EC 744 Wireless Communications
Spring 2007

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Digital Demodulation Techniques

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Syllabus

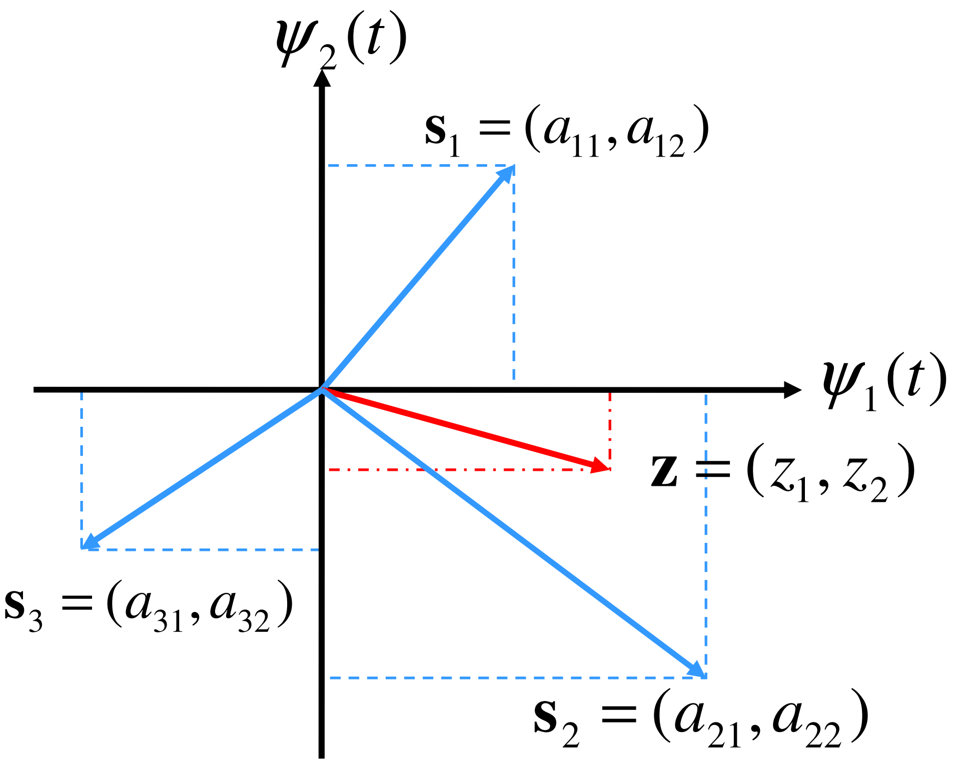
- Tentatively

Week 1	Overview, Probabilities, Random variables, Random process
Week 2	Wireless channels, Statistical Channel modelling, Path loss models
Week 3	Cellular concept and system design fundamentals
Week 4	Modulation techniques, single and multi-carrier
Week 5	Demodulation techniques, Diversity techniques
Week 6	Equalization techniques
Week 7	Mid Term exam
Week 8	802.11 and Mac evaluation
Week 9	Energy models in 802.11
Week 10	Wimax and Mac layer
Week 11	Presentations
Week 12	Presentations
Week 13	Presentations
Week 14	Presentations
Week 15	Final Exam

Signal space

- What is a signal space?
  - Vector representations of signals in an N-dimensional orthogonal space
- Why do we need a signal space?
  - It is a means to convert signals to vectors and vice versa.
  - It is a mean to calculate signals energy and Euclidean distances between signals.
- Why are we interested in Euclidean distances between signals?
  - For detection purposes: The received signal is transformed to a received vectors. The signal which has the minimum distance to the received signal is estimated as the transmitted signal.

Schematic example of a signal space



Transmitted signal alternatives

$$s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12})$$

$$s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22})$$

$$s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32})$$

Received signal at matched filter output

$$z(t) = z_1\psi_1(t) + z_2\psi_2(t) \Leftrightarrow \mathbf{z} = (z_1, z_2)$$

Signal space

- To form a signal space, first we need to know the <u>inner product</u> between two signals (functions):
  - Inner (scalar) product:

$$< x(t), y(t) >= \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

= cross-correlation between x(t) and y(t)

  - Properties of inner product:

$$< ax(t), y(t) >= a < x(t), y(t) >$$

$$< x(t), ay(t) >= a^* < x(t), y(t) >$$

$$< x(t) + y(t), z(t) >=< x(t), z(t) > + < y(t), z(t) >$$

Signal space …

- The distance in signal space is measure by calculating the norm.
- What is norm?
  - Norm of a signal:

$$\|x(t)\| = \sqrt{< x(t), x(t) >} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{E_x}$$

= "length" of x(t)

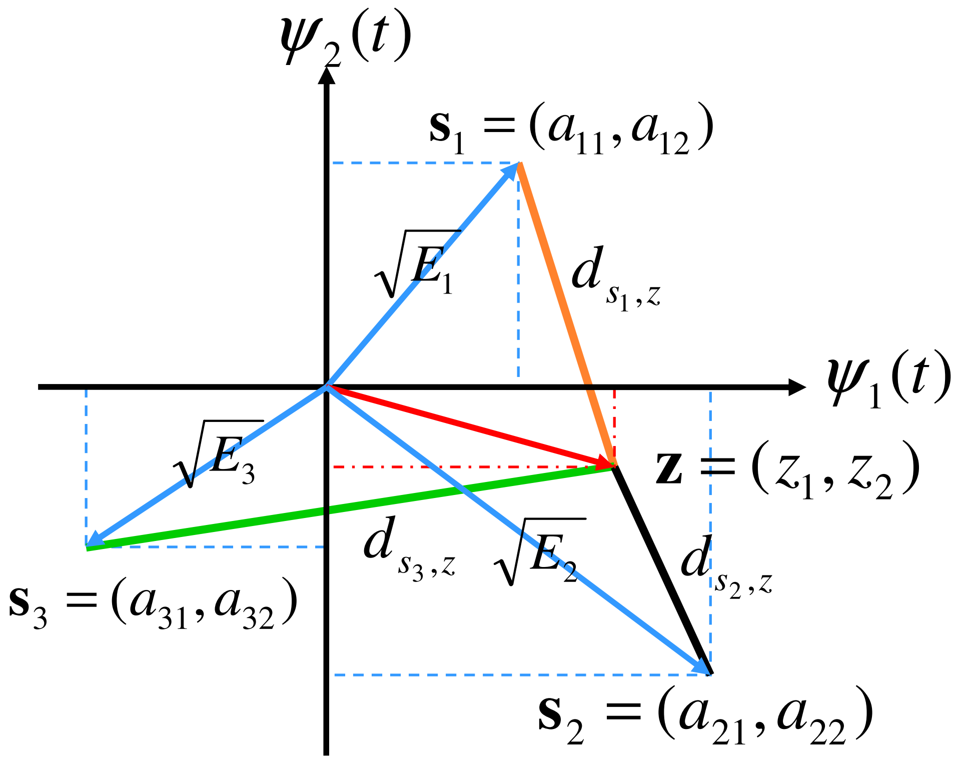
$$\|ax(t)\| = |a| \|x(t)\|$$

  - Norm between two signals:

$$d_{x,y} = \|x(t) - y(t)\|$$

- We refer to the norm between two signals as the Euclidean distance between two signals.

Example of distances in signal space



The Euclidean distance between signals *z(t)* and *s(t)*:

$$d_{s_i,z} = \| s_i(t) - z(t) \| = \sqrt{(a_{i1} - z_1)^2 + (a_{i2} - z_2)^2}$$

$$i = 1,2,3$$

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Signal space …

- Any arbitrary finite set of waveforms $\{s_i(t)\}_{i=1}^{M}$

  where each member of the set is of duration *T*, can be expressed as a linear combination of N orthonogal waveforms

  where $\{\psi_j(t)\}_{j=1}^{N}$ $N \leq M$

$$s_i(t) = \sum_{j=1}^{N} a_{ij} \psi_j(t) \qquad i = 1, \ldots, M \quad N \leq M$$

where

$$a_{ij} = < s_i(t), \psi_j(t) > = \int_0^T s_i(t) \psi_j^*(t) dt \qquad j = 1, \ldots, N \quad i = 1, \ldots, M \qquad 0 \leq t \leq T$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Vector representation of waveform

$$E_i = \sum_{j=1}^{N} |a_{ij}|^2$$

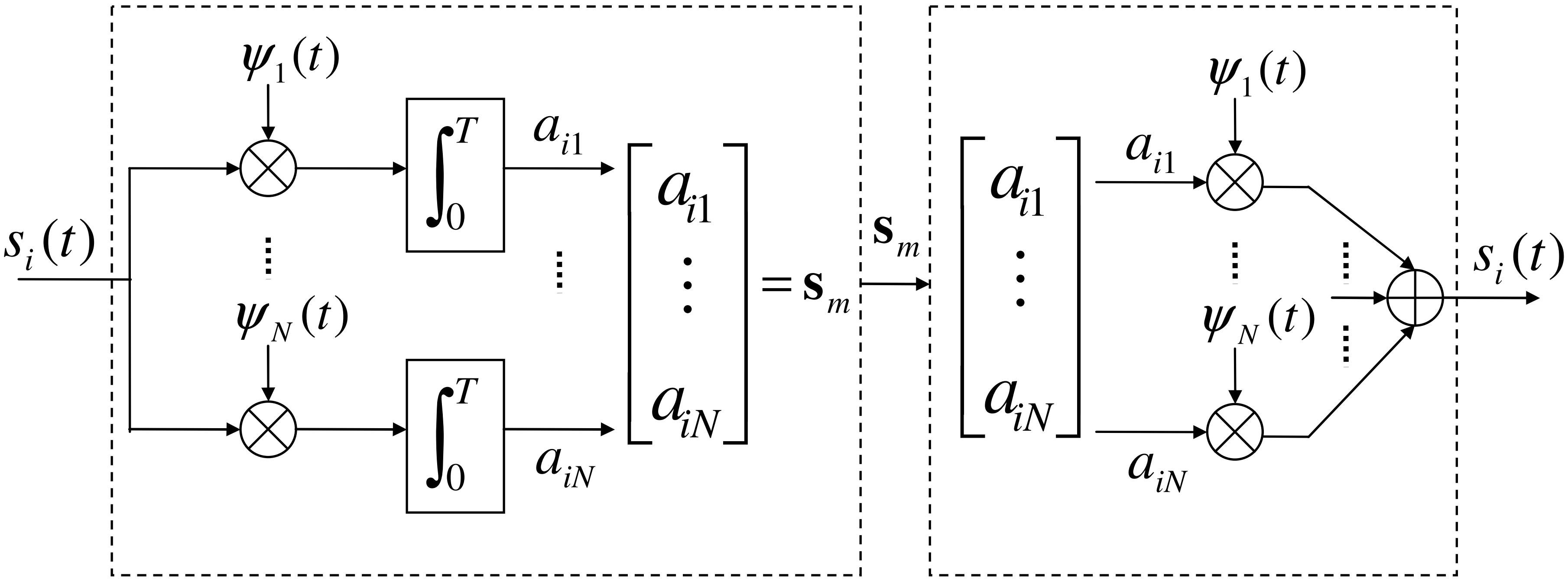
Waveform energy

Signal space …

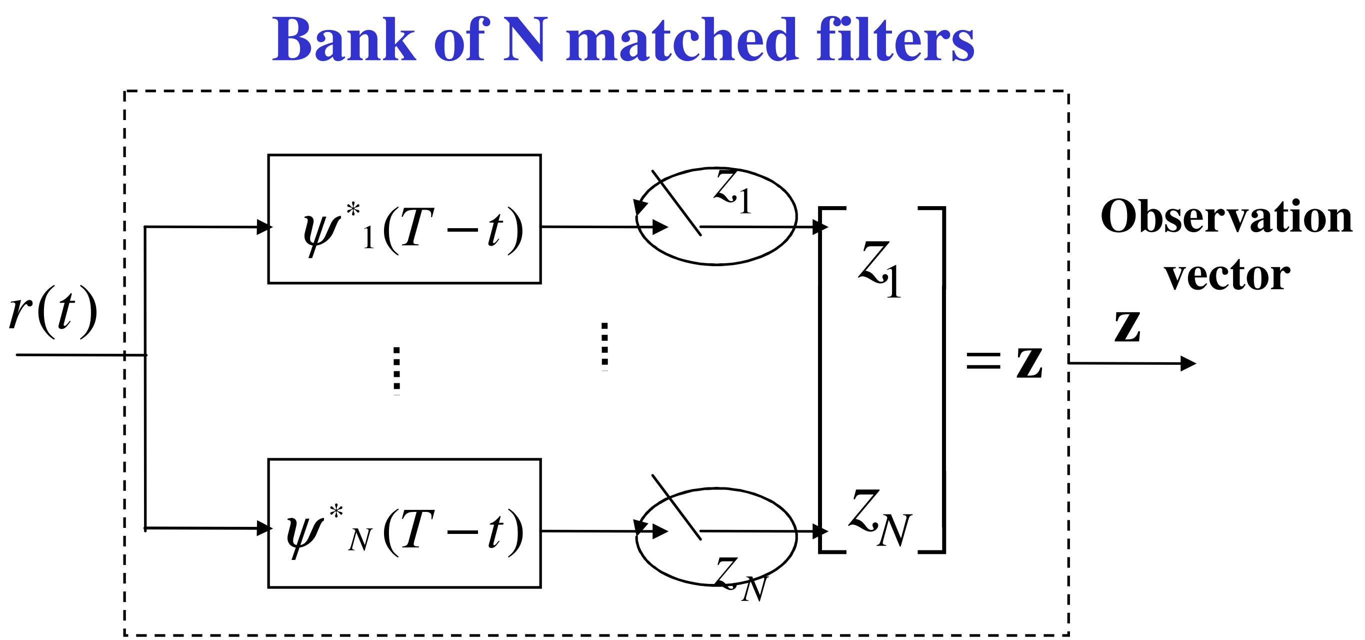
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t)$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Waveform to vector conversion **Vector to waveform conversion**



Implementation of matched filter receiver



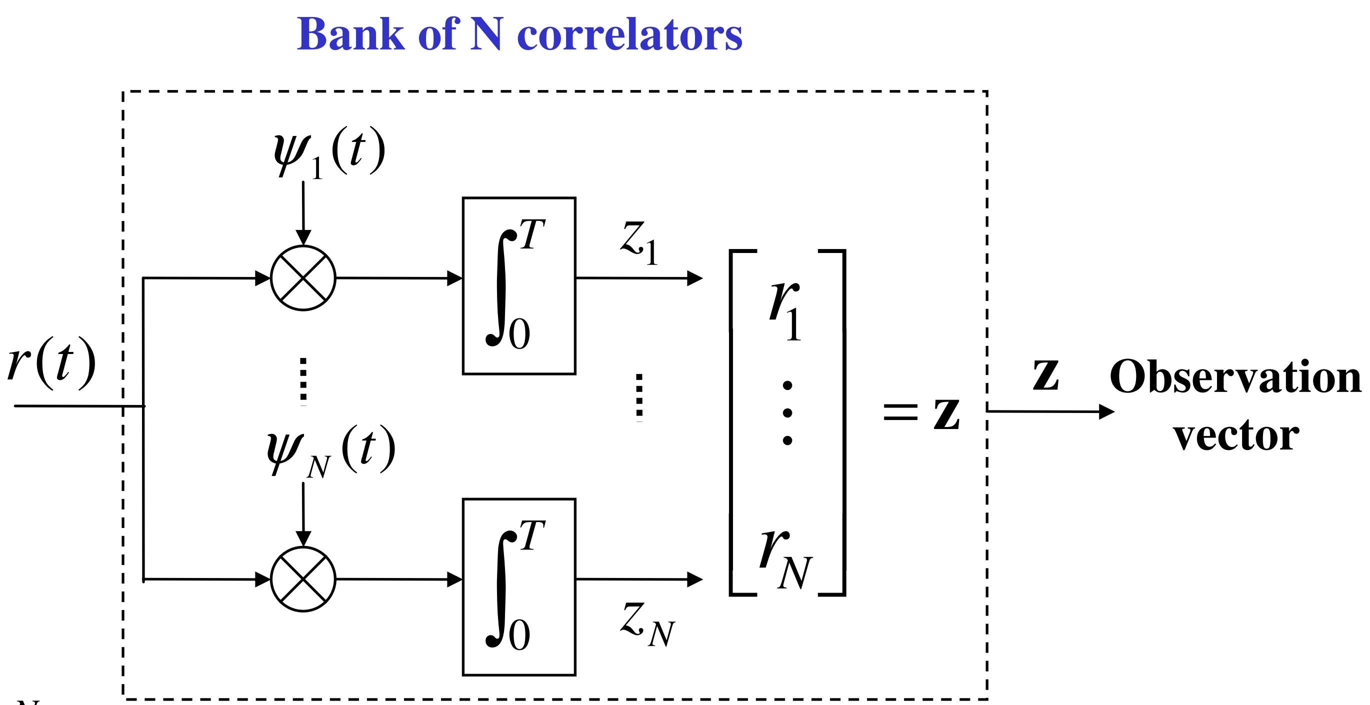
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

$$z_j = r(t) * \psi_j(T-t) \qquad j = 1,\ldots,N$$

$$N \le M$$

Implementation of correlator receiver



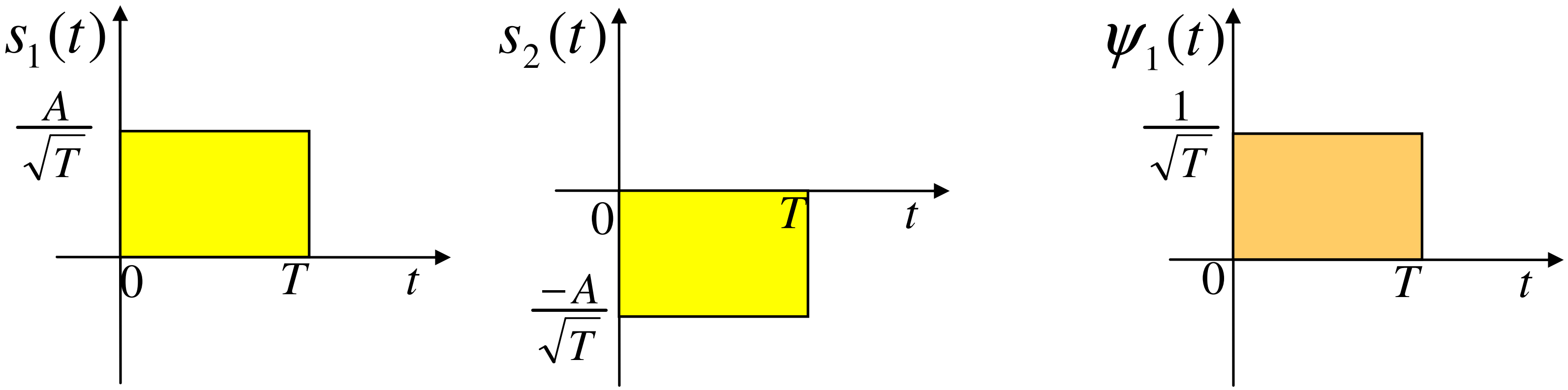
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

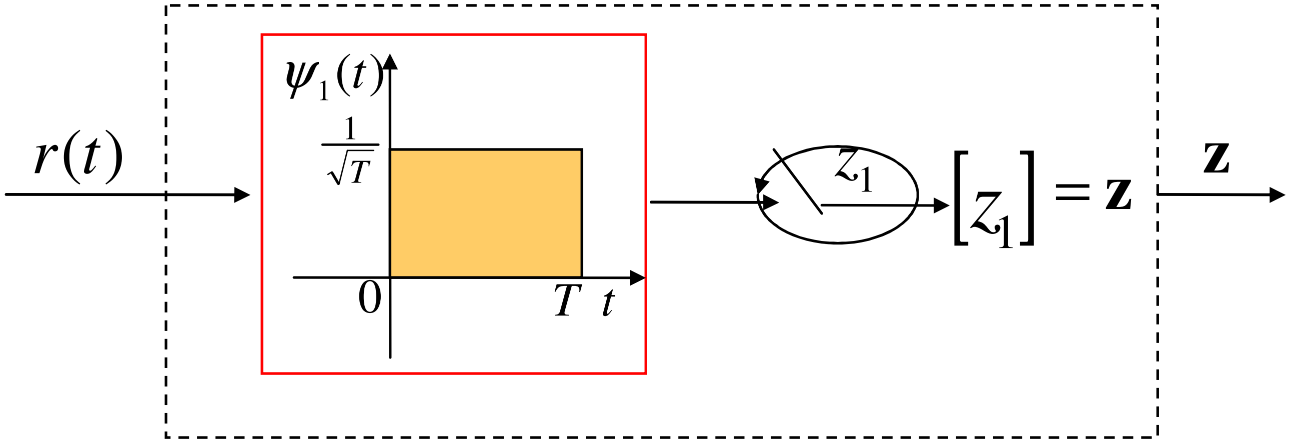
$$z_j = \int_0^T r(t)\psi_j(t)dt \qquad j = 1,\ldots,N$$

$$N \leq M$$

Example of matched filter receivers using basic functions



1 matched filter



- Number of matched filters (or correlators) is reduced by 1 compared to using matched filters (correlators) to the transmitted signal.

White noise in orthonormal signal space

- AWGN $n(t)$ can be expressed as

$$n(t) = \underbrace{\hat{n}(t)}_{\text{Noise projected on the signal space which impacts the detection process.}} + \underbrace{\tilde{n}(t)}_{\text{Noise outside on the signal space}}$$

Noise projected on the signal space which impacts the detection process.

Noise outside on the signal space

$$\begin{cases} \hat{n}(t) = \sum_{j=1}^{N} n_j \psi_j(t) \\ n_j = < n(t), \psi_j(t) > \qquad j = 1, \ldots, N \\ < \tilde{n}(t), \psi_j(t) > = 0 \qquad j = 1, \ldots, N \end{cases}$$

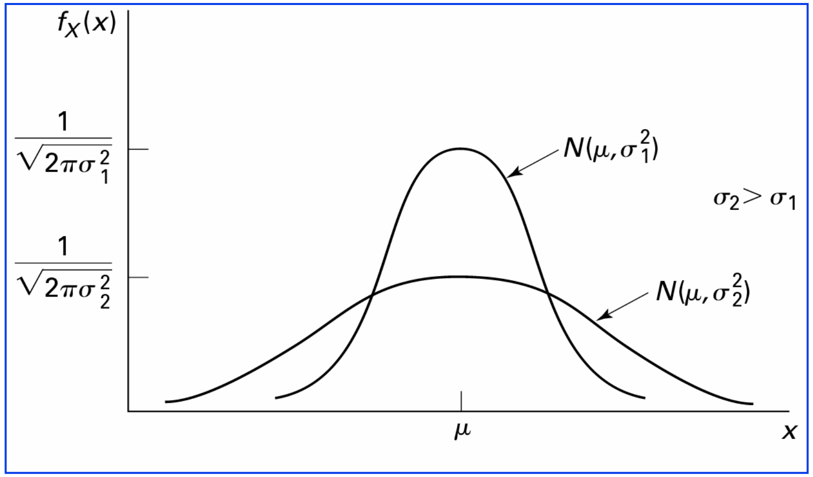
Vector representation of $\hat{n}(t)$

$$\mathbf{n} = (n_1, n_2, \ldots, n_N)$$

$\{n_j\}_{j=1}^{N}$ independent zero-mean Gaussain random variables with variance $\text{var}(n_j) = N_0 / 2$

Gaussian/Normal

- **Normal Distribution:** Completely characterized by mean (μ) and variance (σ^2)

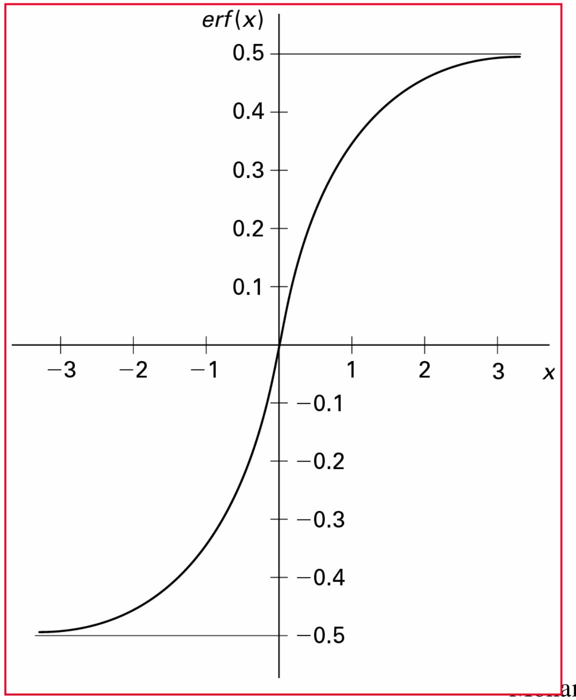


- **Q-function:** one-sided tail of normal pdf

$$Q(z) \triangleq p(x > z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

- **erfc():** two-sided tail.
- So:

$$Q(z) = \frac{1}{2} \text{erfc}\left(\frac{z}{\sqrt{2}}\right)$$



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Normal Distribution: Why?

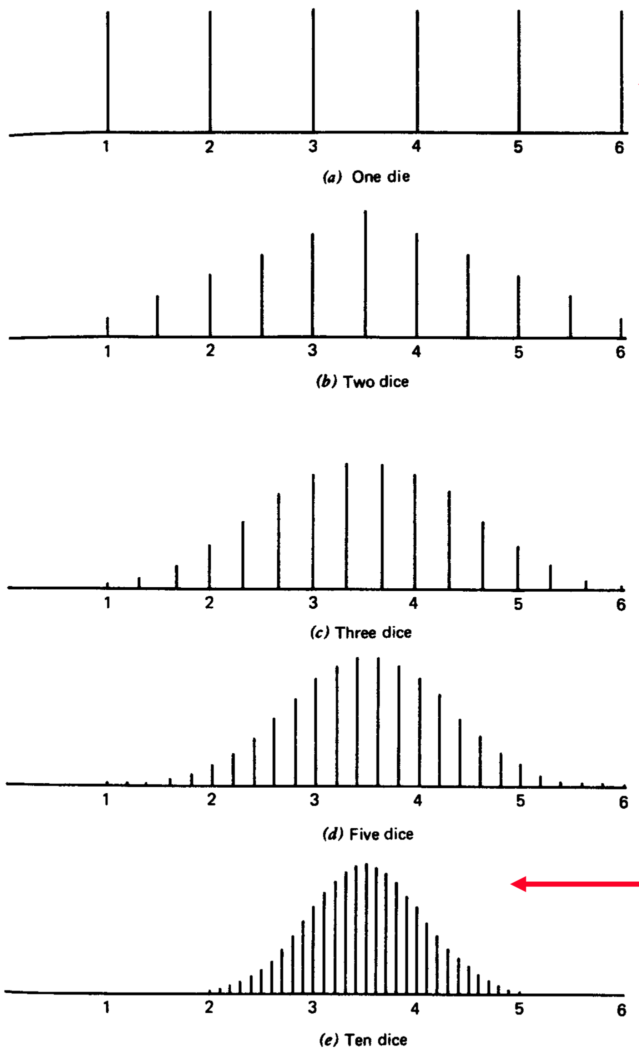


FIGURE 2.10. Distribution of average scores from throwing various numbers of dice.

Uniform distribution looks nothing like bell shaped (gaussian)! Large spread (σ)!

CENTRAL LIMIT TENDENCY!

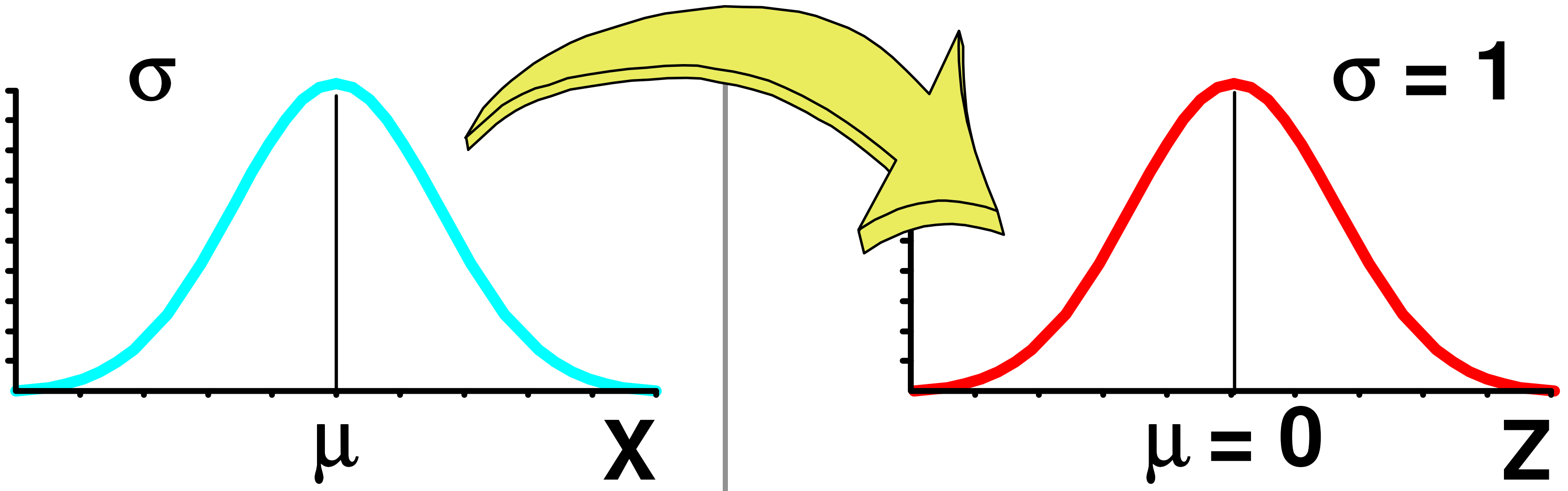
Sum of r.v.s from a uniform distribution after very few samples looks remarkably normal

Standardize the Normal Distribution

$$Z = \frac{X - \mu}{\sigma}$$

Normal Distribution

Standardized Normal Distribution



One table!

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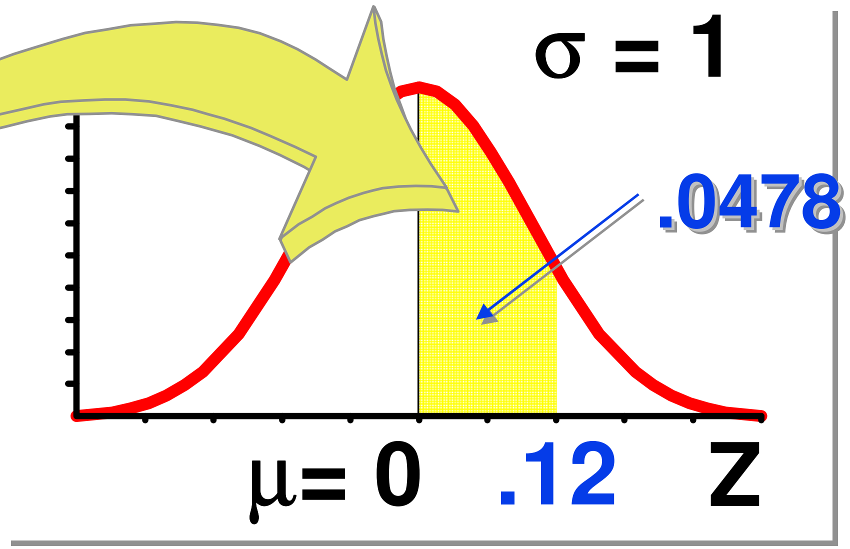
Obtaining the Probability

Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.0000	.0040	.008
0.1	.0398	.0438	**.0478**
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255

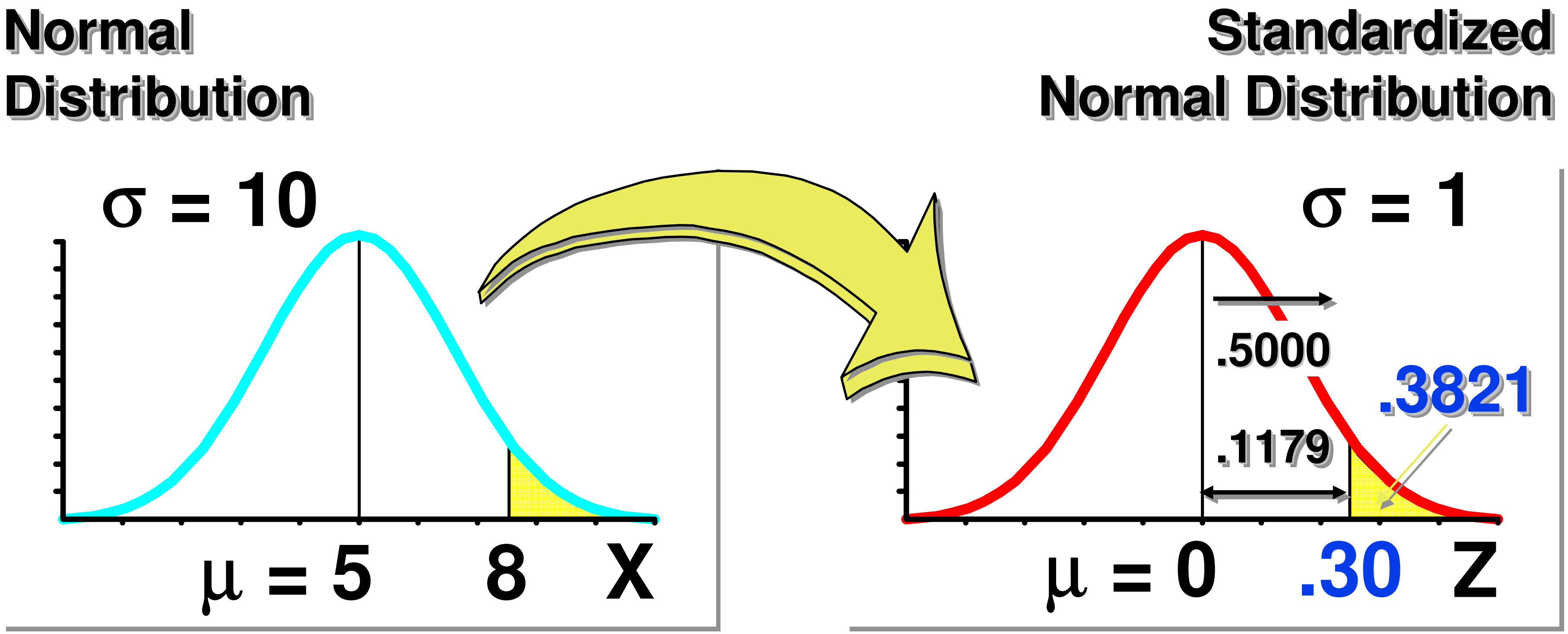
Probabilities



Shaded area exaggerated

Example
P(*X* ≥ 8)

$$Z = \frac{X - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$



Shaded area exaggerated

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Q-function: Tail of Normal Distribution

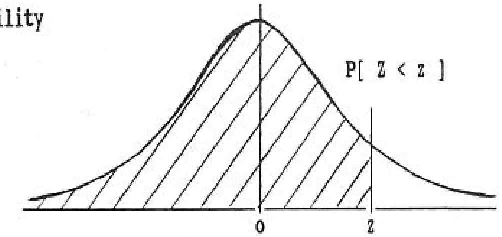
$Q(z) = P(Z > z) = 1 - P[Z < z]$

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\,Z < z\,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$

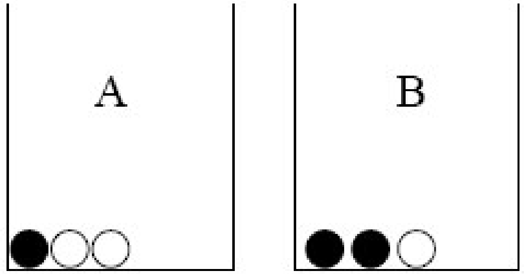


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

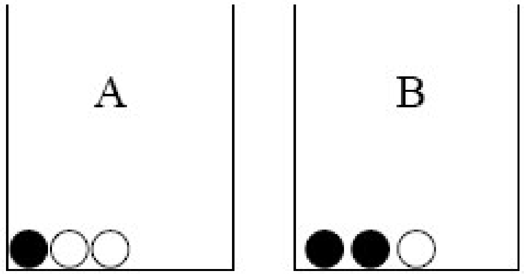
Maximum Likelihood (ML) Detection: Concepts

Likelihood Principle



- Experiment:
  - Pick Urn A or Urn B at random
  - Select a ball from that Urn.
- The ball is black.
- What is the probability that the selected Urn is A?

Likelihood Principle (Contd)



- Write out what you know!
- **P(Black | UrnA) = 1/3**
- **P(Black | UrnB) = 2/3**
- **P(Urn A) = P(Urn B) = 1/2**
- We want **P(Urn A | Black).**
- Gut feeling: Urn B is more likely than Urn A (given that the ball is black). But by how much?
- This is an <u>inverse probability</u> problem.
  - Make sure you understand the inverse nature of the conditional probabilities!
- Solution technique: Use Bayes Theorem.

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Likelihood Principle (Contd)

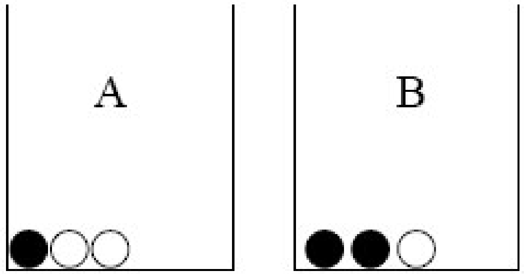
- **<u>Bayes manipulations:</u>**
- **P(Urn A | Black) =**
  - **P(Urn A and Black) /P(Black)**
- Decompose the numerator and denomenator in terms of the probabilities we know.

- **P(Urn A and Black) = P(Black | UrnA)*P(Urn A)**
- **P(Black) = P(Black| Urn A)*P(Urn A) + P(Black| UrnB)*P(UrnB)**

- We know all these values Plug in and crank.
- **P(Urn A and Black) = 1/3 * 1/2**
- **P(Black) = 1/3 * 1/2 + 2/3 * 1/2 = 1/2**
- **P(Urn A and Black) /P(Black) = 1/3 = 0.333**
- Notice that it matches our gut feeling that Urn A is less likely, once we have seen black.

- *<u>The information that the ball is black has CHANGED !</u>*
  - From P(Urn A) = 0.5 to P(Urn A | Black) = 0.333

Likelihood Principle



- Way of thinking…
- Hypotheses: Urn A or Urn B ?
- Observation: "Black"
- Prior probabilities: P(Urn A) and P(Urn B)
- Likelihood of Black given choice of Urn: {aka *forward probability*}
  - P(Black | Urn A) and P(Black | Urn B)
- Posterior Probability: of each hypothesis given evidence
  - P(Urn A | Black) {aka *inverse probability*}
- Likelihood Principle (informal): All inferences depend ONLY on
  - The likelihoods P(Black | Urn A) and P(Black | Urn B), and
  - The priors P(Urn A) and P(Urn B)
- Result is a probability (or distribution) model over the space of possible hypotheses.

Maximum Likelihood (intuition)

- Recall:
- **P(Urn A | Black) = P(Urn A and Black) /P(Black) =**

  P(Black | UrnA)*P(Urn A) / P(Black)

- **P(Urn? | Black)** is maximized when **P(Black | Urn?)** is maximized.
  - Maximization over the hypotheses space (Urn A or Urn B)

- **P(Black | Urn?)** = "likelihood"
- => "*Maximum Likelihood*" approach to maximizing posterior probability

Maximum Likelihood (ML): mechanics

- **Independent Observations** (like Black): $\mathbf{X_1, \ldots, X_n}$
- **Hypothesis θ**
- **Likelihood Function**: $L(\theta) = P(X_1, \ldots, X_n \mid \theta) = \Pi_i P(X_i \mid \theta)$
  - {Independence => multiply individual likelihoods}
- **Log Likelihood** $\mathbf{LL(\theta) = \Sigma_i \log P(X_i \mid \theta)}$

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