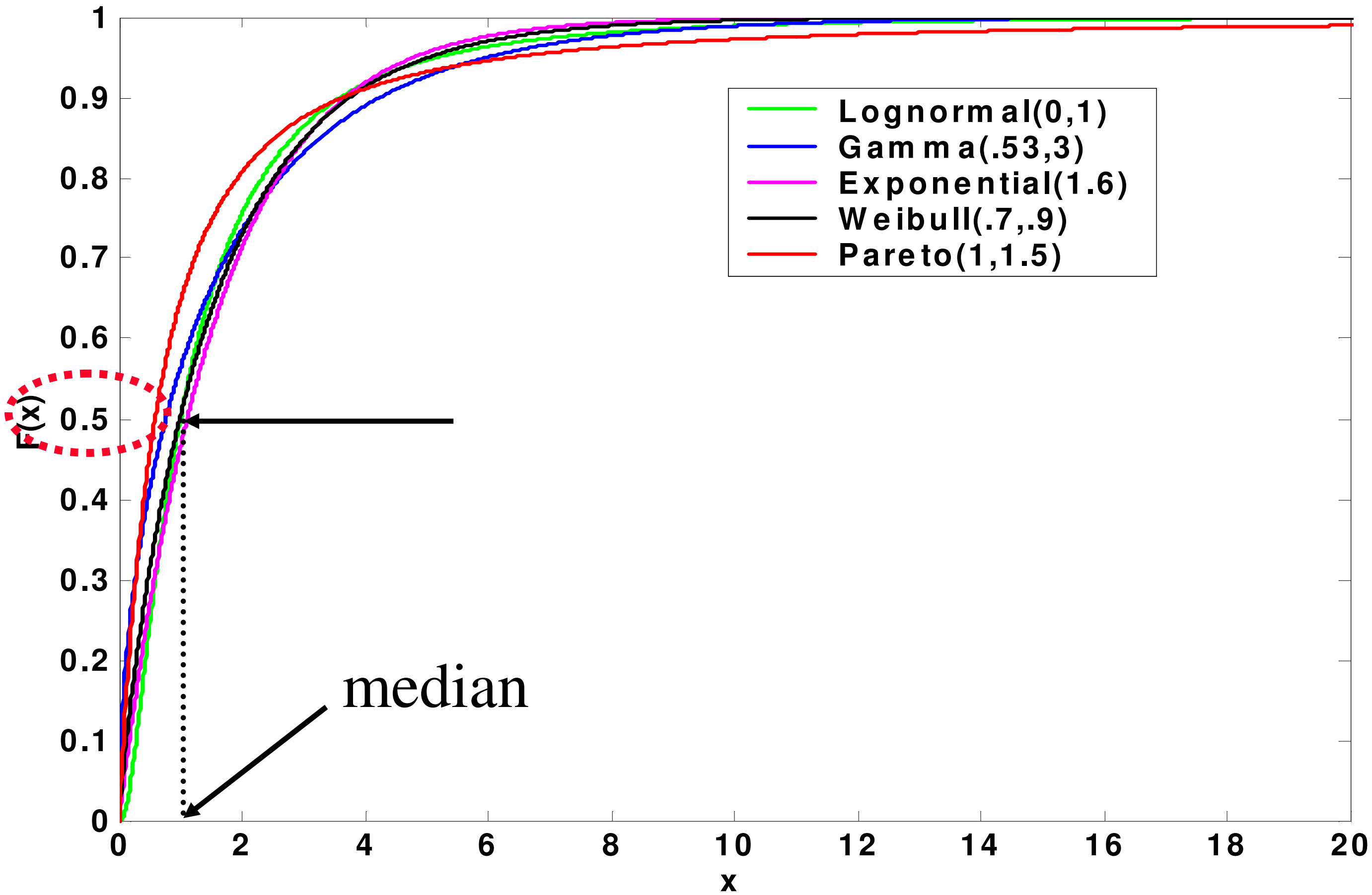
- Also $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to \infty} F_x(x) = 1$

Probability density functions (pdf)



Emphasizes main body of distribution, frequencies, various modes (peaks), variability, skews

Cumulative Distribution Function (CDF)



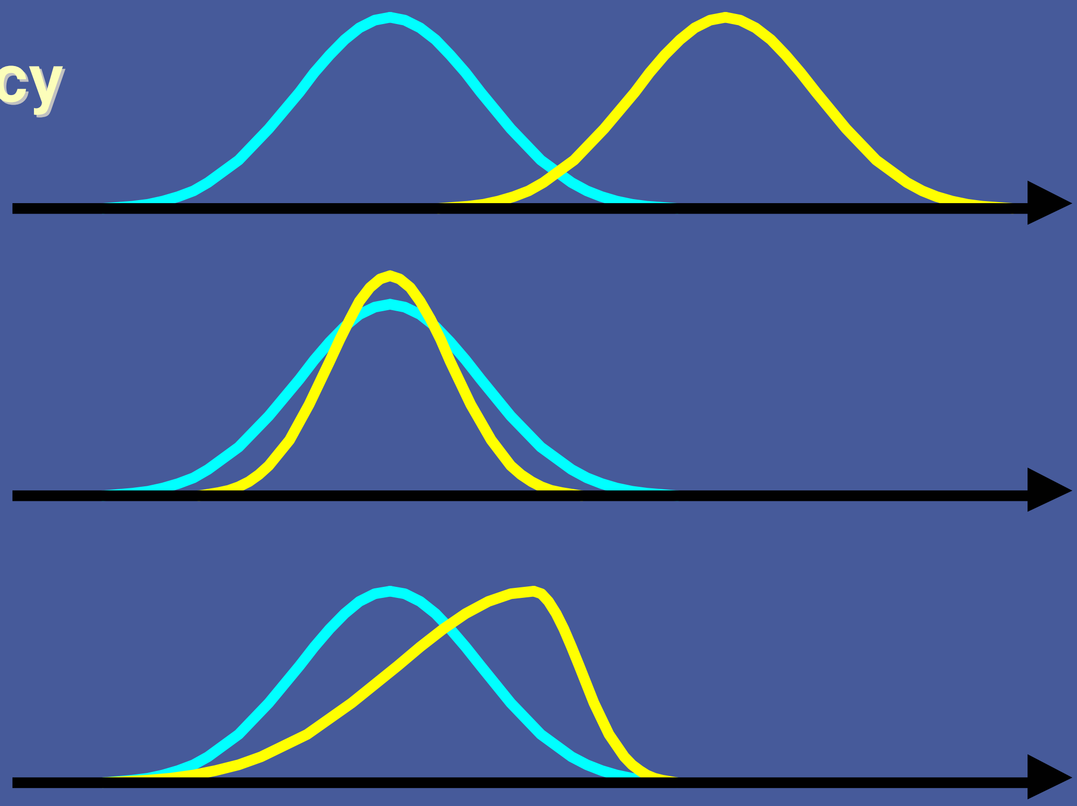
Emphasizes skews, easy identification of median/quartiles, converting uniform rvs to other distribution rvs

Numerical Data Properties

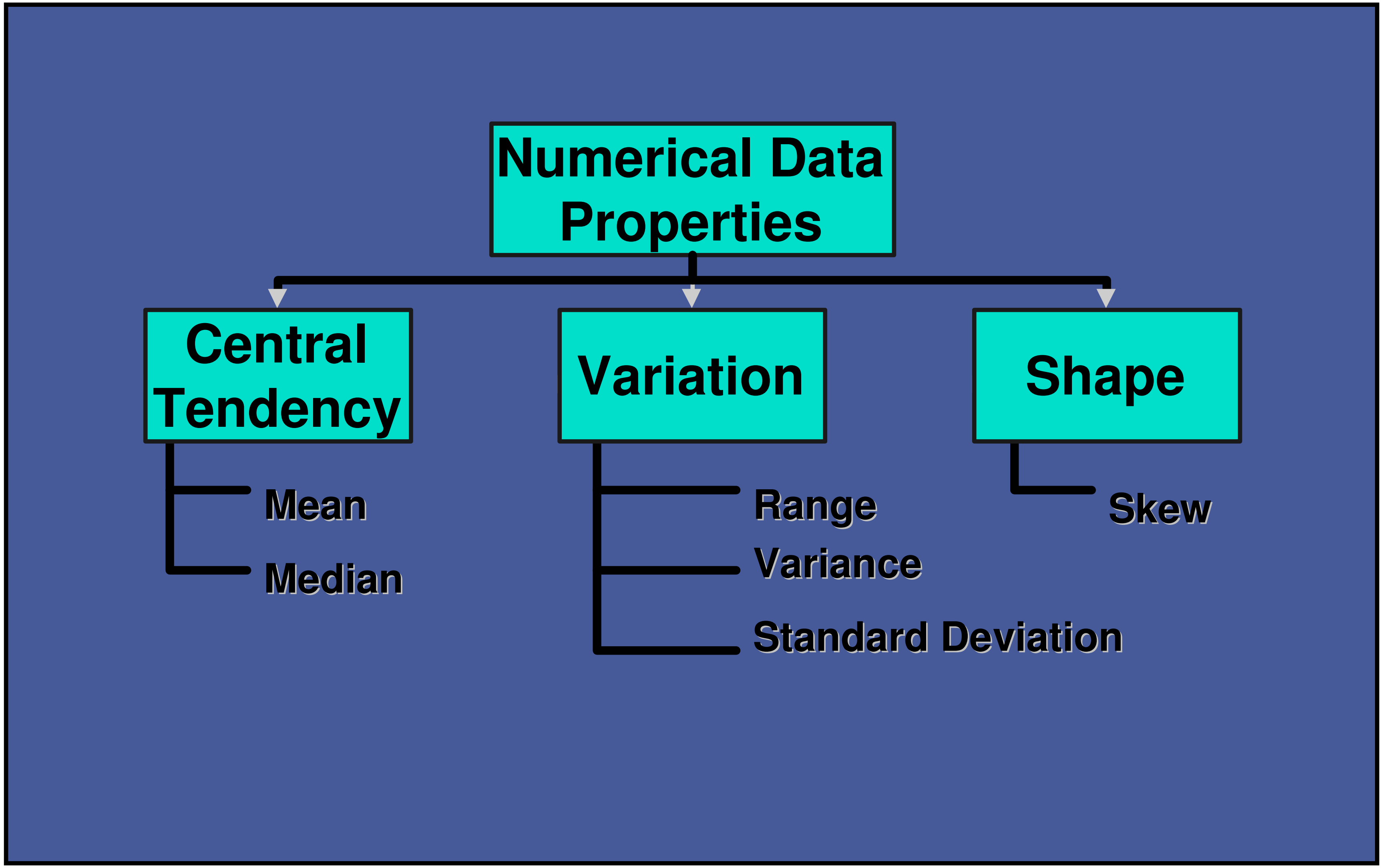
Central Tendency (Location)

Variation (Dispersion)

Shape



Numerical Data Properties & Measures



Expectation of a Random Variable: E[X]

- The <u>expectation</u> (average) of a (discrete-valued) random variable ***X*** is

$$\overline{X} = E(X) = \sum_{x=-\infty}^{\infty} xP(X = x) = \sum_{-\infty}^{\infty} xP_X(x)$$

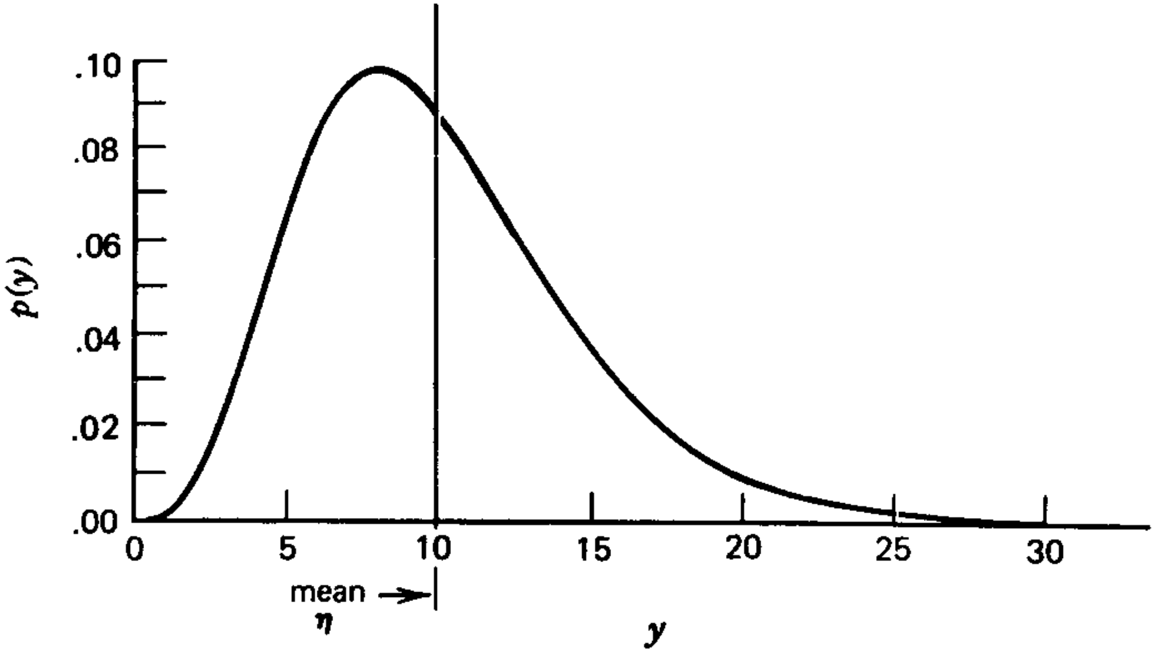


FIGURE 2.7. The mean $\eta = E(y)$ as the center of gravity of a distribution.

Expectation of a Continuous Random Variable

- The expectation (average) of a continuous random variable ***X*** is given by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Note that this is just the continuous equivalent of the discrete expectation

$$E(X) = \sum_{x=-\infty}^{\infty} x P_X(x)$$

Standard Deviation, Coeff. Of Variation, SIQR

- **<u>Variance</u>**: second moment around the mean:
  - $\sigma^2 = E[(X-\mu)^2]$
- **<u>Standard deviation</u>** = σ

$$\text{stdv}(x) = \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\mu'_2 - \mu^2},$$

Covariance and Correlation: Measures of Dependence

- **<u>Covariance</u>**: $\langle (x_i - \mu_i)(x_j - \mu_j) \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle ,$
  - For i = j, covariance = variance!
  - Independence => covariance = 0 (not vice-versa!)

- **<u>Correlation (coefficient)</u>** is a normalized (or scaleless) form of covariance:

$$\text{cor}(x_i, x_j) \equiv \frac{\text{cov}(x_i, x_j)}{\sigma_i \sigma_j},$$

  - Between –1 and +1.
    - Zero => no correlation (uncorrelated).
    - Note: uncorrelated <u>DOES NOT</u> mean independent!

Random Vectors & Sum of R.V.s

- Random Vector = [X_1, …, X_n], where Xi = r.v.
- Covariance Matrix:
  - **K** is an nxn matrix…
  - $K_{ij} = Cov[X_i, X_j]$
  - $K_{ii} = Cov[X_i, X_i] = Var[X_i]$

- Sum of *independent* R.v.s
  - Z = X + Y
  - PDF of Z is the *convolution* of PDFs of X and Y

    $p_Z(z) = p_X(x) * p_Y(y)$. Can use transforms!

Characteristic Functions & Transforms

- Characteristic function: a special kind of expectation

The distribution of a random variable X can be determined from its *characteristic function*, defined as

$$\phi_X(\nu) \triangleq \mathbf{E}[e^{j\nu X}] = \int_{-\infty}^{\infty} p_X(x) e^{j\nu x} dx. \quad \text{(B.10)}$$

- Captures all the *moments*, and is related to the *IFT of pdf:*

We see from (B.10) that the characteristic function $\phi_X(\nu)$ of $X(t)$ is the inverse Fourier transform of the distribution $p_X(x)$ evaluated at $f = \nu/(2\pi)$. Thus we can obtain $p_X(x)$ from $\phi_X(\nu)$ as

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\nu) e^{-j\nu x} dx. \quad \text{(B.11)}$$

This will become significant in finding the distribution for sums of random variables.

Important (Discrete) Random Variable: Bernoulli

- The simplest possible measurement on an experiment:
  - **Success** (***X = 1)*** or **failure *(X = 0)***.

- Usual notation:

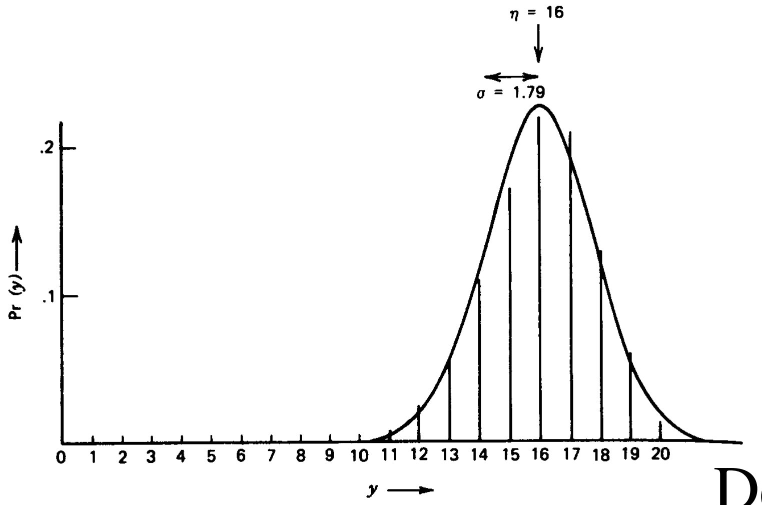
$$P_X(1) = P(X = 1) = p \qquad P_X(0) = P(X = 0) = 1 - p$$

  - ***E(X)*=**

Binomial can be skewed or normal

$$p(Y = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

$$\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}.$$

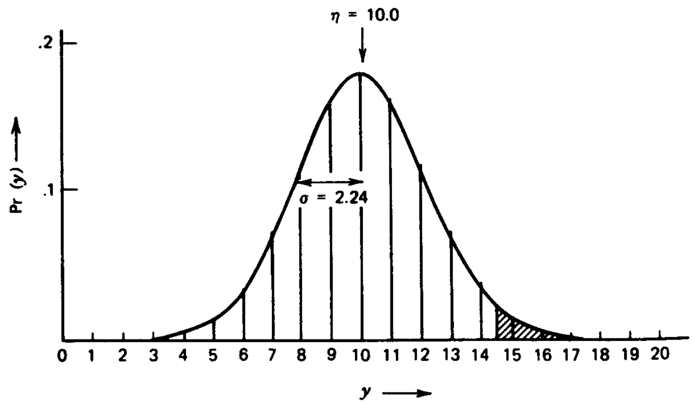


(c) Binomial distribution with mean p = 0.8 and n = 20.

Depends upon p and n !

Mean

$$\mu = E(x) = np$$



(d) Binomial distribution with mean p = 0.5 and n = 20.

FIGURE 5.4. (continued)

Standard Deviation

$$\sigma = \sqrt{np(1-p)}$$

Important Random Variable: Poisson

- A Poisson random variable ***X*** is defined by its PMF: (limit of binomial)

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

Where λ > 0 is a constant

***E(X)* =** λ

- Poisson random variables are good for *counting frequency of occurrence*: like the number of customers that arrive to a bank in one hour, or the number of packets that arrive to a router in one second.

Important Continuous Random Variable: Exponential

- Used to represent time, e.g. until the next arrival
- Has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

for some $\lambda > 0$

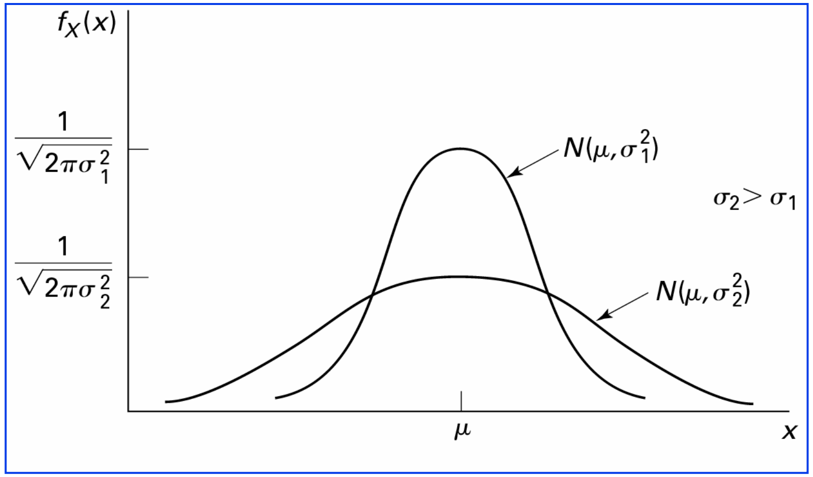
- Properties:

$$\int_0^\infty f_X(x)dx = 1 \quad \text{and} \quad E(X) = \frac{1}{\lambda}$$

  - Need to use integration by Parts!

Gaussian/Normal

- **Normal Distribution:** Completely characterized by mean (μ) and variance (σ^2)

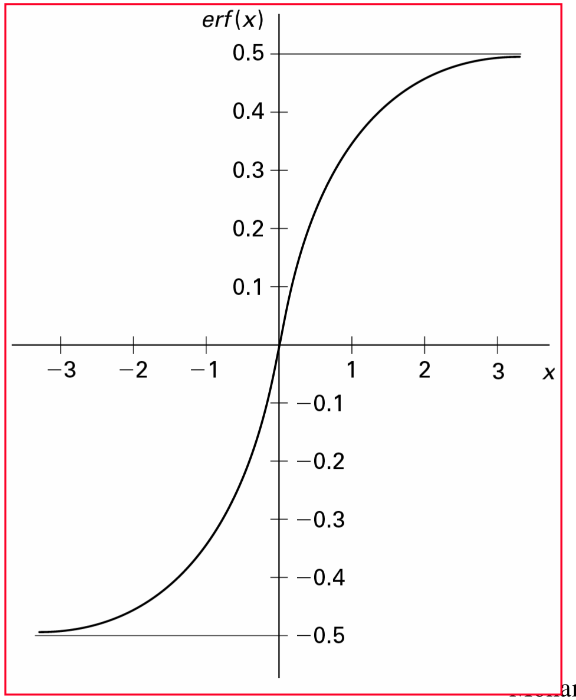


- **Q-function:** one-sided tail of normal pdf

$$Q(z) \triangleq p(x > z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

- **erfc():** two-sided tail.
- So:

$$Q(z) = \frac{1}{2}\text{erfc}\left(\frac{z}{\sqrt{2}}\right)$$



Normal Distribution: Why?

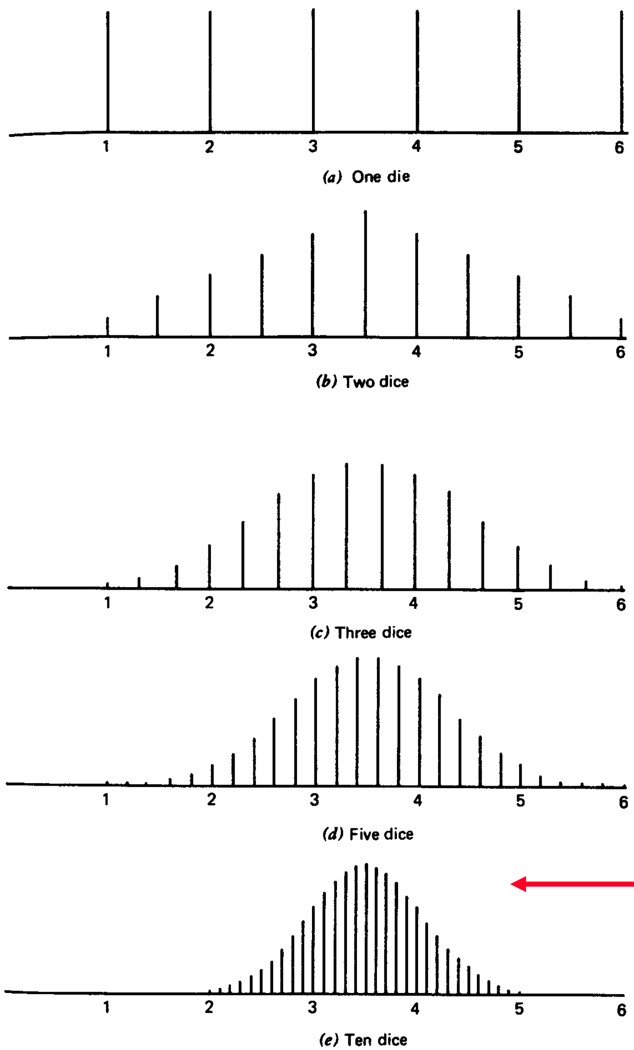


FIGURE 2.10. Distribution of average scores from throwing various numbers of dice.

Uniform distribution looks nothing like bell shaped (gaussian)! Large spread (σ)!

CENTRAL LIMIT TENDENCY!

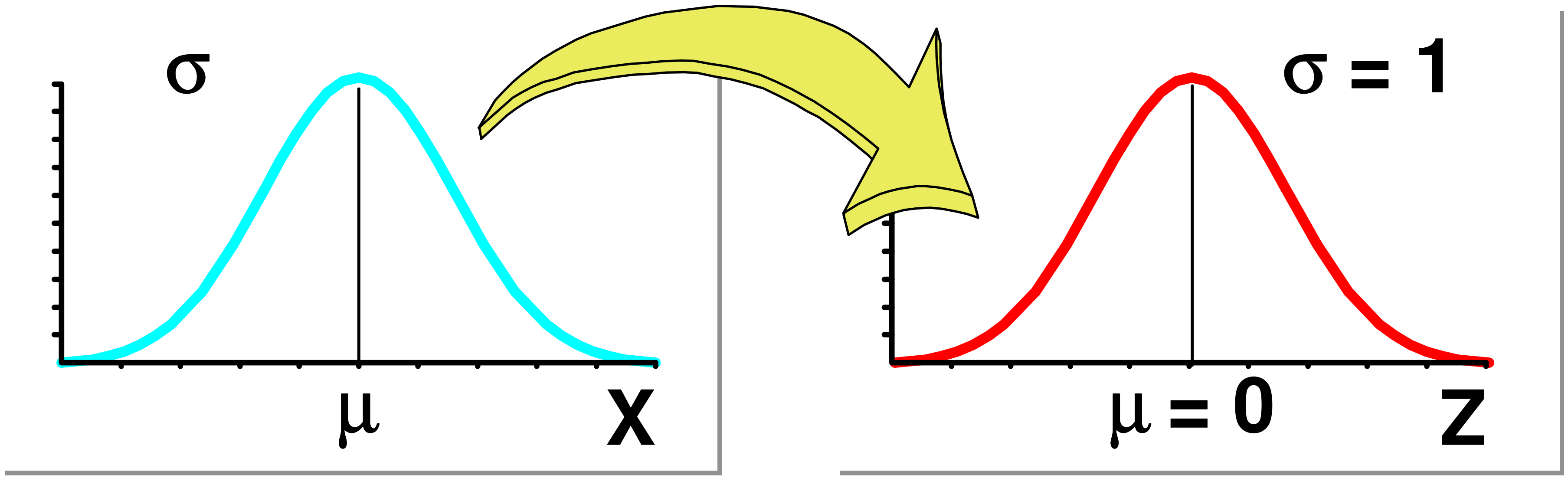
Sum of r.v.s from a uniform distribution after very few samples looks remarkably normal

Standardize the Normal Distribution

$$Z = \frac{X - \mu}{\sigma}$$

Normal Distribution

Standardized Normal Distribution



One table!

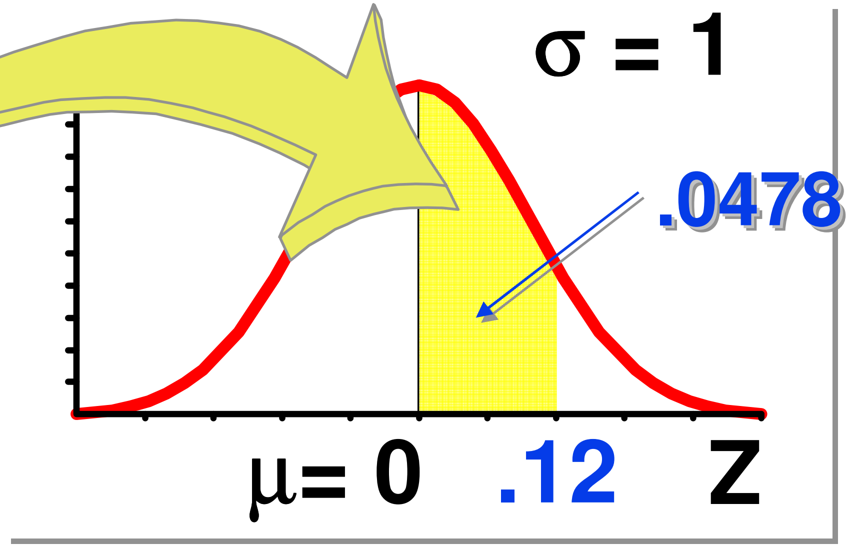
Mohamed Khedr

Obtaining the Probability

Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.0000	.0040	.008[illegible]
0.1	.0398	.0438	.0478
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255

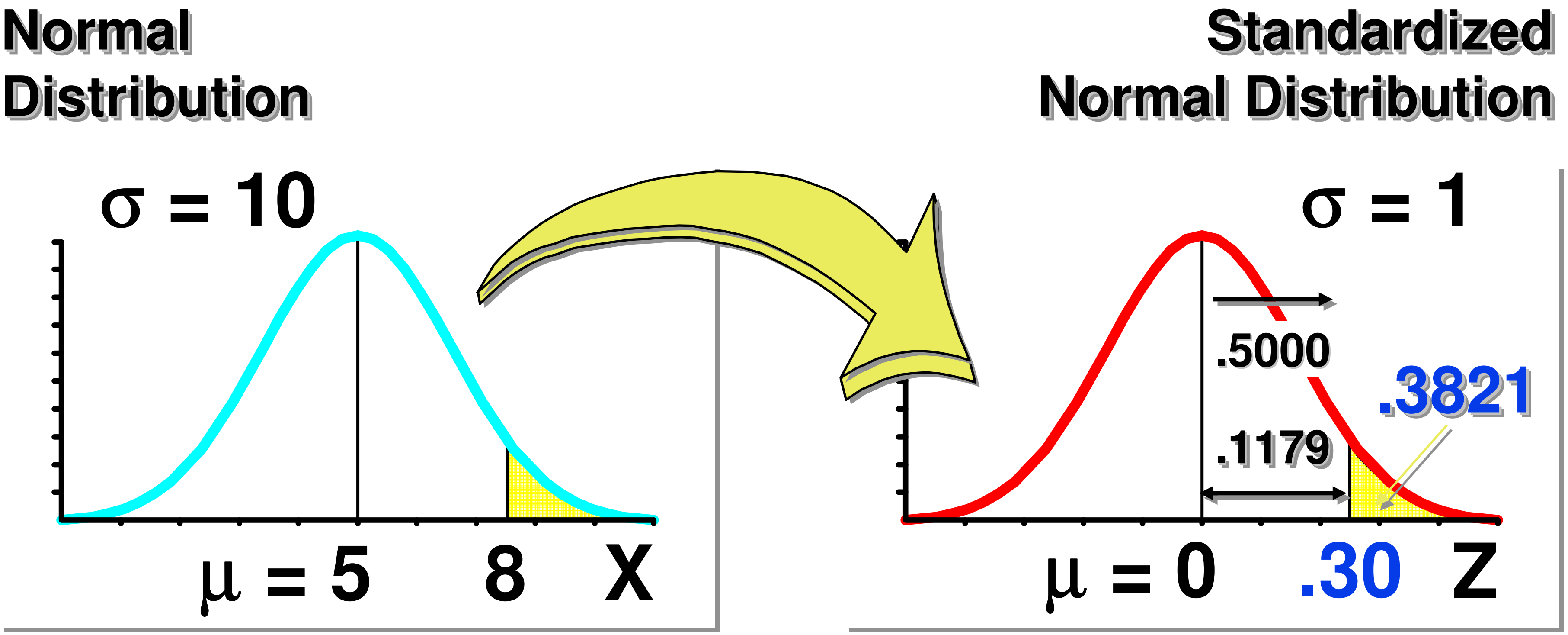
Probabilities



Shaded area exaggerated

Example
$P(X \geq 8)$

$$Z = \frac{X - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$



Shaded area exaggerated

Q-function: Tail of Normal Distribution

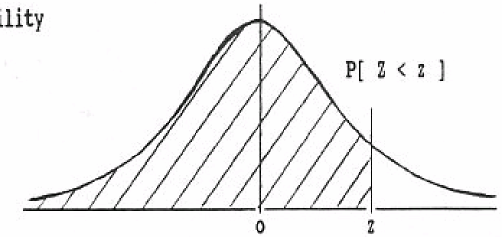
$Q(z) = P(Z > z) = 1 - P[Z < z]$

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

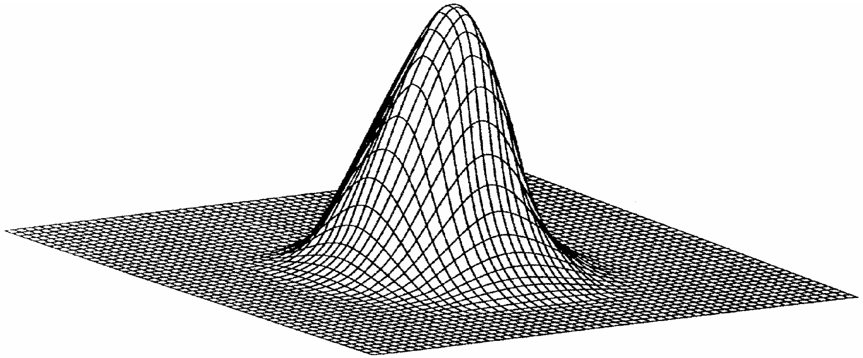
$$P[\,Z < z\,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



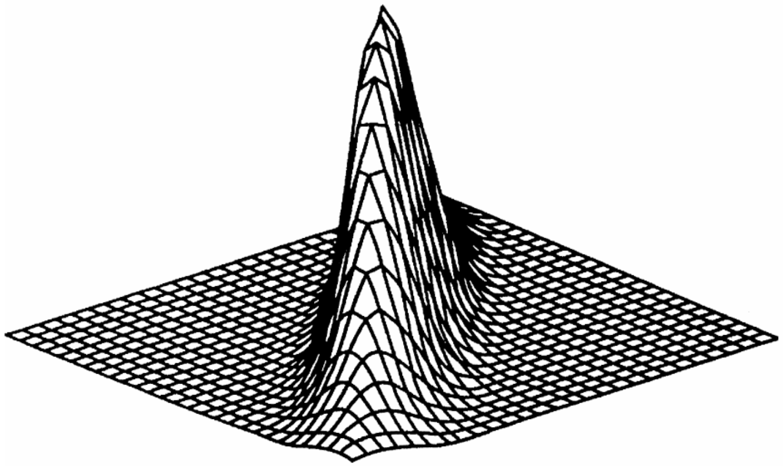
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

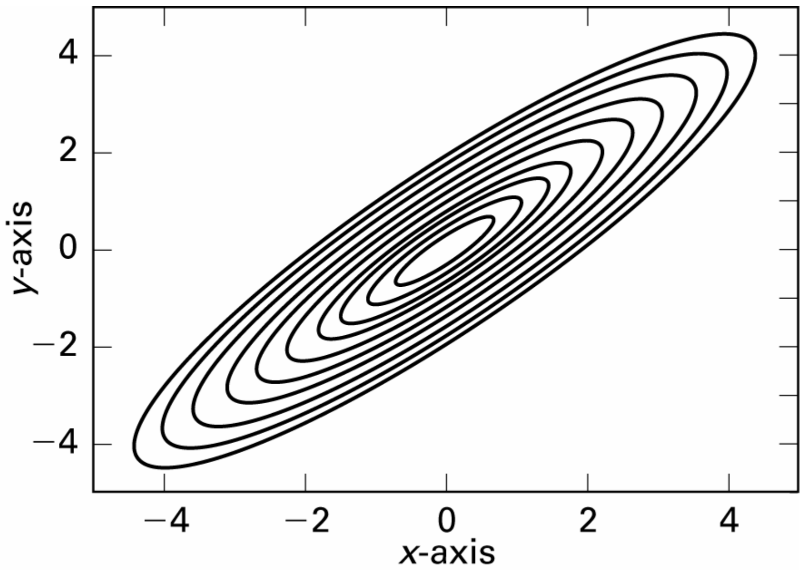
Gaussian Random Vectors (uncorrelated vs correlated)



Graph of the joint Gaussian density



(a)



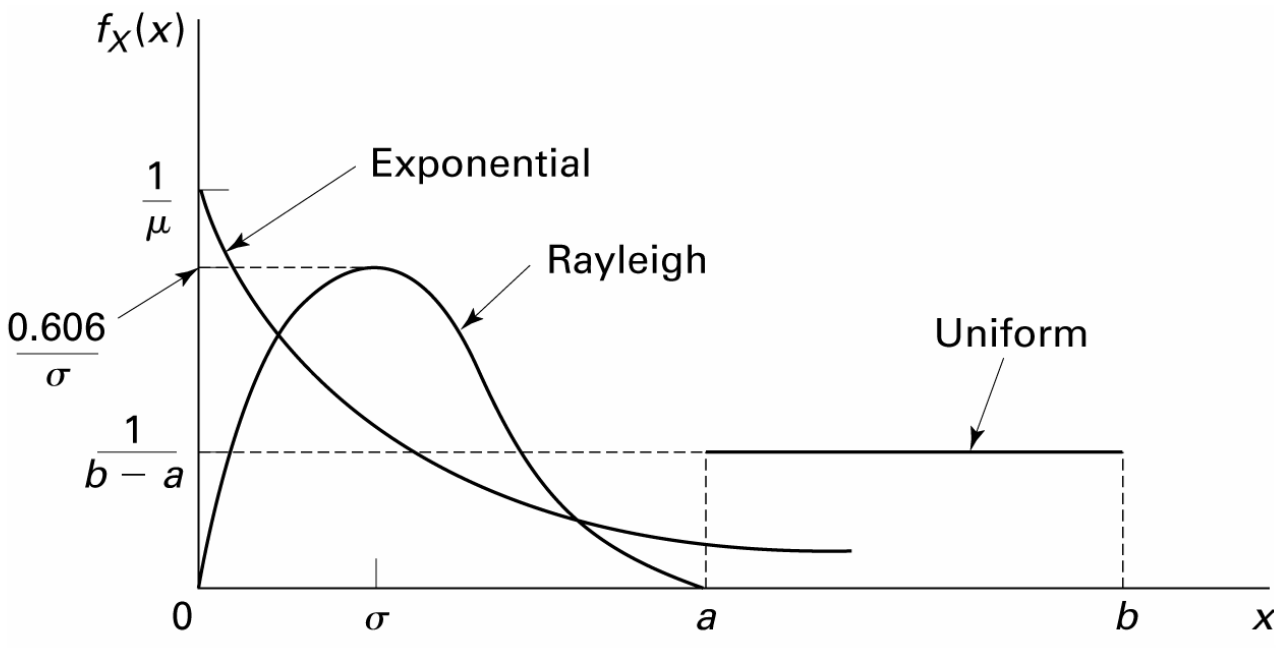
(b)

(a) Gaussian pdf with $\bar{X} = \bar{Y} = 0$, $\sigma_X = \sigma_Y = 2$, and $\rho = 0.9$; (b) Contours of constant density.

Complex Gaussian R.V: Circular Symmetry

- A complex Gaussian random variable X whose real and imaginary components are i.i.d. gaussian $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$
- … satisfies a *circular symmetry* property:
  - $e^{j\phi}X$ has the same distribution as X for any ϕ.
  - $e^{j\phi}$ multiplication: rotation in the complex plane.
- We shall call such a random variable *circularly symmetric complex Gaussian*,
  - …denoted by $\mathcal{CN}(0, \sigma^2)$, where $\sigma^2 = E[|X|^2]$.

Related Distributions



The rayleigh, exponential, and uniform pdf 's.

$X = [X_1, \ldots, X_n]$ is **Normal**

$\|X\|$ is **Rayleigh** { eg: *magnitude* of a complex gaussian channel $X_1 + jX_2$ }

$\|X\|^2$ is **Chi-Squared w/ *n*-degrees of freedom**

When n = 2, chi-squared becomes **exponential**. {eg: *power* in complex gaussian channel: sum of squares…}