EC 7xx Advanced Digital Communications
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Random Process and Optimum Detection

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Syllabus

- Tentatively

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Random Sequences and Random Processes

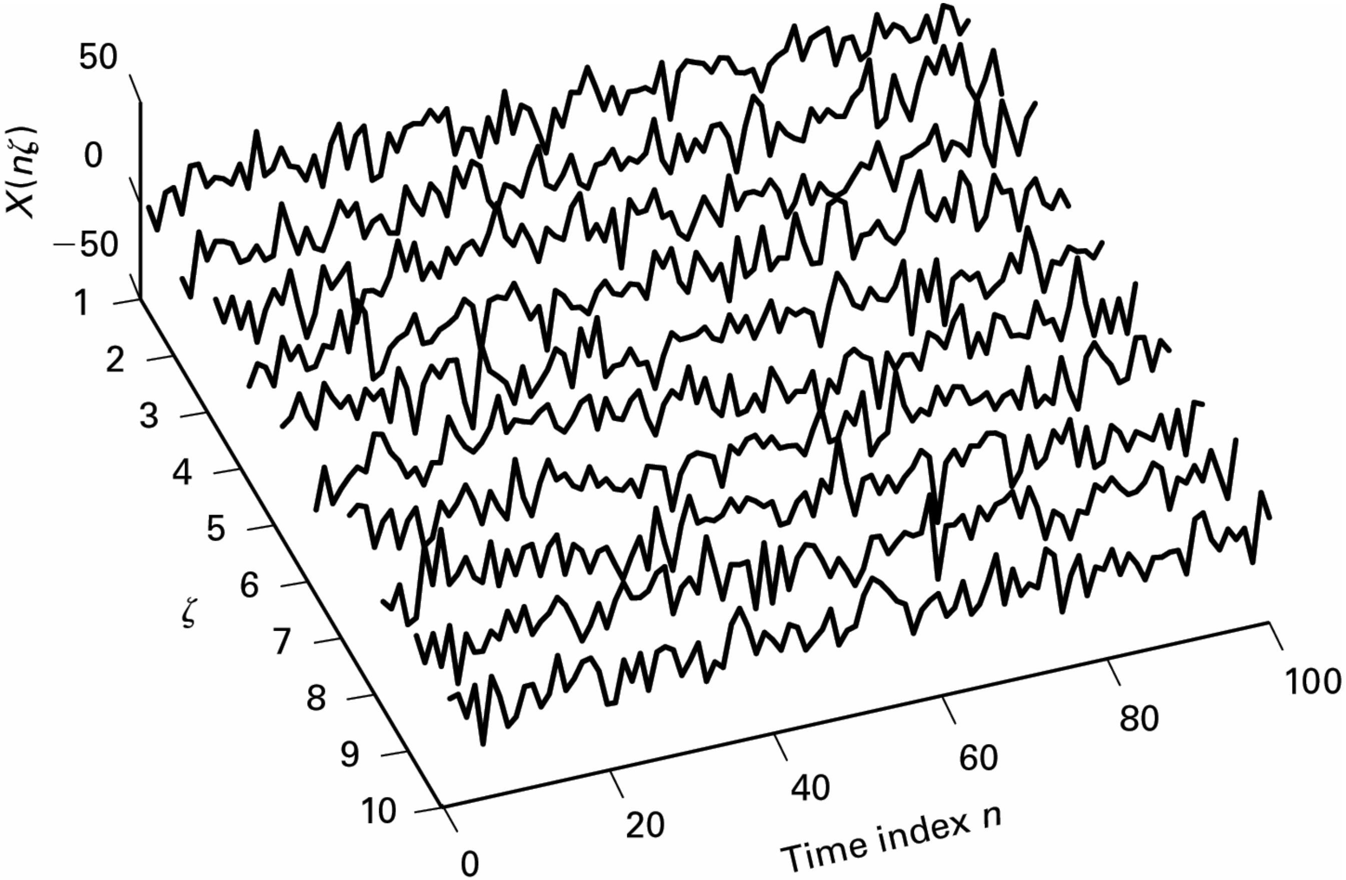
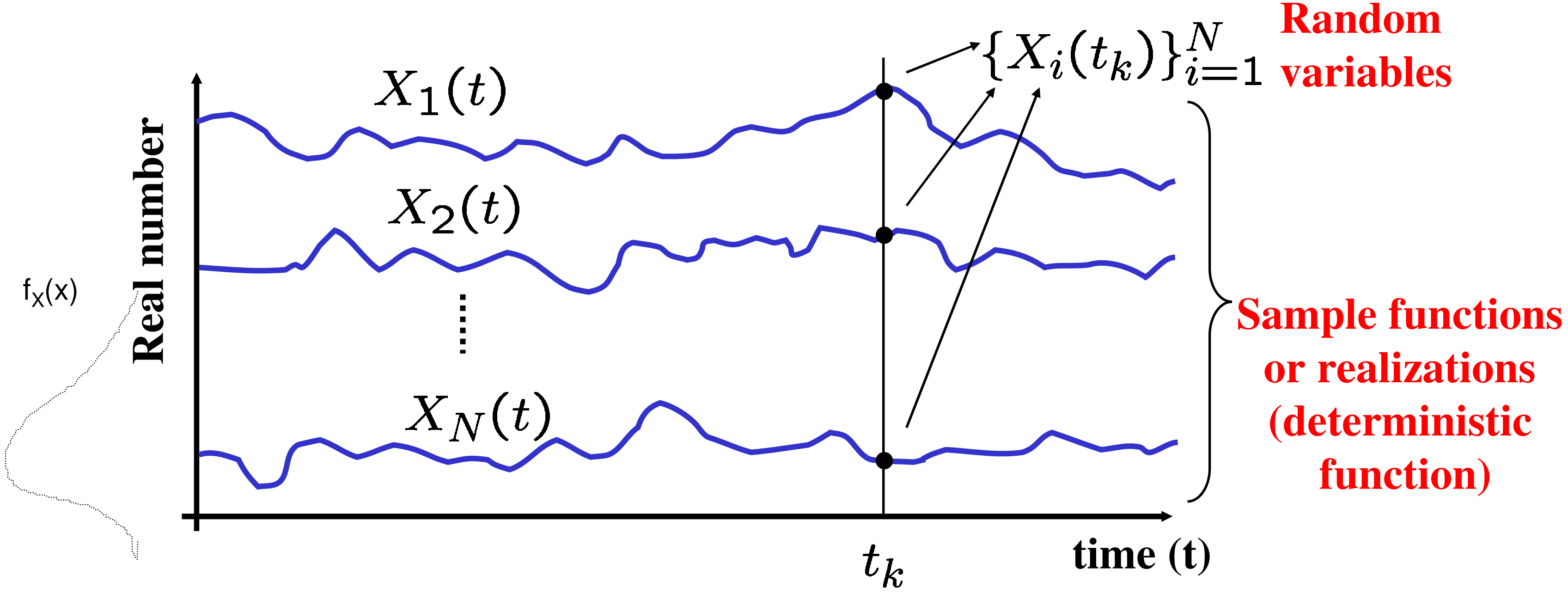


Illustration of the concept of *random sequence* $X(n, \zeta)$ where the ζ domain (i.e., the sample space Ω) consists of just 10 values. (Samples connected for plot.)

Random process

- A random process is a collection of time functions, or signals, corresponding to various outcomes of a random experiment. For each outcome, there exists a deterministic function, which is called a sample function or a realization.

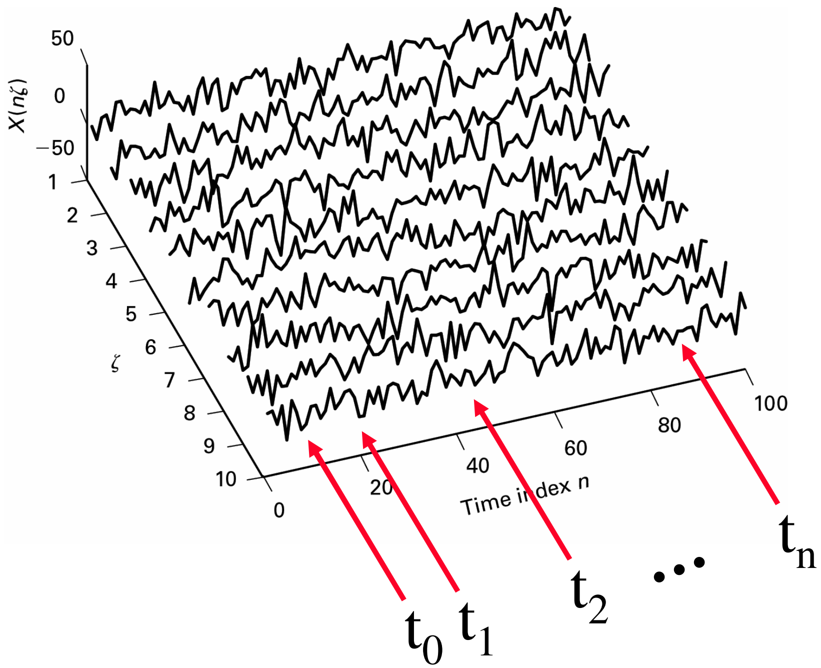


Specifying a Random Process

- A random process is defined by all its joint CDFs

$$p(X(t_0) \leq x_0, X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n).$$

for *all possible sets of sample times* $\{t_0, t_1, \ldots, t_n\}$



Stationarity

- If time-shifts (any value T) do not affect its joint CDF

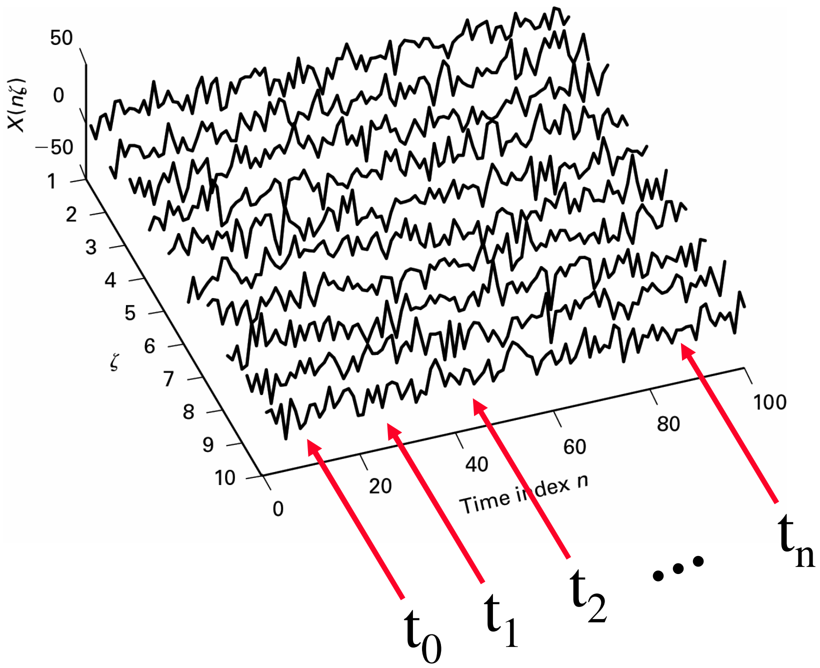
$$p(X(t_0) \leq x_0, X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n) =$$

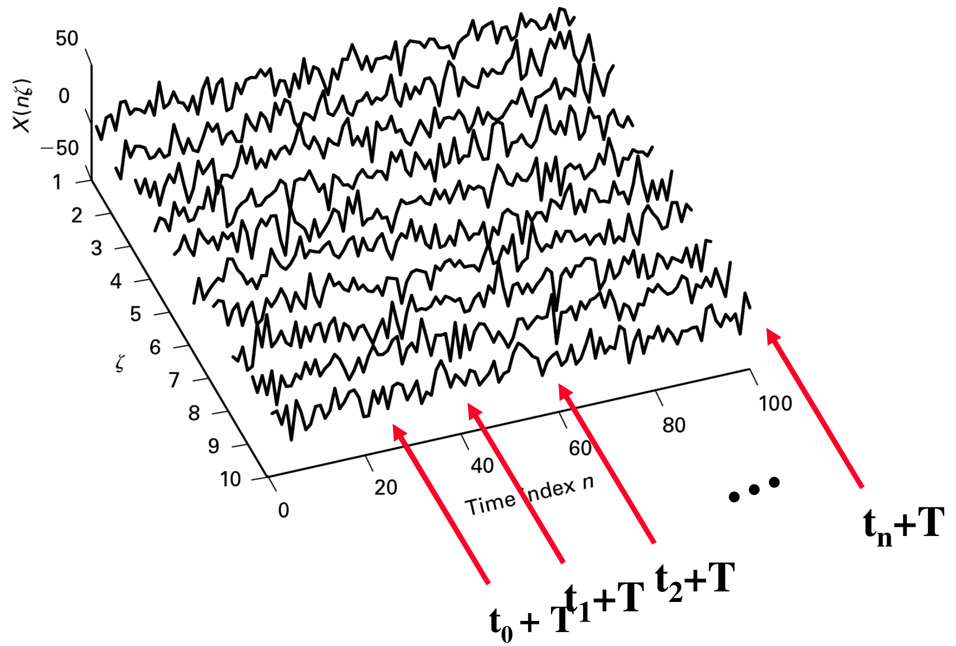
$$p(X(t_0+T) \leq x_0, X(t_1+T) \leq x_1, \ldots, X(t_n+T) \leq x_n).$$

$$A_X(t, t+\tau) =$$

$$\mathbf{E}[X(t)] = \mathbf{E}[X(t-t)] = \mathbf{E}[X(0)] = \mu_X.$$

$$\mathbf{E}[X(t-t)X(t+\tau-t)] = \mathbf{E}[X(0)X(\tau)] \triangleq A_X(\tau)$$





Wide Sense Stationarity (wss)

$$\mathbf{E}[X(t)] = \mathbf{E}[X(t-t)] = \mathbf{E}[X(0)] = \mu_X.$$

$$A_X(t, t+\tau) = \mathbf{E}[X(t-t)X(t+\tau-t)] = \mathbf{E}[X(0)X(\tau)] \triangleq A_X(\tau)$$

- Keep only above two properties (2nd order stationarity)…
  - Don't insist that higher-order moments or higher order joint CDFs be unaffected by lag T
- With LTI systems, we will see that WSS inputs lead to WSS outputs,
  - In particular, if a WSS process with PSD $S_X(f)$ is passed through a linear time-invariant filter with frequency response $H(f)$, then the filter output is also a WSS process with power spectral density $|H(f)|^2 S_X(f)$.
- Gaussian *w.s.s.* = Gaussian stationary process (since it only has 2nd order moments)

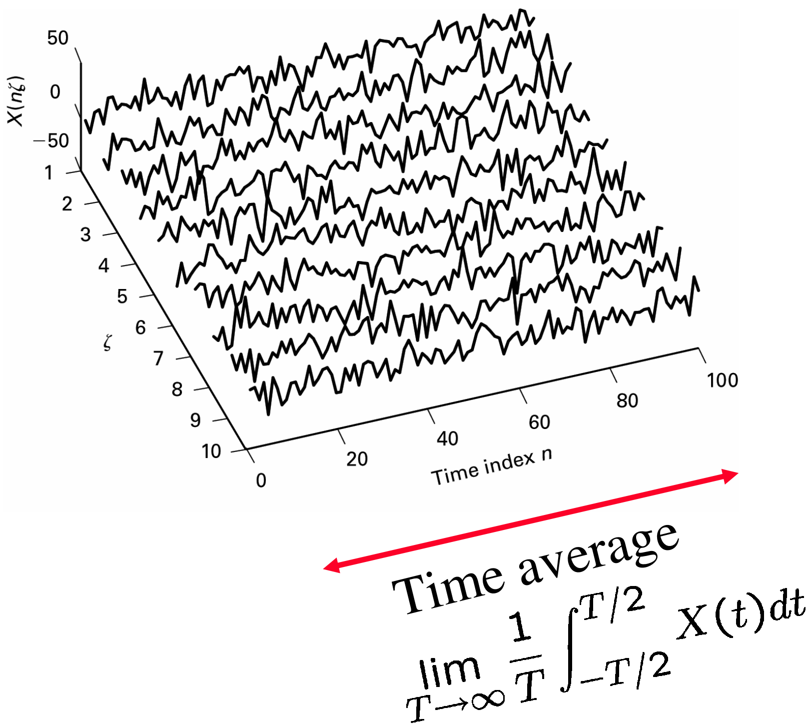
Ergodicity

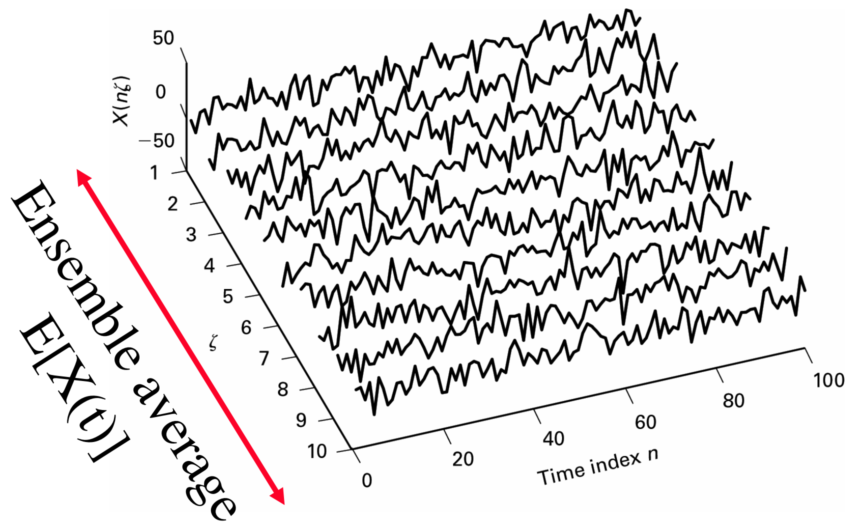
- Time averages = Ensemble averages

[i.e. "***ensemble***" averages like mean/autocorrelation can be computed as "***time-averages***" over a single realization of the random process]

- A random process: ergodic in *mean* and *autocorrelation* (like w.s.s.) if

$$m_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$$ and $$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)X^*(t-\tau)dt$$





Autocorrelation: Summary

- Autocorrelation of an energy signal

$$R_x(\tau) = x(\tau) \star x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a power signal

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

  - For a periodic signal:

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a random signal

$$R_X(t_i, t_j) = \mathsf{E}[X(t_i)X^*(t_j)]$$

  - For a WSS process:

$$R_X(\tau) = \mathsf{E}[X(t)X^*(t-\tau)]$$

Power Spectral Density (PSD)

The power spectral density (PSD) of a WSS process is defined as the Fourier transform of its autocorrelation function with respect to τ:

$$S_X(f) = \int_{-\infty}^{\infty} A_X(\tau) e^{-j2\pi f \tau} d\tau. \quad \text{(B.26)}$$

The autocorrelation can be obtained from the PSD through the inverse transform:

$$A_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df. \quad \text{(B.27)}$$

The PSD takes its name from the fact that the expected power of a random process $X(t)$ is the integral of its PSD:

$$\mathbf{E}[X^2(t)] = A_X(0) = \int_{-\infty}^{\infty} S_X(f) df, \quad \text{(B.28)}$$

1. $S_X(f)$ is real and $S_X(f) \geq 0$
2. $S_X(-f) = S_X(f)$
3. $A_X(0) = \int S_X(\omega)\, d\omega$

Spectral density: Summary

- **<u>Energy signals</u>**:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad X(f) = \mathcal{F}[x(t)]$$

  - Energy spectral density (ESD): $\Psi_x(f) = |X(f)|^2$

- **<u>Power signals:</u>**

$$P_x = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \{c_n\} = \mathcal{F}[x(t)]$$

  - ***<u>Power spectral density (PSD):</u>***

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0) \qquad f_0 = 1/T_0$$

- **<u>Random process</u>**:
  - Power spectral density (PSD):

$$G_X(f) = \mathcal{F}[R_X(\tau)]$$

<u>Note</u>: we have used *f* for *ω* and *Gx* for *Sx*

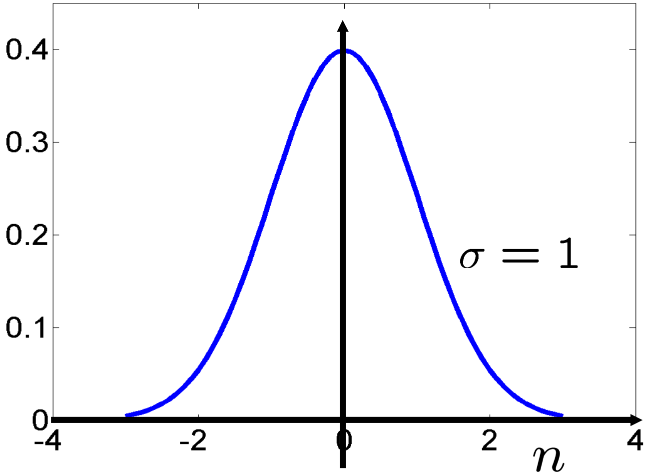
Properties of an autocorrelation function

- For real-valued (and WSS for random signals):
  1. Autocorrelation and spectral density form a Fourier transform pair. $R_X(\tau) \leftrightarrow S_X(\omega)$
  2. Autocorrelation is symmetric around zero. $R_X(-\tau) = R_X(\tau)$
  3. Its maximum value occurs at the origin. $|R_X(\tau)| \leq R_X(0)$
  4. Its value at the origin is equal to the average power or energy. $\mathbf{E}[X^2(t)] = A_X(0) = \int_{-\infty}^{\infty} S_X(f)df,$

Noise in communication systems

- Thermal noise is described by a zero-mean Gaussian random process, n(t).
- Its PSD is flat, hence, it is called white noise. IID gaussian.

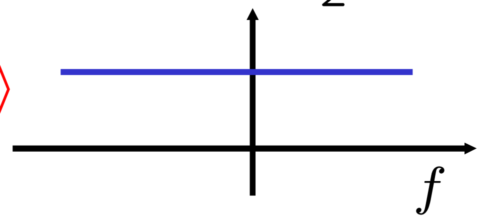
$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma^2}\right]$$



Probability density function

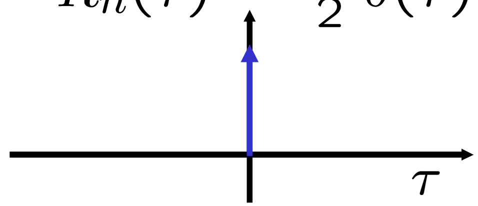
$$G_n(f) = \frac{N_0}{2} \text{ [w/Hz]}$$

Power spectral density



$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

Autocorrelation function



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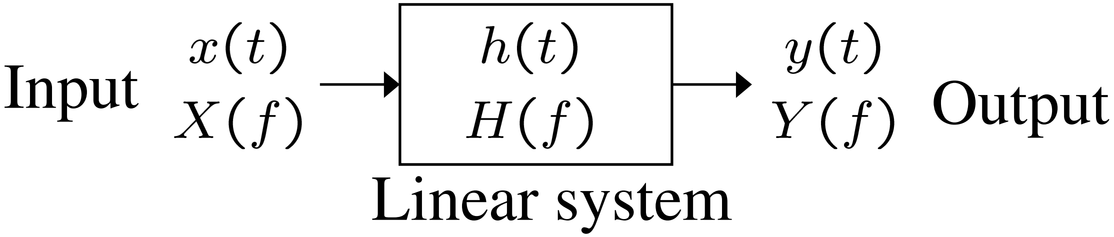
White Gaussian Noise

- White:
  - *Power spectral density (PSD)* is the same, i.e. *flat*, for all frequencies of interest (from dc to 10^{12} Hz)
  - Autocorrelation is a *delta* function => two samples no matter however close are uncorrelated.
    - $N_0/2$ to indicate two-sided PSD
    - Zero-mean gaussian completely characterized by its variance (σ^2)
    - Variance of filtered noise is finite = $N_0/2$
  - Similar to "white light" contains equal amounts of all frequencies in the visible band of EM spectrum
- Gaussian + uncorrelated => i.i.d.
  - Affects each symbol independently: memoryless channel
- Practically: if b/w of noise is much larger than that of the system: good enough

Signal transmission w/ linear systems (filters)



- Deterministic signals: $Y(f) = X(f)H(f)$
- Random signals: $G_Y(f) = G_X(f)|H(f)|^2$

<u>Ideal distortion less transmission:</u>

• All the frequency components of the signal not only arrive with an <u>*identical time delay*</u>, but also <u>*amplified or attenuated equally*</u>.

$$y(t) = Kx(t - t_0) \text{ or } H(f) = Ke^{-j2\pi f t_0}$$

(Deterministic) Systems with Stochastic Inputs

A deterministic system[1] transforms each input waveform $X(t,\xi_i)$ into an output waveform $Y(t,\xi_i) = T[X(t,\xi_i)]$ by operating only on the time variable t. Thus a set of realizations at the input corresponding to a process $X(t)$ generates a new set of realizations $\{Y(t,\xi)\}$ at the output associated with a new process $Y(t)$.

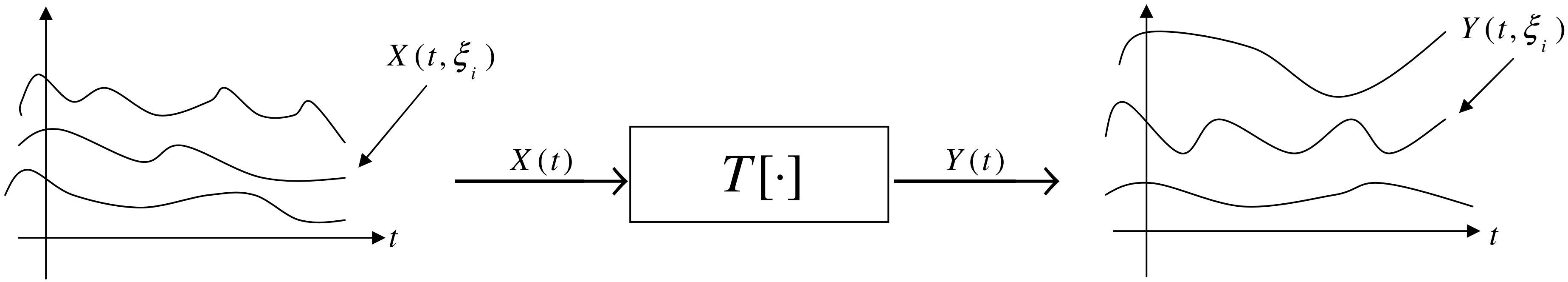


Fig. 14.3

Our goal is to study the output process statistics in terms of the input process statistics and the system function.

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[1]A stochastic system on the other hand operates on both the variables t and ξ.

Deterministic Systems

Memoryless Systems

$$Y(t) = g[X(t)]$$

Systems with Memory

Time-varying systems

Time-Invariant systems

Linear systems

$$Y(t) = L[X(t)]$$

Linear-Time Invariant (LTI) systems

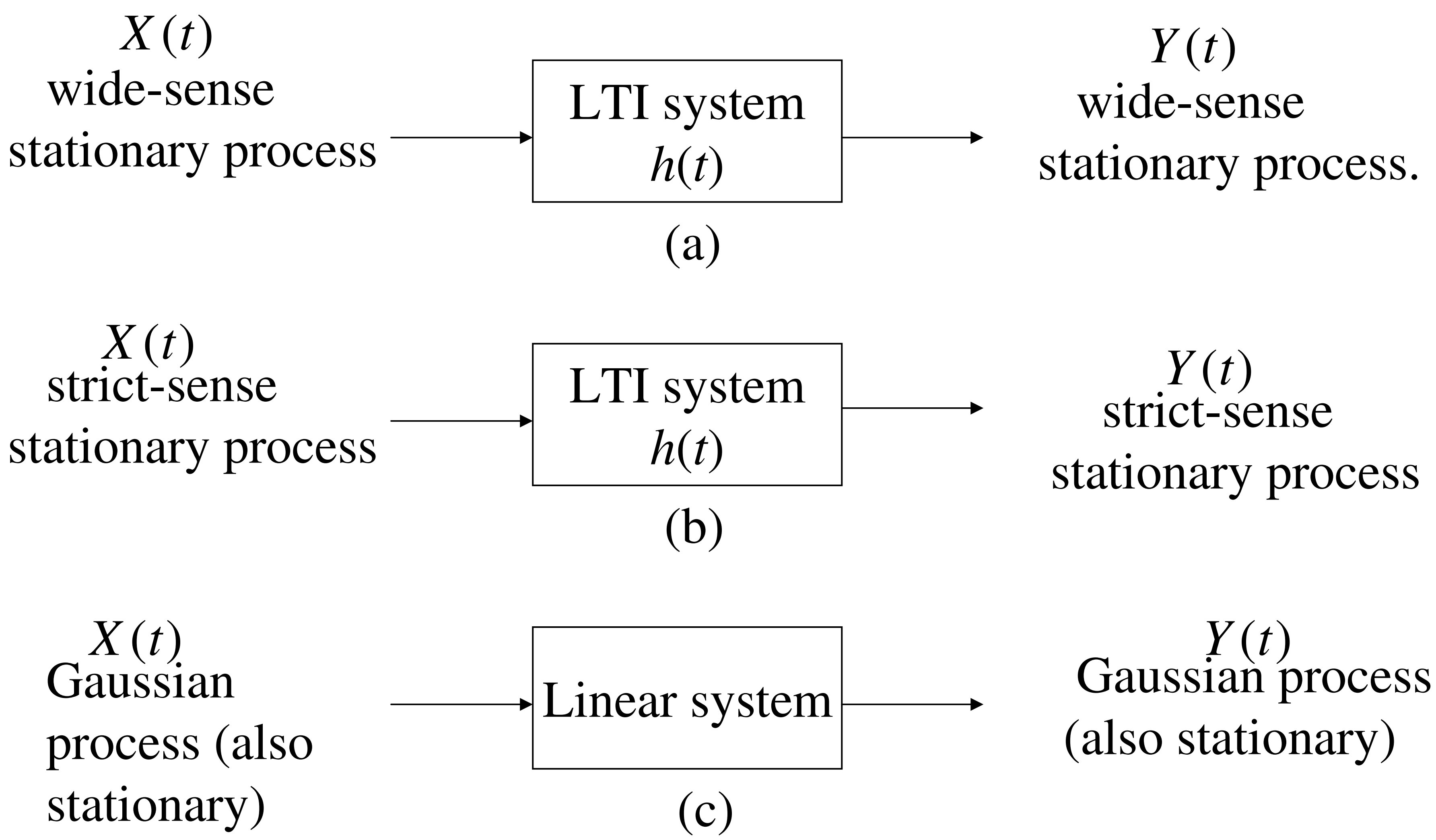
$X(t) \longrightarrow$ $h(t)$ $\longrightarrow$

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau.$$

LTI system

LTI Systems: WSS input good enough



White Noise Process & LTI Systems

$W(t)$ is said to be a white noise process if

$$R_{WW}(t_1, t_2) = q(t_1)\delta(t_1 - t_2),$$

i.e., $E[W(t_1)\ W^*(t_2)] = 0$ unless $t_1 = t_2$.

$W(t)$ is said to be *wide-sense stationary (w.s.s) white noise*
if **$E[W(t)]$ = constant**, and

$$R_{WW}(t_1, t_2) = q\delta(t_1 - t_2) = q\delta(\tau).$$

If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables

Narrowband Noise Representation

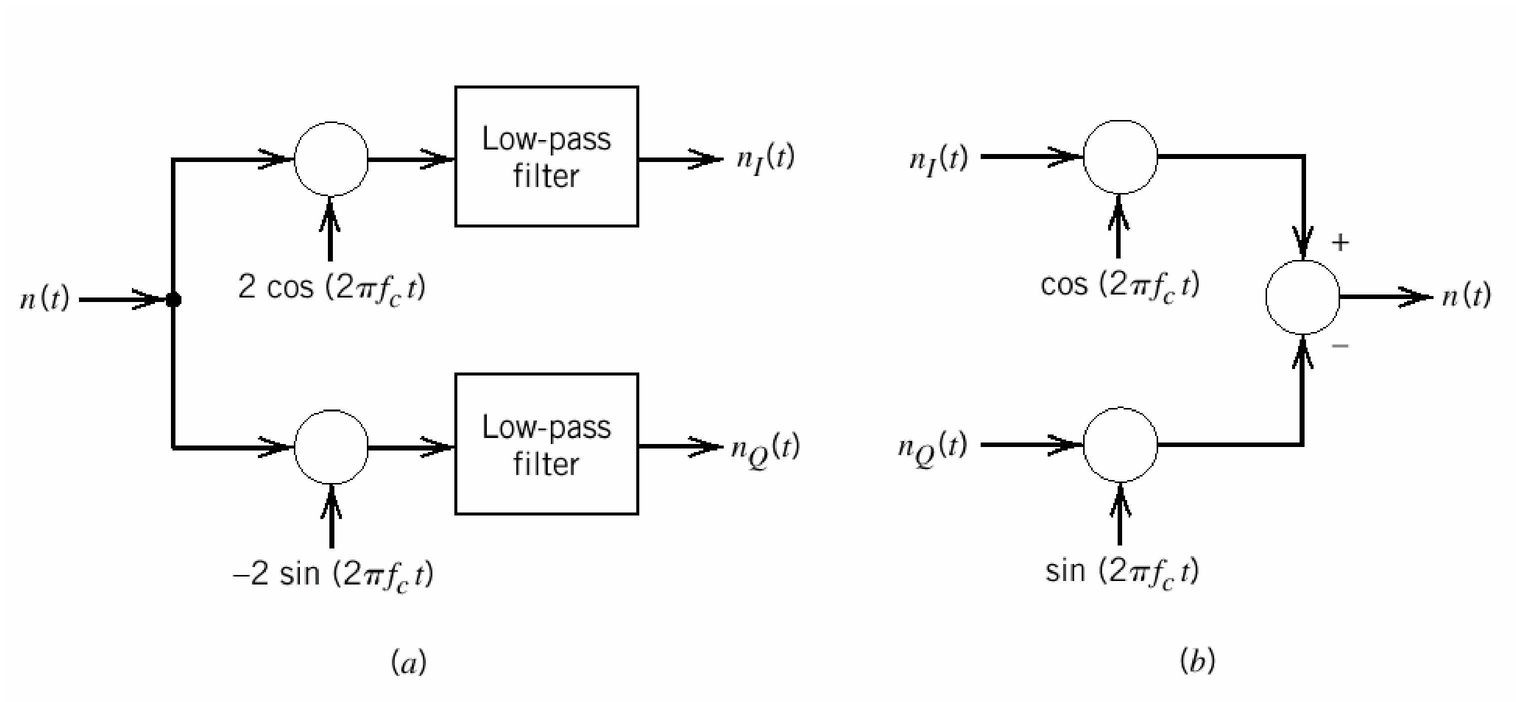
- The noise process appearing at the output of a narrowband filter is called ***narrowband noise***.
- Representations of narrowband noise
  - A pair of component called the ***in-phase*** and ***quadrature*** components.
  - Two other components called the ***envelop*** and ***phase***.

Representation of Narrowband Noise in Terms of In-Phase and Quadrature Components

- Consider a narrowband noise $n(t)$ of bandwidth $2B$ centered on frequency.
- We may represent n(t) in the canonical (standard) form:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

where, $n_I(t)$ is ***in-phase*** component of $n(t)$ and $n_Q(t)$ is ***quadrature*** component of $n(t)$.



- $n_I(t)$ and $n_Q(t)$ have important properties:
  - $n_I(t)$ and $n_Q(t)$ have zero mean.
  - $n(t)$ is Gaussian, then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian.
  - $n(t)$ is stationary, then $n_I(t)$ and $n_Q(t)$ are jointly stationary.
  - Both $n_I(t)$ and $n_Q(t)$ have the same power spectral density.

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & otherwise \end{cases}$$

  - $n_I(t)$ and $n_Q(t)$ have the same variance as $n(t)$

- The cross-spectral density of the $n_I(t)$ and $n_Q(t)$ is purely imaginary:

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f)$$
$$= \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & -B \le f \le B \\ 0, & otherwise \end{cases}$$

- If $n(t)$ is Gaussian and its power spectral density $S_N(t)$ is symmetric about the mid-band frequency f_c ,

then $n_I(t)$ and $n_Q(t)$ are statistically independent.

Representation of Narrowband Noise in Terms of Envelope and Phase Components

- Here we represent $n(t)$ in terms of envelope and phase components:

$$n(t) = r(t)\cos[2\pi f_c t + \psi(t)]$$

where, $r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$ and $\psi(t) = \tan^{-1}\left[\dfrac{n_Q(t)}{n_I(t)}\right]$

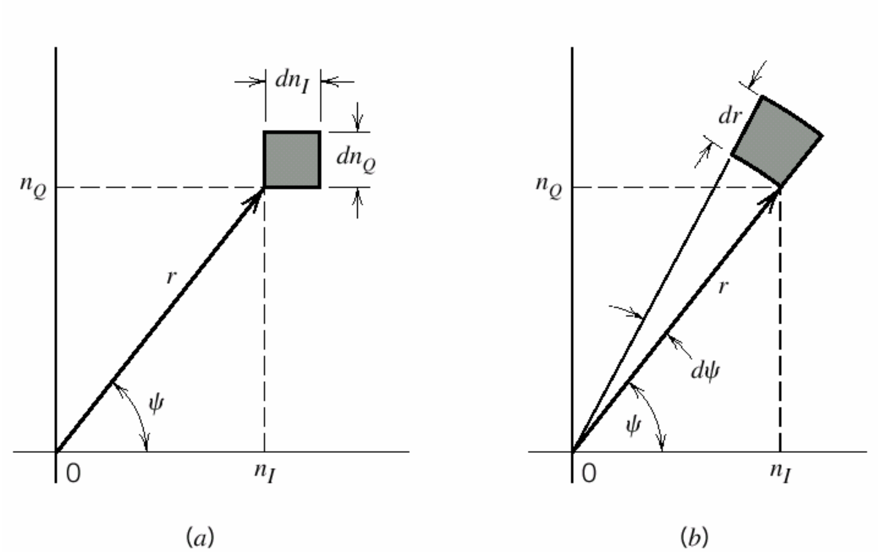
- $r(t)$ is called the ***envelope*** of $n(t)$, and the $\psi(t)$ is called the ***phase*** of $n(t)$.

- The probability distributions of $r(t)$ and $\psi(t)$ may be obtained from those of $n_I(t)$ and $n_Q(t)$ as follows.

- Let N_I and N_Q denote the random variables obtained by the sample functions $n_I(t)$ and $n_Q(t)$, respectively.
- Then, N_I and N_Q are independent Gaussian random variables of zero mean and variance σ^2 .
- So, we may express their joint probability density function by

$$f_{N_I,N_Q}(n_I,n_Q) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{n_I^{\;2}+n_Q^{\;2}}{2\sigma^2}\right)$$

- **Figure** Illustrating the coordinate system for representation of narrowband noise: (*a*) in terms of in-phase and quadrature components, and (*b*) in terms of envelope and phase.



$$n_I = r\cos\psi$$
$$n_Q = r\sin\psi$$

$$dn_I dn_Q = r dr d\psi$$

- Now, let R and Ψ denote the random variables obtained by the sample functions $r(t)$ and $\psi(t)$, respectively.

- Then we find the joint probability density function of R and is Ψ

$$f_{R,\Psi}(r,\psi) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad (1.113)$$

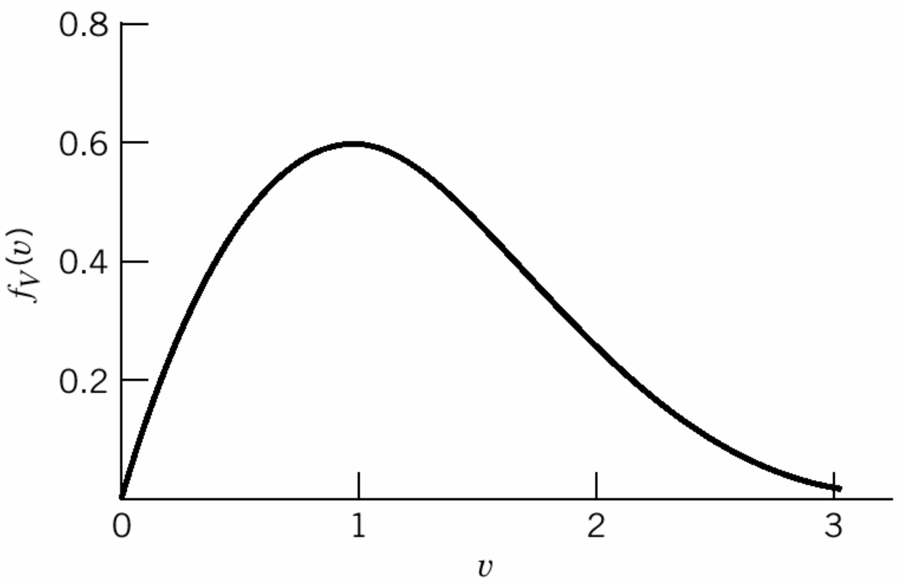
  - From (1.113), the random variables R and Ψ are statistically independent.
  - Therefore,

$$f_{\Psi}(\psi) = \begin{cases} \frac{1}{2\pi}, & 0 \le \psi \le 2\pi \\ 0, & elsewhere \end{cases}$$

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), & r \ge 0 \\ 0, & elsewhere \end{cases}$$

- ***Rayleigh distribution*** (Figure 1.22) : A random variable having the probability density function of Equation (1.115).
- Let, $\upsilon = r / \sigma$ then the normalized form is

$$f_V(v) = \begin{cases} v \exp\left(-\dfrac{v2}{2}\right), & v \geq 0 \\ 0, & elsewhere \end{cases}$$



Sine Wave Plus Narrowband Noise

- Add the sinusoidal wave $A\cos(2\pi f_c t)$ to the narrowband noise $n(t)$.

$$x(t) = A\cos(2\pi f_c t) + n(t)$$

- Use in-phase and quadrature components for $n(t)$

$$x(t) = n_I'(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

where, $n_I'(t) = A + n_I(t)$

- We assume that $n(t)$ is Gaussian with zero mean and variance σ^2, then we find that:

- Joint probability density function of N_I' and N_Q

$$f_{N_I',N_Q}(n_I',n_Q)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{(n_I'-A)^2+n_Q{}^2}{2\sigma^2}\right)$$

- Joint probability density function of R and Ψ

$$f_{R,\Psi}(r,\psi)=\frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+A^2-2Ar\cos\psi}{2\sigma^2}\right)$$

- Now we are interested in the probability density function of R

$$f_R(r) = \int_0^{2\pi} f_{R,\Psi}(r,\psi)d\psi$$

$$= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)\underline{\int_0^{2\pi}\exp\left(\frac{Ar}{\sigma^2}\cos\psi\right)d\psi} \qquad (1.126)$$

modified Bessel function of the first kind of zero order

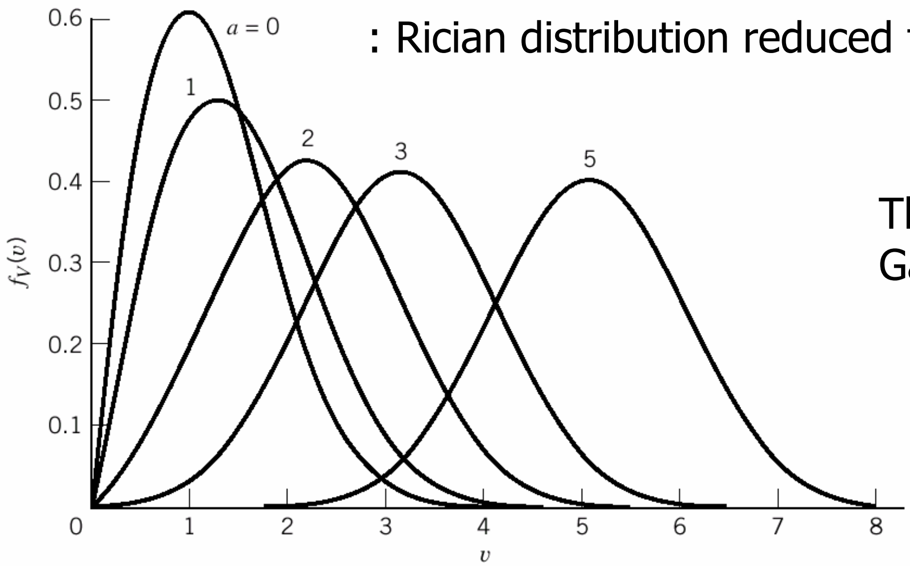
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos\psi)d\psi \quad , x = Ar/\sigma^2$$

- Rewrite (1.126)

$$f_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right)$$

- This is called the ***Rician distribution***. (Figure 1.23)
- Let $v = r/\sigma,\ \ a = A/\sigma$, then the normalized form is

$$f_V(v) = v\exp\left(-\frac{v^2 + a^2}{2}\right) I_0(av)$$



: Rician distribution reduced to the Rayleigh distribution

The envelope distribution is approximately Gaussian when a is large

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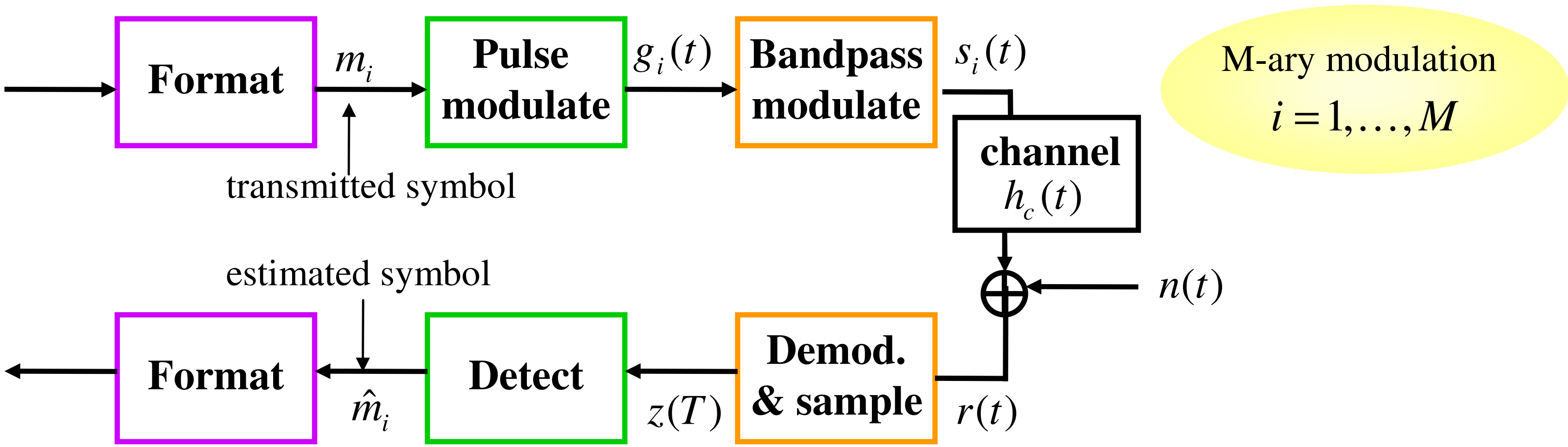
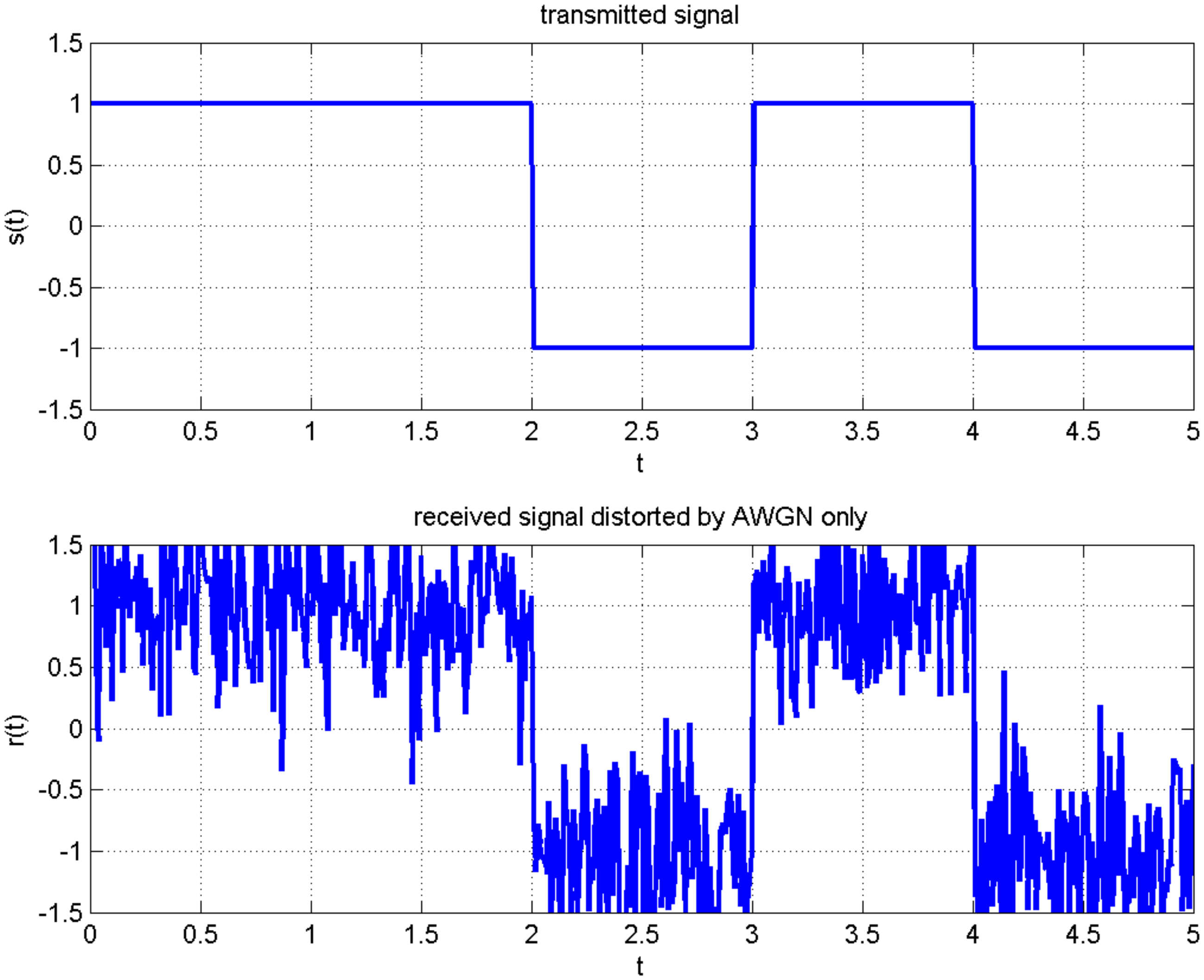
Demodulation and Detection

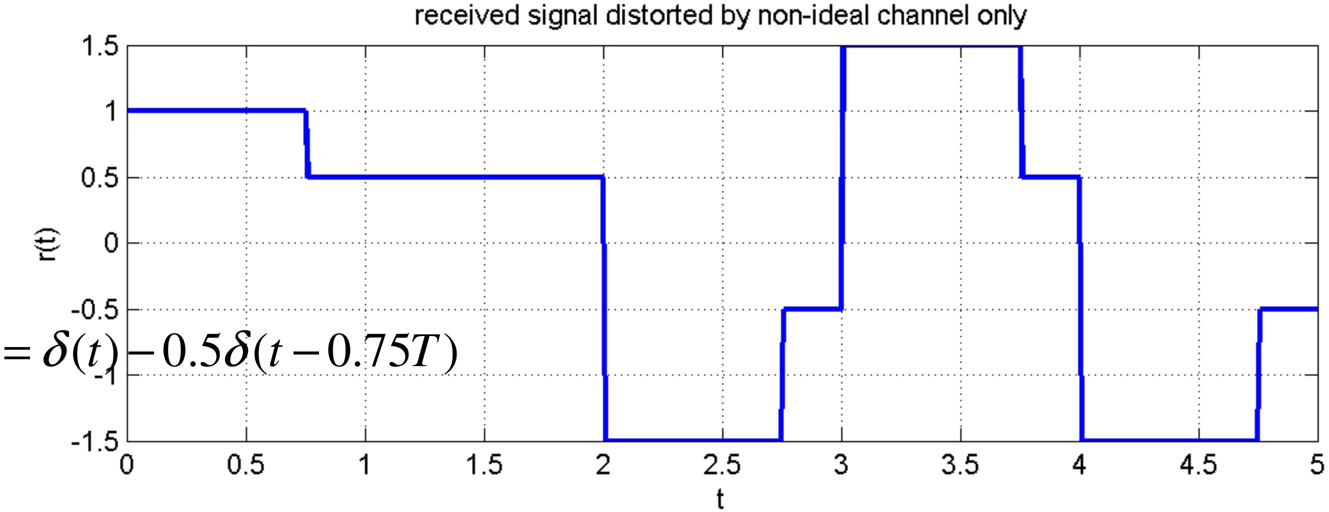


- Major sources of errors:
  - Thermal noise (AWGN)
    - disturbs the signal in an additive fashion (Additive)
    - has flat spectral density for all frequencies of interest (White)
    - is modeled by Gaussian random process (Gaussian Noise)
  - Inter-Symbol Interference (ISI)
    - Due to the filtering effect of transmitter, channel and receiver, symbols are "smeared".

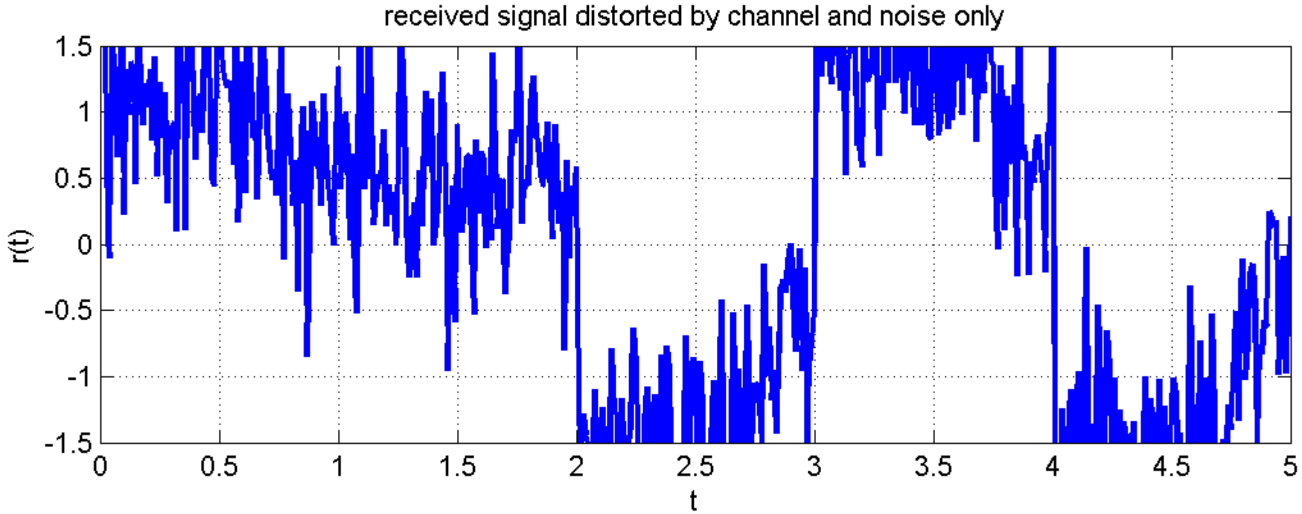
Example: Impact of the channel



Example: Channel impact …



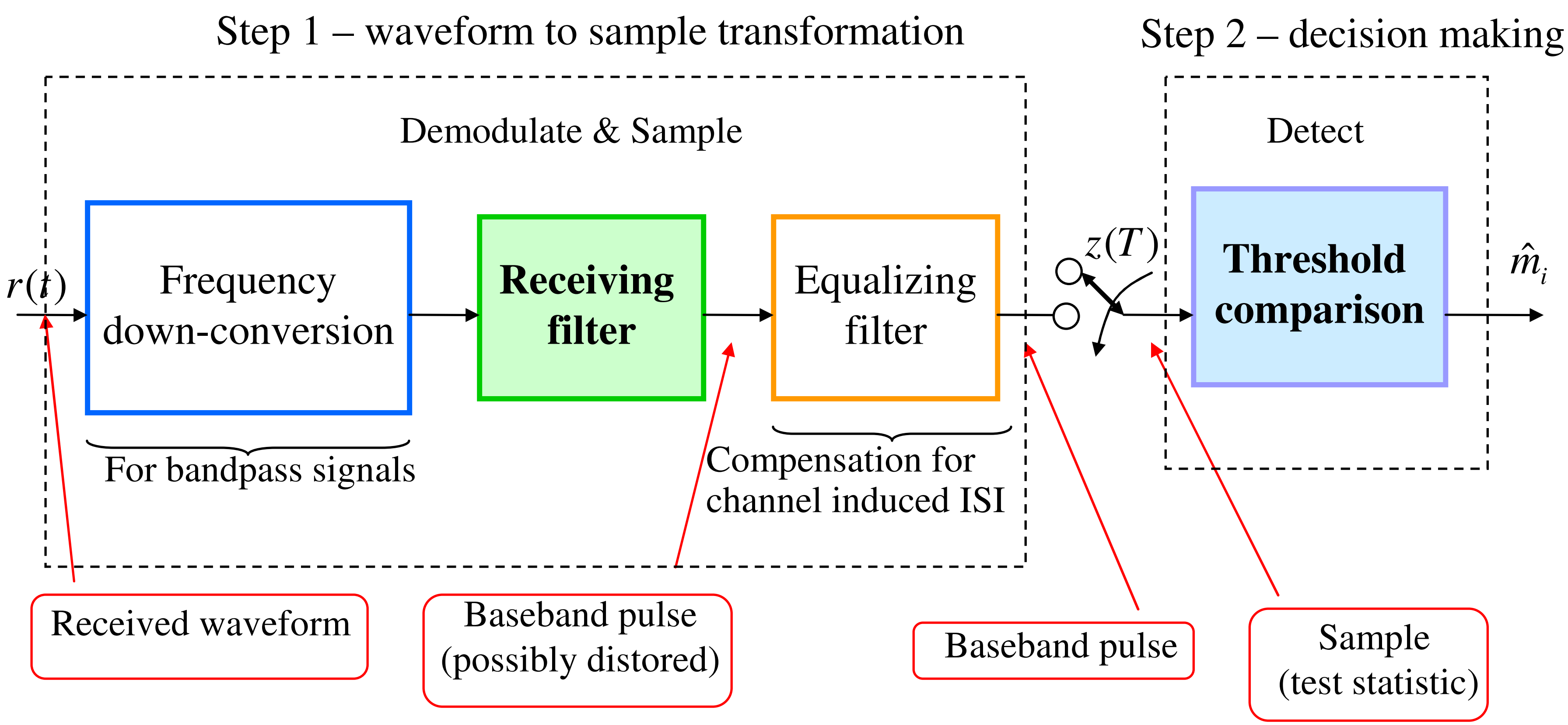
$$h_c(t) = \delta(t) - 0.5\delta(t - 0.75T)$$



Receiver tasks

- Demodulation and sampling:
  - Waveform recovery and preparing the received signal for detection:
    - Improving the signal power to the noise power (SNR) using matched filter
    - Reducing ISI using equalizer
    - Sampling the recovered waveform
- Detection:
  - Estimate the transmitted symbol based on the received sample

Receiver structure



Baseband and Bandpass

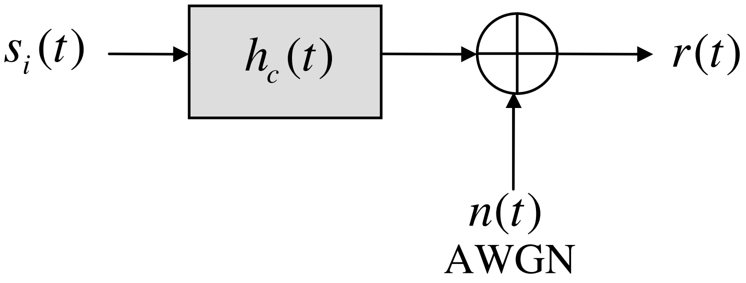
- Bandpass model of detection process is equivalent to baseband model because:
  - The received bandpass waveform is first transformed to a baseband waveform.
  - Equivalence theorem:
    - Performing bandpass linear signal processing followed by heterodyning the signal to the baseband, yields the same results as heterodyning the bandpass signal to the baseband , followed by a baseband linear signal processing.

Steps in designing the receiver

- Find optimum solution for receiver design with the following goals:
  1. Maximize SNR
  2. Minimize ISI
- Steps in design:
  - Model the received signal
  - Find separate solutions for each of the goals.
- First, we focus on designing a receiver which maximizes the SNR.

Design the receiver filter to maximize the SNR

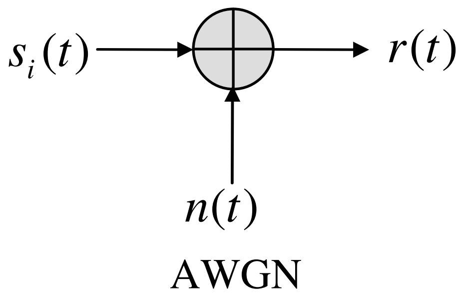
- Model the received signal



$$r(t) = s_i(t) * h_c(t) + n(t)$$

- Simplify the model:
  - Received signal in AWGN

Ideal channels
$h_c(t) = \delta(t)$



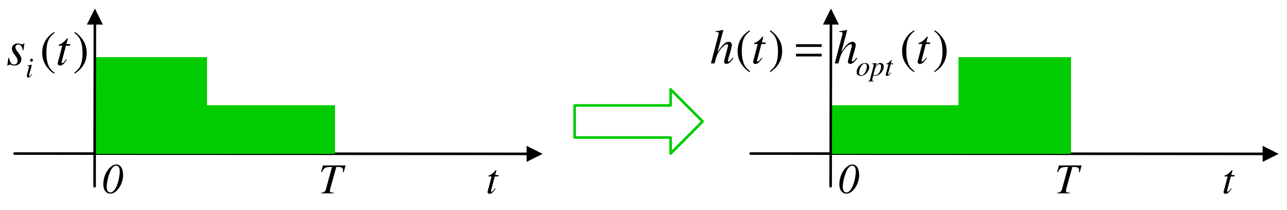
$$r(t) = s_i(t) + n(t)$$

Matched filter receiver

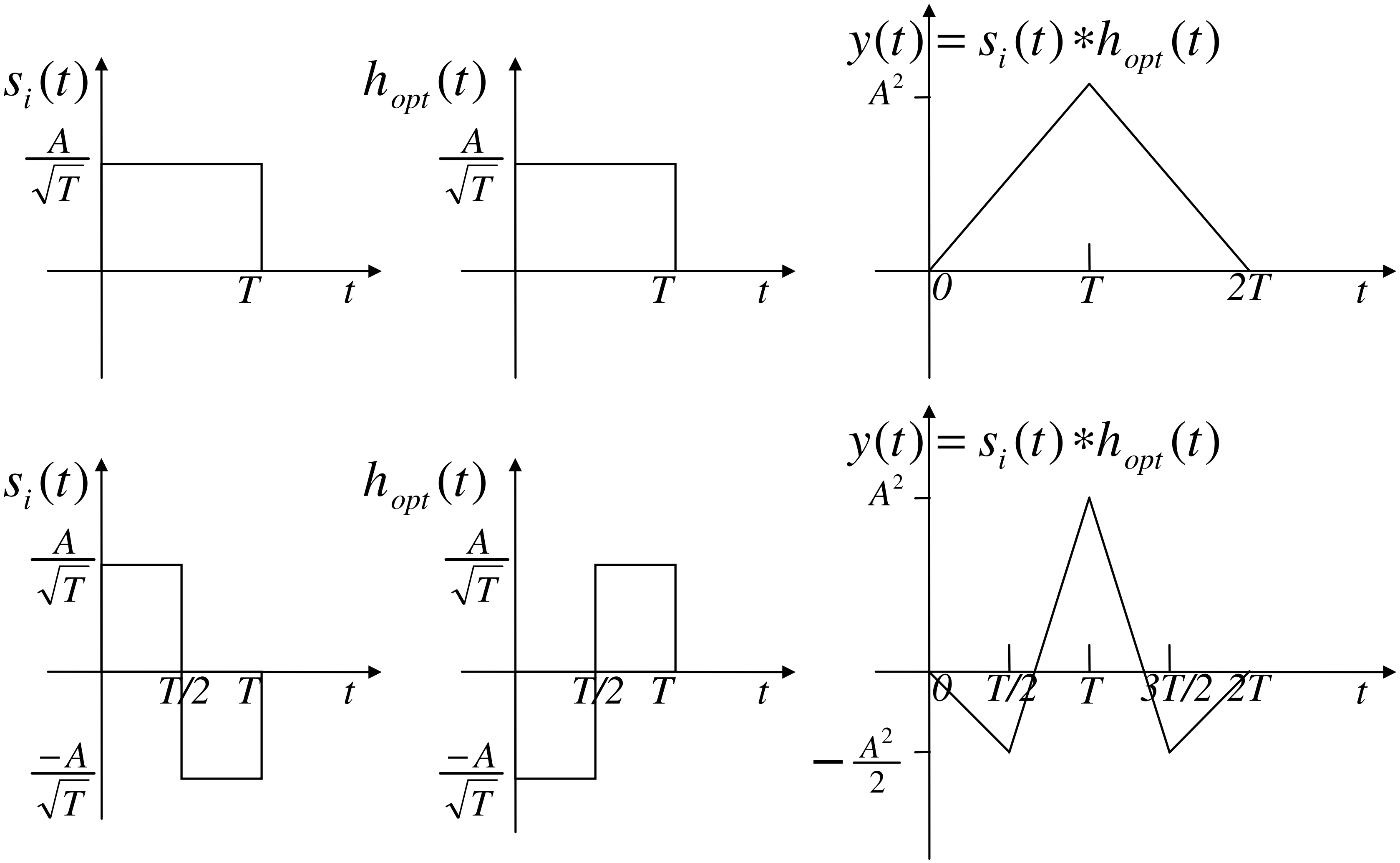
- Problem:
  - Design the receiver filter $h(t)$ such that the SNR is maximized at the <u>sampling time</u> when $s_i(t), i = 1,\ldots, M$ is transmitted.
- Solution:
  - The optimum filter, is the Matched filter, given by

$$h(t) = h_{opt}(t) = s_i^*(T - t)$$
$$H(f) = H_{opt}(f) = S_i^*(f)\exp(-j2\pi fT)$$

which is the time-reversed and delayed version of the conjugate of the transmitted signal



Example of matched filter



Properties of the matched filter

The Fourier transform of a matched filter output with the matched signal as input is, except for a time delay factor, proportional to the ESD of the input signal.

$$Z(f) = |S(f)|^2 \exp(-j2\pi fT)$$

The output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the input signal to which the filter is matched.

$$z(t) = R_s(t-T) \Rightarrow z(T) = R_s(0) = E_s$$

The output SNR of a matched filter depends only on the ratio of the signal energy to the PSD of the white noise at the filter input.

$$\max\left(\frac{S}{N}\right)_T = \frac{E_s}{N_0/2}$$

Two matching conditions in the matched-filtering operation:

spectral phase matching that gives the desired output peak at time *T*.

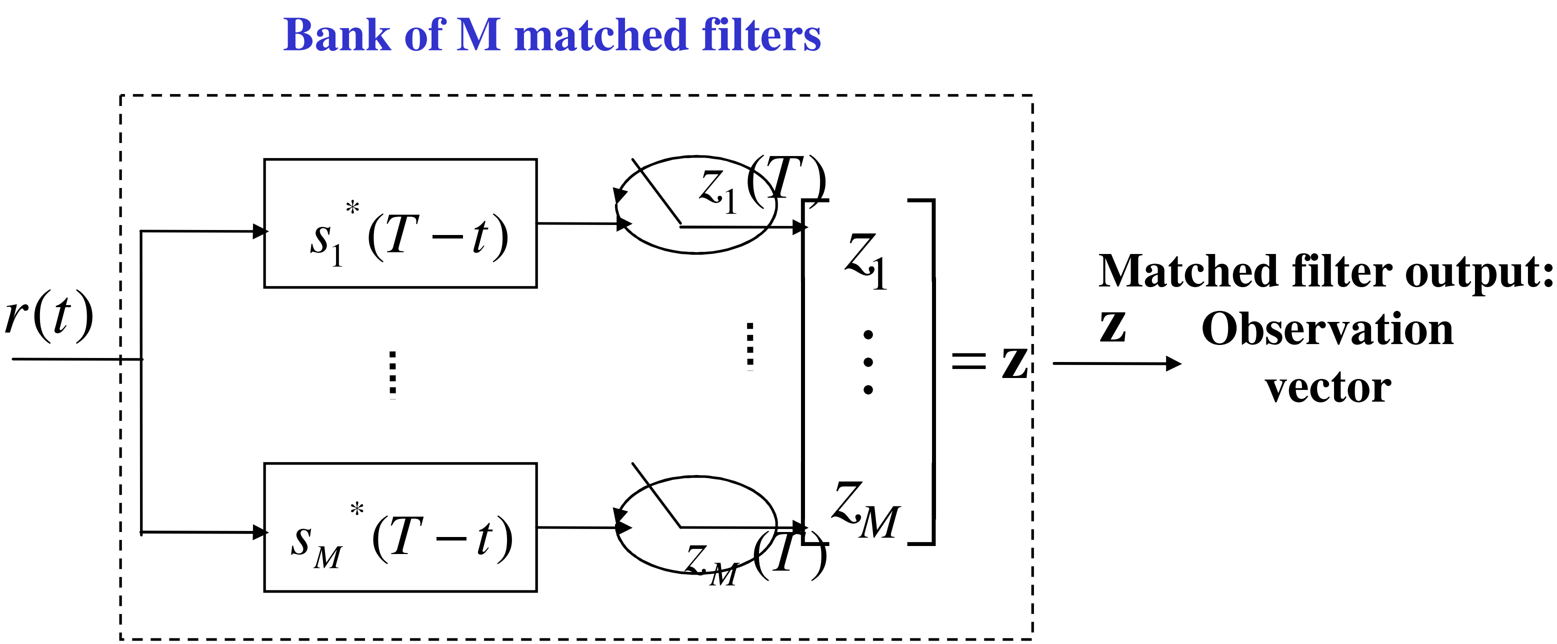
spectral amplitude matching that gives optimum SNR to the peak value.

Correlator receiver

- The matched filter output at the <u>sampling time</u>, can be realized as the correlator output.

$$z(T) = h_{opt}(T) * r(T)$$
$$= \int_0^T r(\tau) s_i^*(\tau) d\tau = < r(t), s(t) >$$

Implementation of matched filter receiver

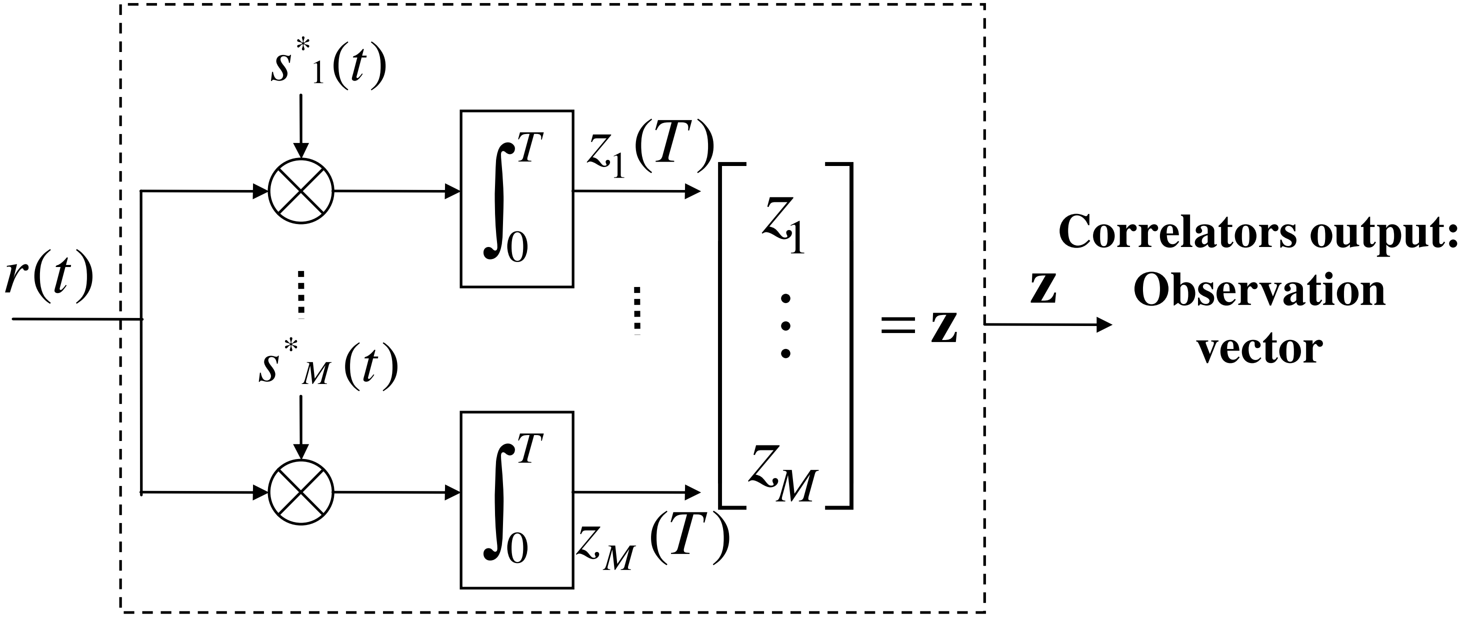


$$z_i = r(t) * s^*_i(T-t) \quad i = 1, \ldots, M$$

$$\mathbf{z} = (z_1(T), z_2(T), \ldots, z_M(T)) = (z_1, z_2, \ldots, z_M)$$

Implementation of correlator receiver

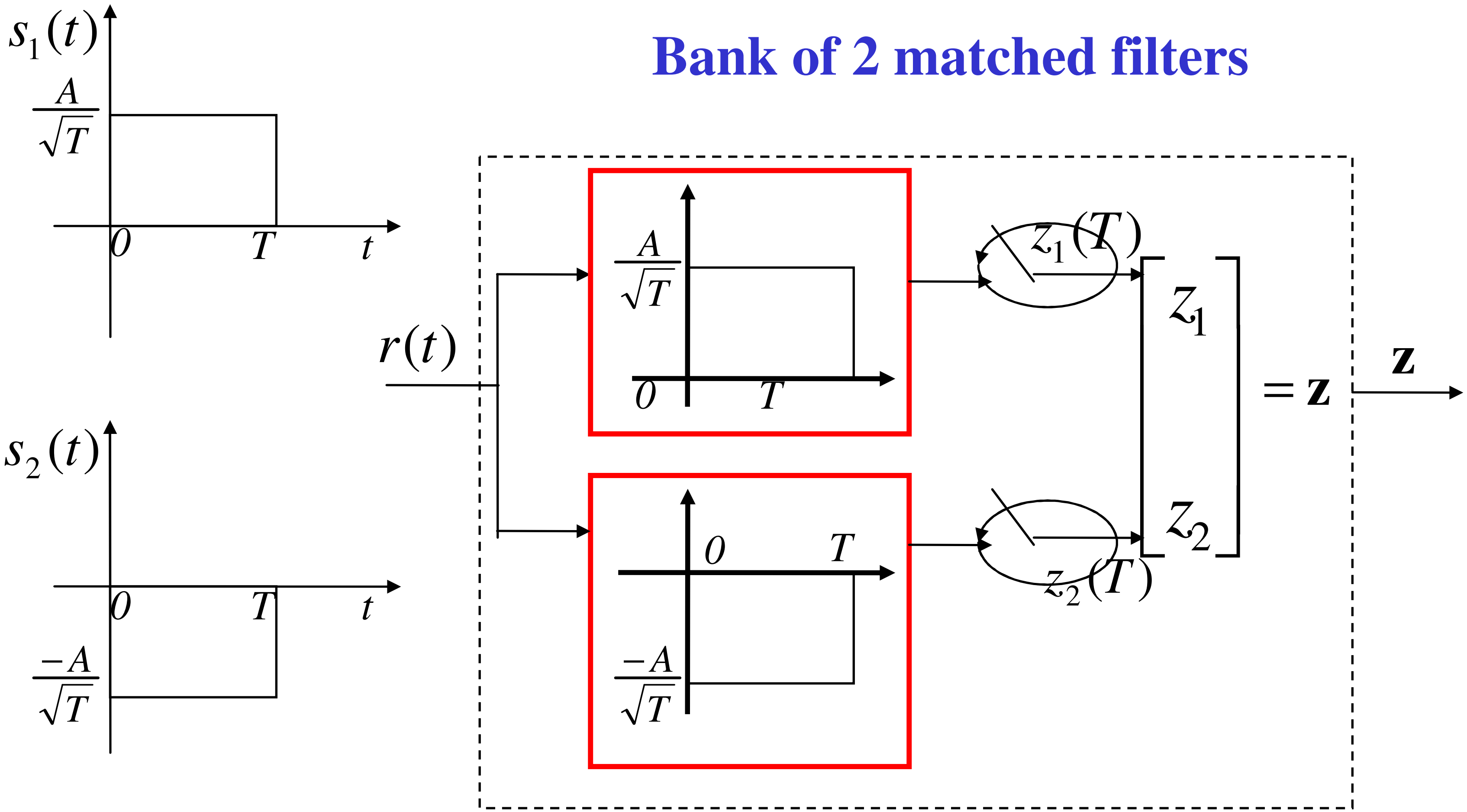
Bank of M correlators



$$\mathbf{z} = (z_1(T), z_2(T), \ldots, z_M(T)) = (z_1, z_2, \ldots, z_M)$$

$$z_i = \int_0^T r(t) s_i(t) dt \qquad i = 1, \ldots, M$$

Implementation example of matched filter receivers



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Questions?