EC 7xx Advanced Digital Communications
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Digital Signal Representation

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Syllabus

- Tentatively

Week 1	Overview, Probabilities, Random variables
Week 2	Random Process, Optimum Detection
Week 3	Digital Signal Representation
Week 4	
Week 5	
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Week 7	
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Week 10	
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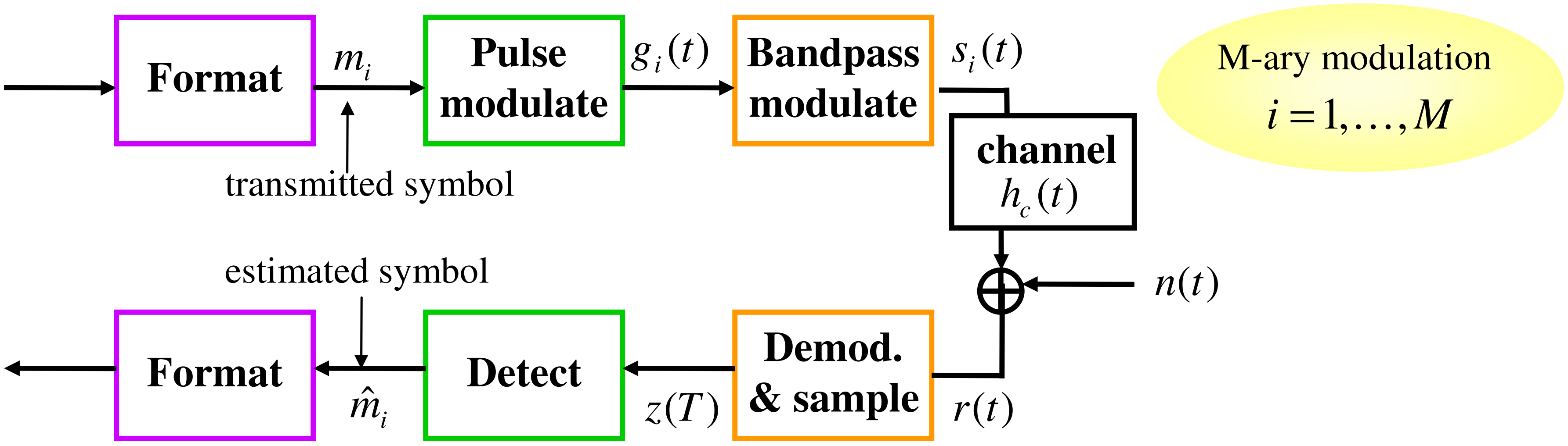
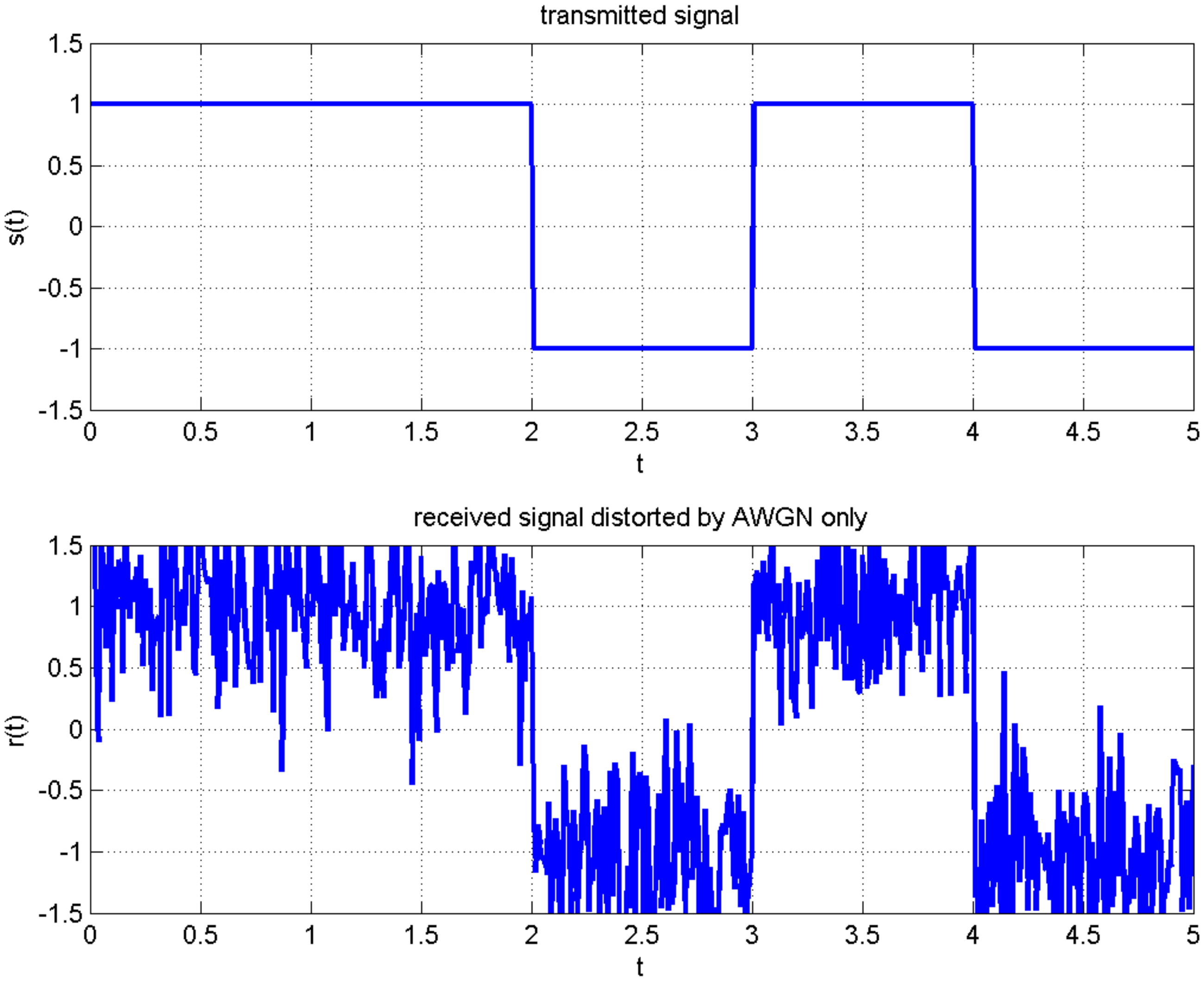
Week 15	

Demodulation and Detection

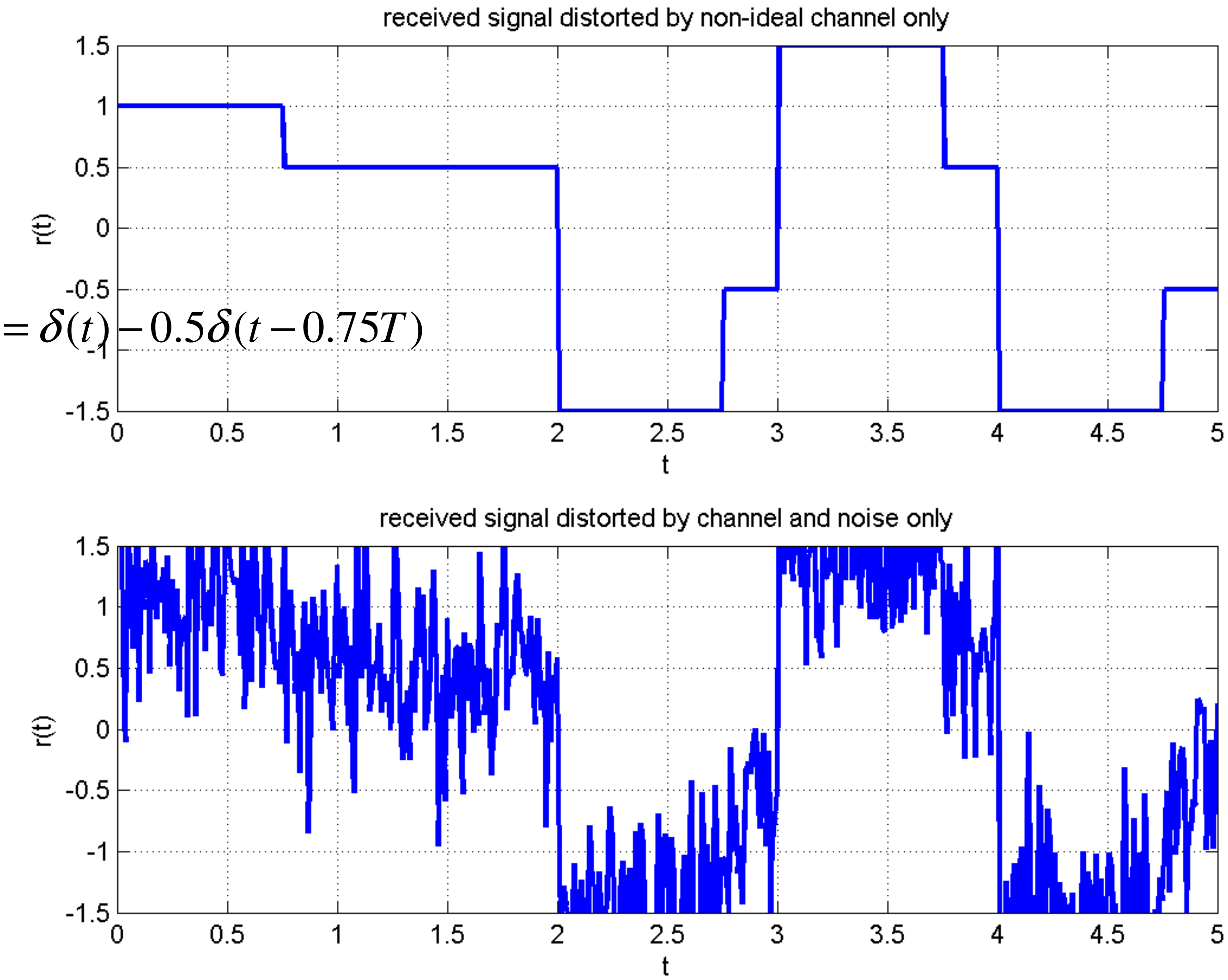


- Major sources of errors:
  - Thermal noise (AWGN)
    - disturbs the signal in an additive fashion (Additive)
    - has flat spectral density for all frequencies of interest (White)
    - is modeled by Gaussian random process (Gaussian Noise)
  - Inter-Symbol Interference (ISI)
    - Due to the filtering effect of transmitter, channel and receiver, symbols are “smeared”.

Example: Impact of the channel



Example: Channel impact …

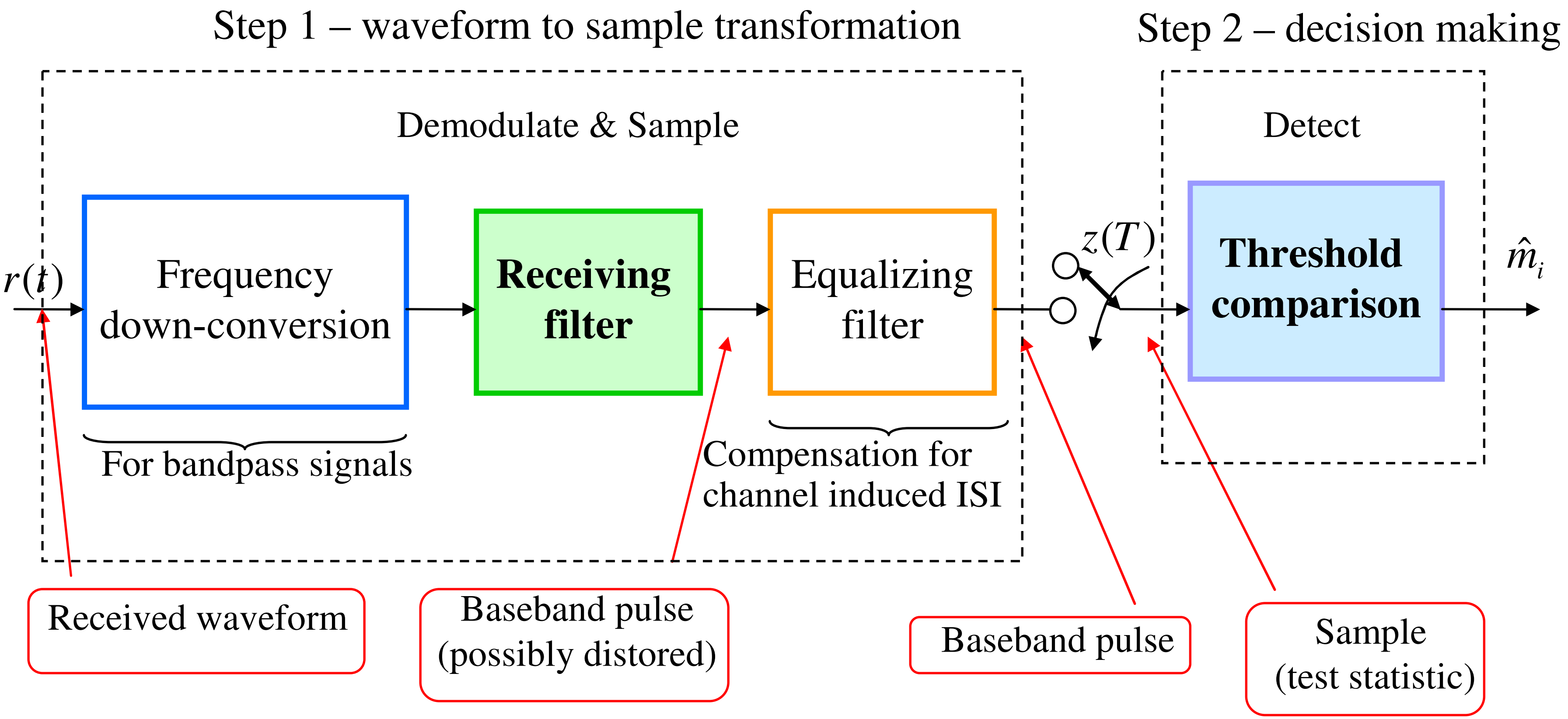


$$h_c(t) = \delta(t) - 0.5\delta(t - 0.75T)$$

Receiver tasks

- Demodulation and sampling:
  - Waveform recovery and preparing the received signal for detection:
    - Improving the signal power to the noise power (SNR) using matched filter
    - Reducing ISI using equalizer
    - Sampling the recovered waveform
- Detection:
  - Estimate the transmitted symbol based on the received sample

Receiver structure



Step 1 – waveform to sample transformation

Step 2 – decision making

Demodulate & Sample

Detect

$r(t)$

Frequency down-conversion

Receiving filter

Equalizing filter

$z(T)$

Threshold comparison

$\hat{m}_i$

For bandpass signals

Compensation for channel induced ISI

Received waveform

Baseband pulse (possibly distored)

Baseband pulse

Sample (test statistic)

Baseband and Bandpass

- Bandpass model of detection process is equivalent to baseband model because:
  - The received bandpass waveform is first transformed to a baseband waveform.

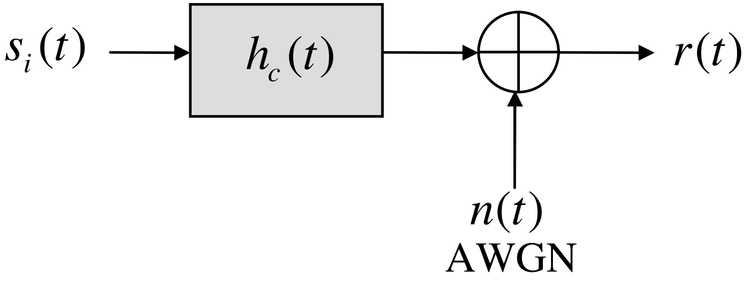
  - Equivalence theorem:
    - Performing bandpass linear signal processing followed by heterodyning the signal to the baseband, yields the same results as heterodyning the bandpass signal to the baseband , followed by a baseband linear signal processing.

Steps in designing the receiver

- Find optimum solution for receiver design with the following goals:
  1. Maximize SNR
  2. Minimize ISI
- Steps in design:
  - Model the received signal
  - Find separate solutions for each of the goals.
- First, we focus on designing a receiver which maximizes the SNR.

Design the receiver filter to maximize the SNR

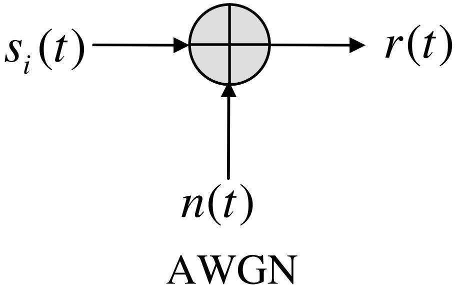
- Model the received signal



$$\boxed{r(t) = s_i(t) * h_c(t) + n(t)}$$

- Simplify the model:
  - Received signal in AWGN

Ideal channels
$h_c(t) = \delta(t)$



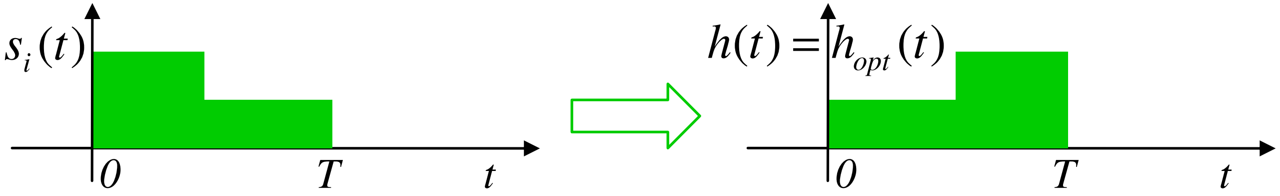
$$\boxed{r(t) = s_i(t) + n(t)}$$

Matched filter receiver

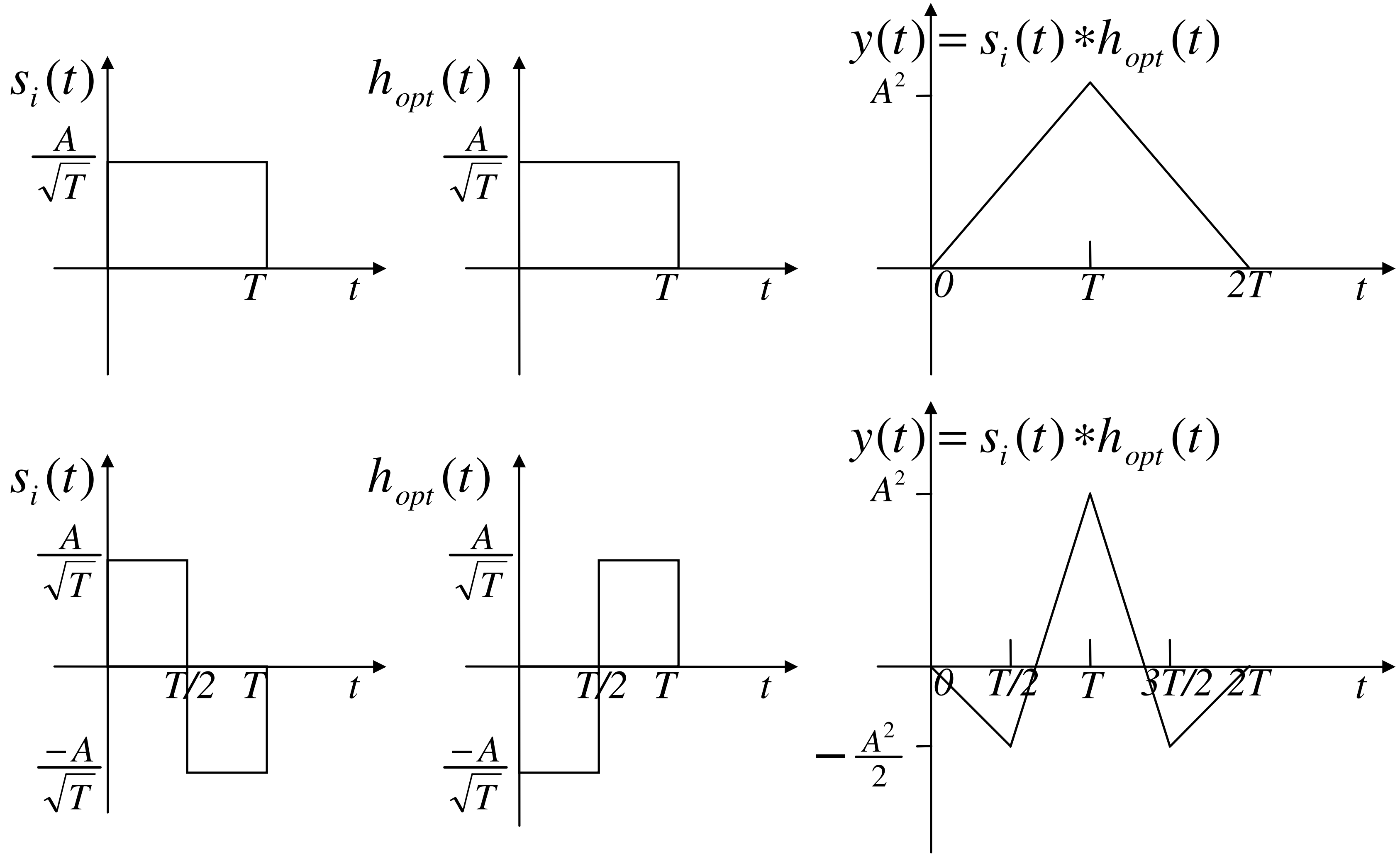
- Problem:
  - Design the receiver filter $h(t)$ such that the SNR is maximized at the sampling time when $s_i(t), i = 1,\ldots, M$ is transmitted.
- Solution:
  - The optimum filter, is the Matched filter, given by

$$\boxed{\begin{aligned} h(t) &= h_{opt}(t) = s_i^*(T-t) \\ H(f) &= H_{opt}(f) = S_i^*(f)\exp(-j2\pi fT) \end{aligned}}$$

which is the time-reversed and delayed version of the conjugate of the transmitted signal



Example of matched filter

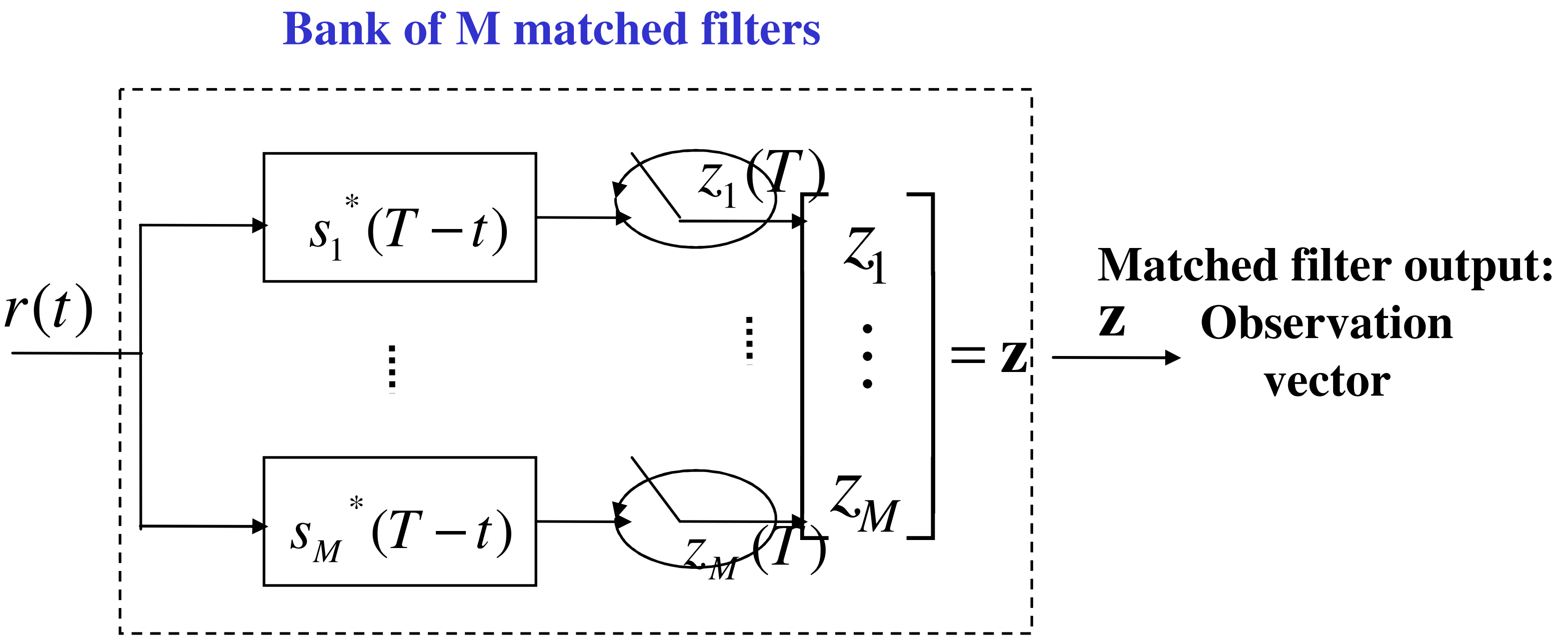


Correlator receiver

- The matched filter output at the sampling time, can be realized as the correlator output.

$$\boxed{\begin{aligned} z(T) &= h_{opt}(T) * r(T) \\ &= \int_0^T r(\tau) s_i^*(\tau) d\tau =< r(t), s(t) > \end{aligned}}$$

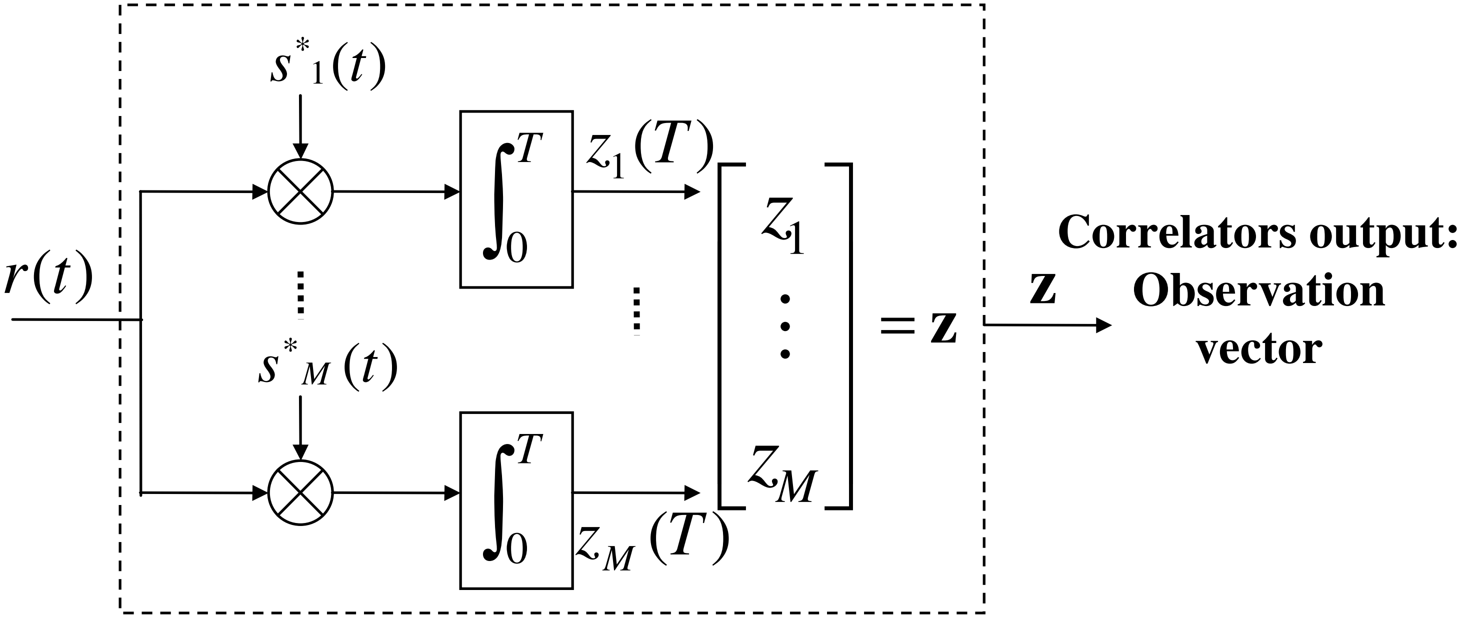
Implementation of matched filter receiver



$$z_i = r(t) * s^*_i(T-t) \quad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1(T), z_2(T),\ldots, z_M(T)) = (z_1, z_2,\ldots, z_M)$$

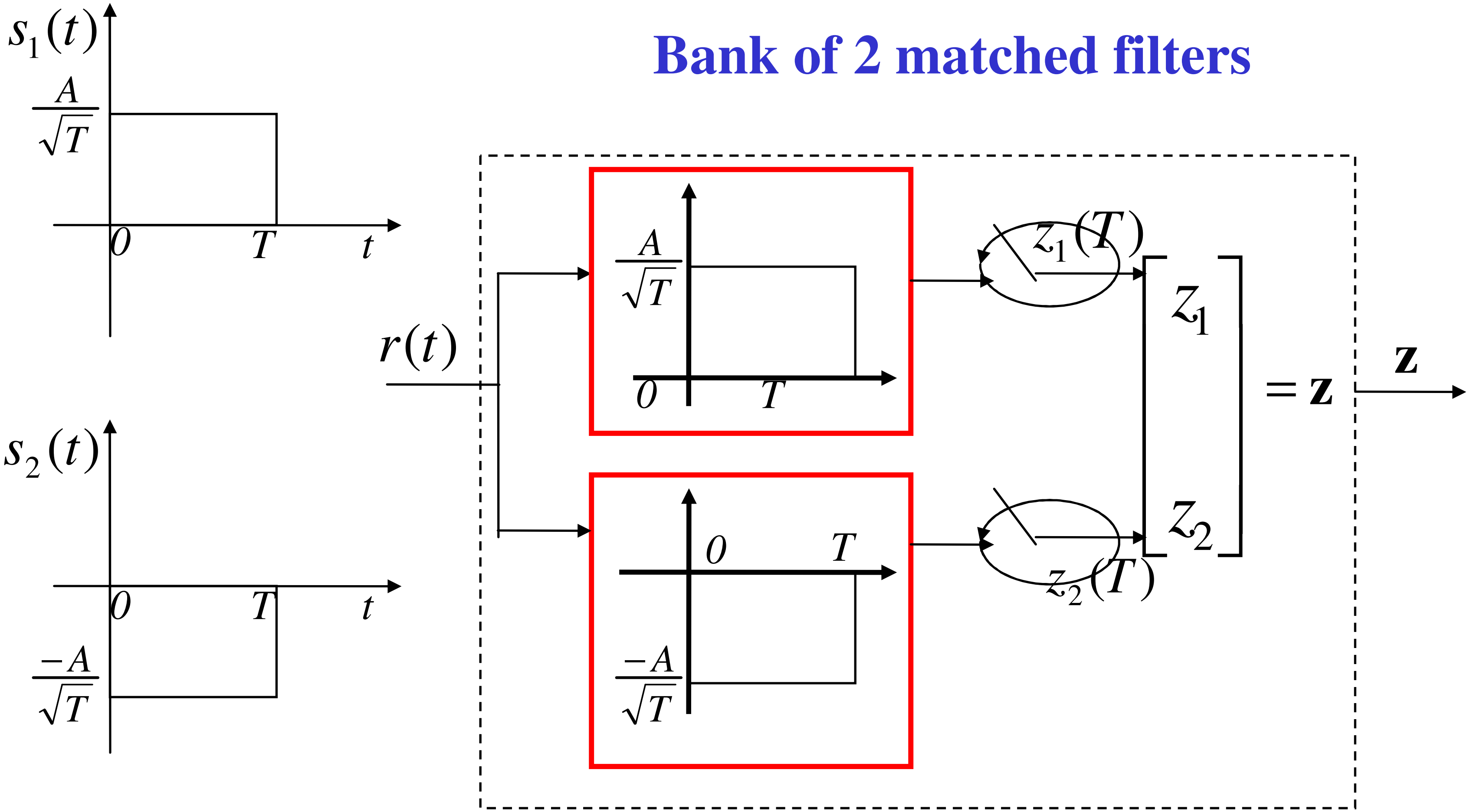
Implementation of correlator receiver



$$\mathbf{z} = (z_1(T), z_2(T), \ldots, z_M(T)) = (z_1, z_2, \ldots, z_M)$$

$$z_i = \int_0^T r(t) s_i(t) dt \qquad i = 1, \ldots, M$$

Implementation example of matched filter receivers



GRAM – SCHMIDT ORTHOGONALIZATION PROCEDURE:

In case of Gram-Schmidt Orthogonalization procedure, any set of ‘m’ energy signals $\{S_i(t)\}$ can be represented by a linear combination of ‘N’ orthonormal basis functions where $N \le m$. That is we may represent the given set of real valued energy signals $S_1(t)$, $S_2(t)$. . . . . . . $S_m(t)$ each of duration T seconds in the form

$$S_1(t)=S_{11}\phi_1(t)+S_{12}\phi_2(t)........+S_{1N}\phi_N(t)$$

$$S_2(t)=S_{21}\phi_1(t)+S_{22}\phi_2(t)........+S_{2N}\phi_N(t)$$

$$S_m(t)=S_{m1}\phi_1(t)+S_{m2}\phi_2(t)........+S_{mN}\phi_N(t)$$

$$S_i(t)=\sum_{j=1}^{N} S_{ij}\phi_j(t) \begin{cases} 0\le t\le T \\ i=1,2,3......m \end{cases}$$

$$S_{ij}(t)=\int_0^T S_i(t)\phi_j(t) \begin{cases} i=1,2,3.\ ....m \\ j=1,2,3......n \end{cases}$$

The co-efficient S_{ij} may be viewed as the j^{th} element of the N – dimensional Vector S_i

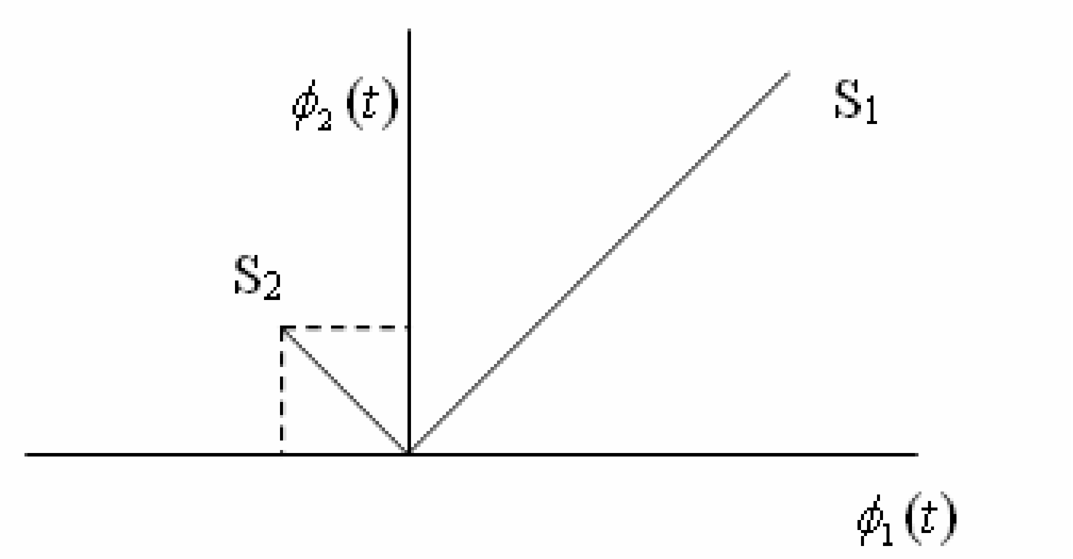
$$S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \\ ' \\ ' \\ ' \\ ' \\ S_{iN} \end{bmatrix}$$

i = 1,2,3 . . . . . . m

Let $$S_1 = 3\phi_1(t) + 4\phi_2(t)$$

$$S_2 = -\phi_1(t) + 2\phi_2(t)$$

Vector $S_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



- To find an orthonormal basis functions for a given set of signals, the Gram-Schmidt procedure can be used.
- Gram-Schmidt procedure:
  - Given a signal set $\{s_i(t)\}_{i=1}^{M}$, compute an orthonormal basis $\{\psi_j(t)\}_{j=1}^{N}$
    1. Define $\psi_1(t) = s_1(t)/\sqrt{E_1} = s_1(t)/\|s_1(t)\|$
    2. For $i = 2,\ldots,M$ compute $d_i(t) = s_i(t) - \sum_{j=1}^{i-1} < s_i(t), \psi_j(t) > \psi_j(t)$

       If $d_i(t) \neq 0$ let $\psi_i(t) = d_i(t)/\|d_i(t)\|$

       If $d_i(t) = 0$, do not assign any basis function.
    3. Renumber the basis functions such that basis is

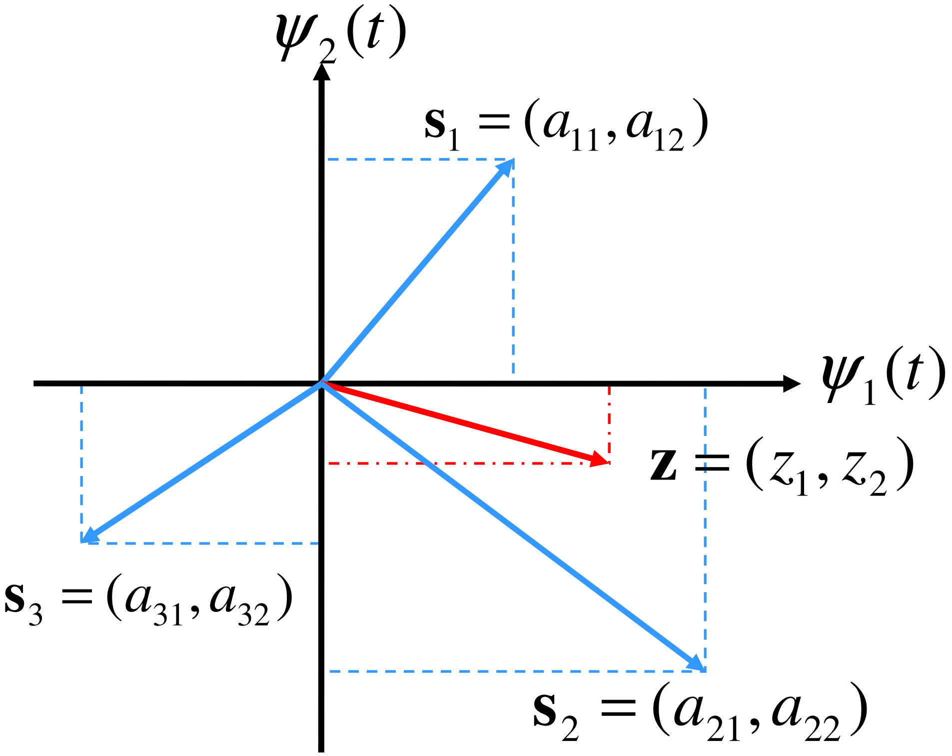
       $\{\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\}$

- This is only necessary if $d_i(t) = 0$ for any i in step 2.
- Note that $N \leq M$

Signal space

- What is a signal space?
  - Vector representations of signals in an N-dimensional orthogonal space
- Why do we need a signal space?
  - It is a means to convert signals to vectors and vice versa.
  - It is a means to calculate signals energy and Euclidean distances between signals.
- Why are we interested in Euclidean distances between signals?
  - For detection purposes: The received signal is transformed to a received vectors. The signal which has the minimum distance to the received signal is estimated as the transmitted signal.

Schematic example of a signal space



Transmitted signal alternatives

$$s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12})$$

$$s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22})$$

$$s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32})$$

Received signal at matched filter output →

$$z(t) = z_1\psi_1(t) + z_2\psi_2(t) \Leftrightarrow \mathbf{z} = (z_1, z_2)$$

Signal space

- To form a signal space, first we need to know the inner product between two signals (functions):
  - Inner (scalar) product:

$$< x(t), y(t) >= \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

= cross-correlation between x(t) and y(t)

  - Properties of inner product:

$$< ax(t), y(t) >= a < x(t), y(t) >$$

$$< x(t), ay(t) >= a^* < x(t), y(t) >$$

$$< x(t) + y(t), z(t) >=< x(t), z(t) > + < y(t), z(t) >$$

Signal space …

- The distance in signal space is measure by calculating the norm.
- What is norm?
  - Norm of a signal:

$$\|x(t)\| = \sqrt{< x(t), x(t) >} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{E_x}$$

= "length" of x(t)

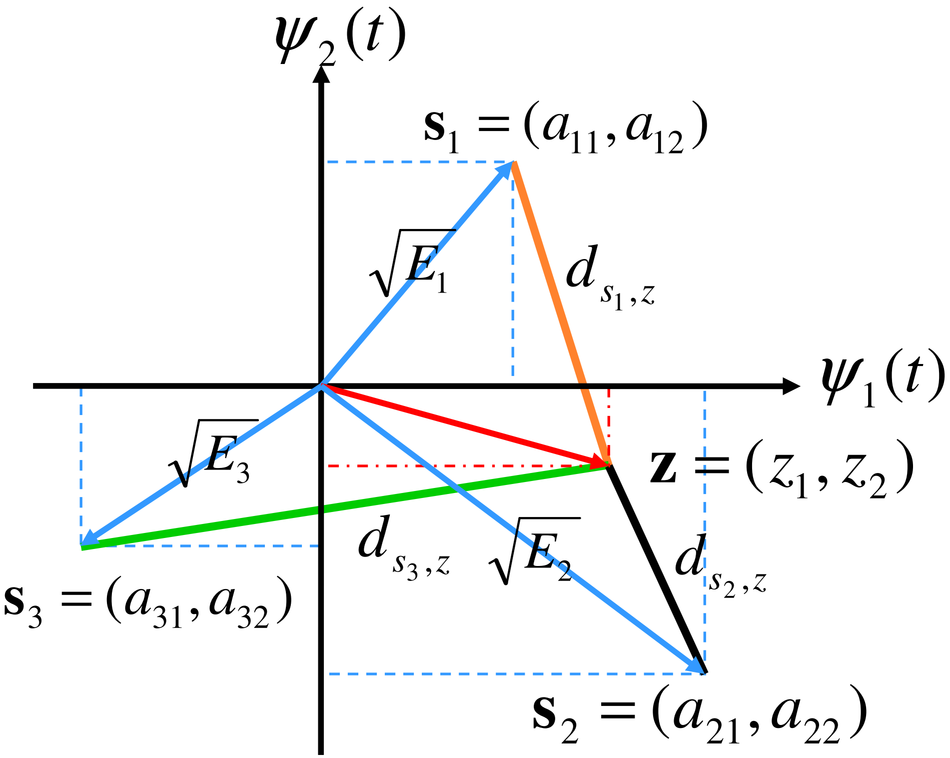
$$\|ax(t)\| = |a| \|x(t)\|$$

  - Norm between two signals:

$$d_{x,y} = \|x(t) - y(t)\|$$

- We refer to the norm between two signals as the Euclidean distance between two signals.

Example of distances in signal space



The Euclidean distance between signals *z(t)* and *s(t)*:

$$d_{s_i,z} = \left\| s_i(t) - z(t) \right\| = \sqrt{(a_{i1} - z_1)^2 + (a_{i2} - z_2)^2}$$

$$i = 1,2,3$$

Orthogonal signal space

- N-dimensional orthogonal signal space is characterized by N linearly independent functions $\{\psi_j(t)\}_{j=1}^{N}$ called basis functions. The basis functions must satisfy the orthogonality condition

$$\boxed{<\psi_i(t),\psi_j(t)>=\int_0^T \psi_i(t)\psi_j^*(t)dt = K_i\delta_{ji} \qquad \begin{matrix} 0 \le t \le T \\ j,i = 1,\dots,N \end{matrix}}$$

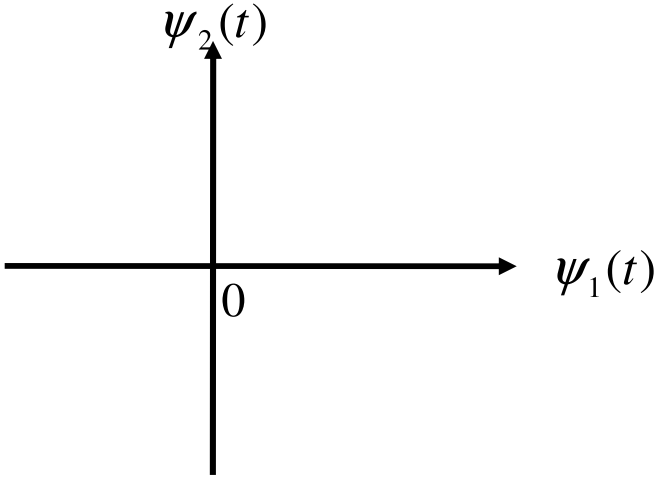
where $\delta_{ij} = \begin{cases} 1 \rightarrow i = j \\ 0 \rightarrow i \neq j \end{cases}$

- If all $K_i = 1$, the signal space is orthonormal.

Example of an orthonormal basis

- ❑ Example: 2-dimensional orthonormal signal space

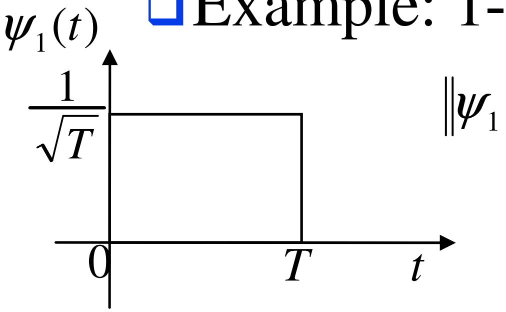
$$\begin{cases} \psi_1(t) = \sqrt{\dfrac{2}{T}} \cos(2\pi t / T) & 0 \le t < T \\ \psi_2(t) = \sqrt{\dfrac{2}{T}} \sin(2\pi t / T) & 0 \le t < T \end{cases}$$



$$< \psi_1(t), \psi_2(t) >= \int_0^T \psi_1(t) \psi_2(t) dt = 0$$

$$\|\psi_1(t)\| = \|\psi_2(t)\| = 1$$

- ❑ Example: 1-dimensional orthonormal signal space



$$\|\psi_1(t)\| = 1$$

Signal space …

- Any arbitrary finite set of waveforms $\{s_i(t)\}_{i=1}^{M}$

where each member of the set is of duration *T*, can be expressed as a linear combination of N orthonogal waveforms where $\{\psi_j(t)\}_{j=1}^{N}$. $N \leq M$

$$\boxed{s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad \begin{matrix} i = 1,\dots,M \\ N \leq M \end{matrix}}$$

where

$$a_{ij} = \frac{1}{K_j} < s_i(t), \psi_j(t) >= \frac{1}{K_j}\int_0^T s_i(t)\psi_j^*(t)dt \qquad \begin{matrix} j = 1,\dots,N \\ i = 1,\dots,M \end{matrix} \qquad 0 \leq t \leq T$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \dots, a_{iN})$$

Vector representation of waveform

$$E_i = \sum_{j=1}^{N} K_j \left|a_{ij}\right|^2$$

Waveform energy

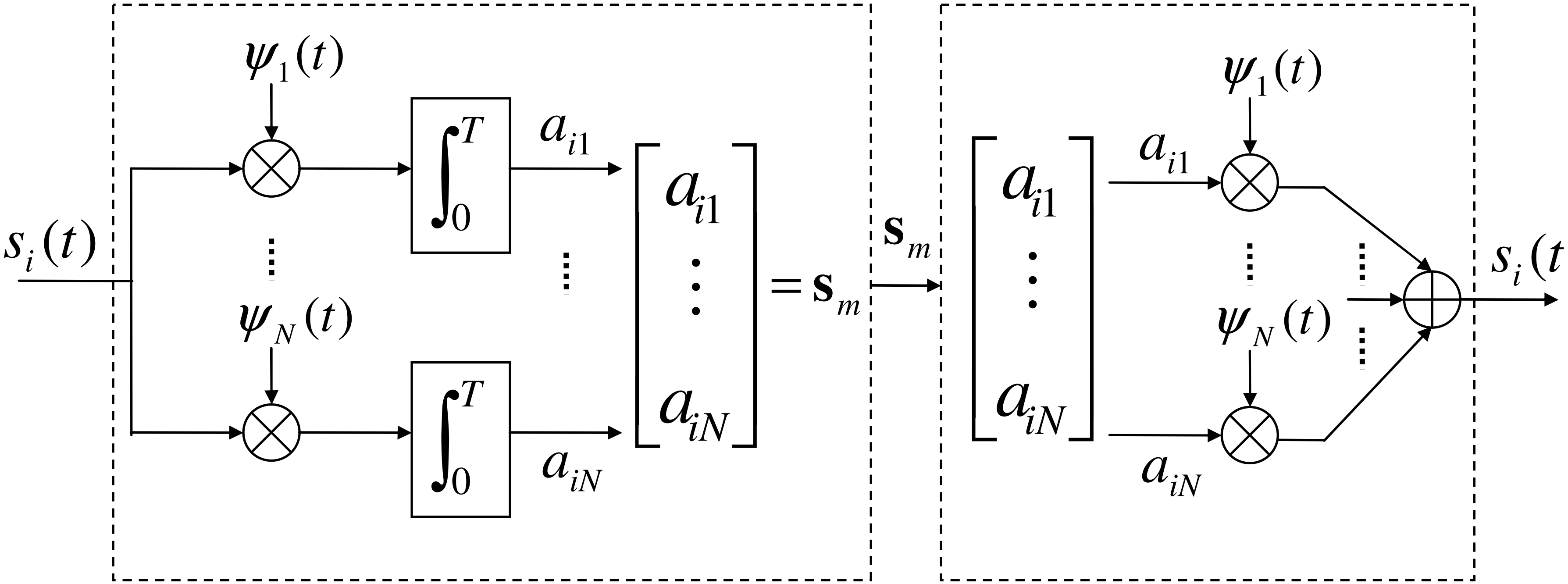
Signal space …

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t)$$

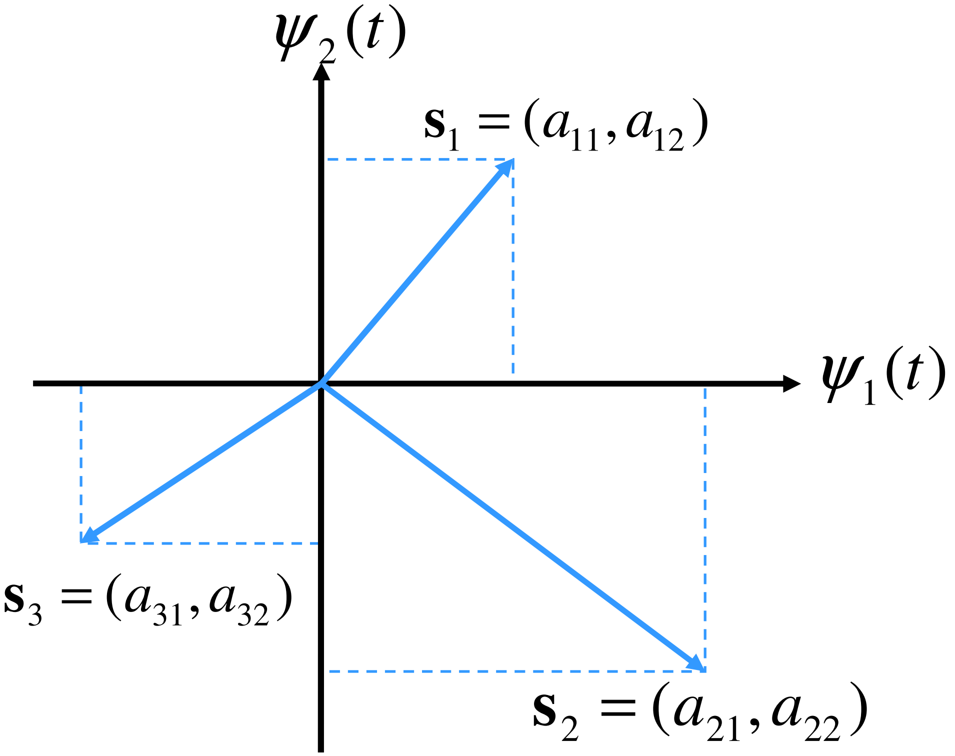
$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Waveform to vector conversion

Vector to waveform conversion



Example of projecting signals to an orthonormal signal space



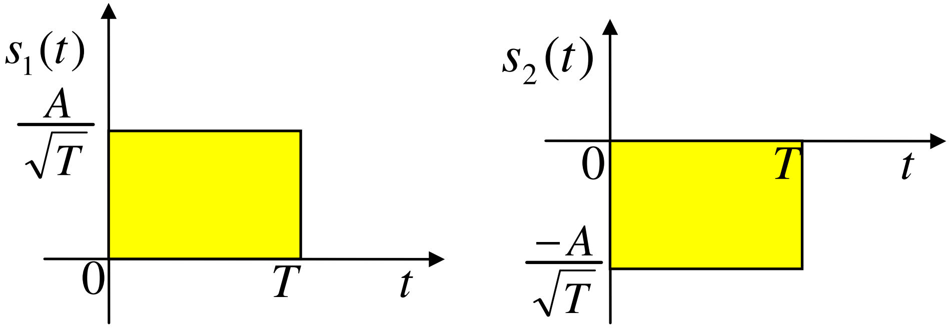
Transmitted signal alternatives

$$\begin{cases} s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12}) \\ s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22}) \\ s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32}) \end{cases}$$

$$a_{ij} = \int_0^T s_i(t)\psi_j(t)dt \quad j = 1,\ldots,N \quad i = 1,\ldots,M \quad 0 \leq t \leq T$$

Example of Gram-Schmidt procedure

- Find the basis functions and plot the signal space for the following transmitted signals:



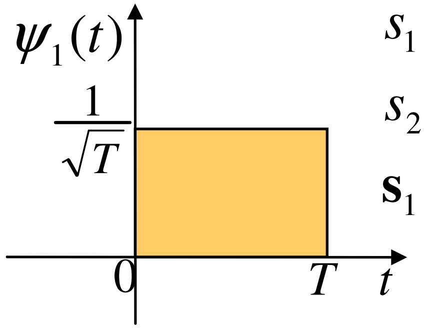
- Using Gram-Schmidt procedure:

① $E_1 = \int_0^T |s_1(t)|^2 dt = A^2$

$\psi_1(t) = s_1(t) / \sqrt{E_1} = s_1(t) / A$

② $< s_2(t), \psi_1(t) > = \int_0^T s_2(t)\psi_1(t) dt = -A$

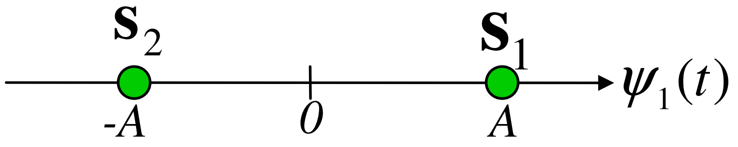
$d_2(t) = s_2(t) - (-A)\psi_1(t) = 0$



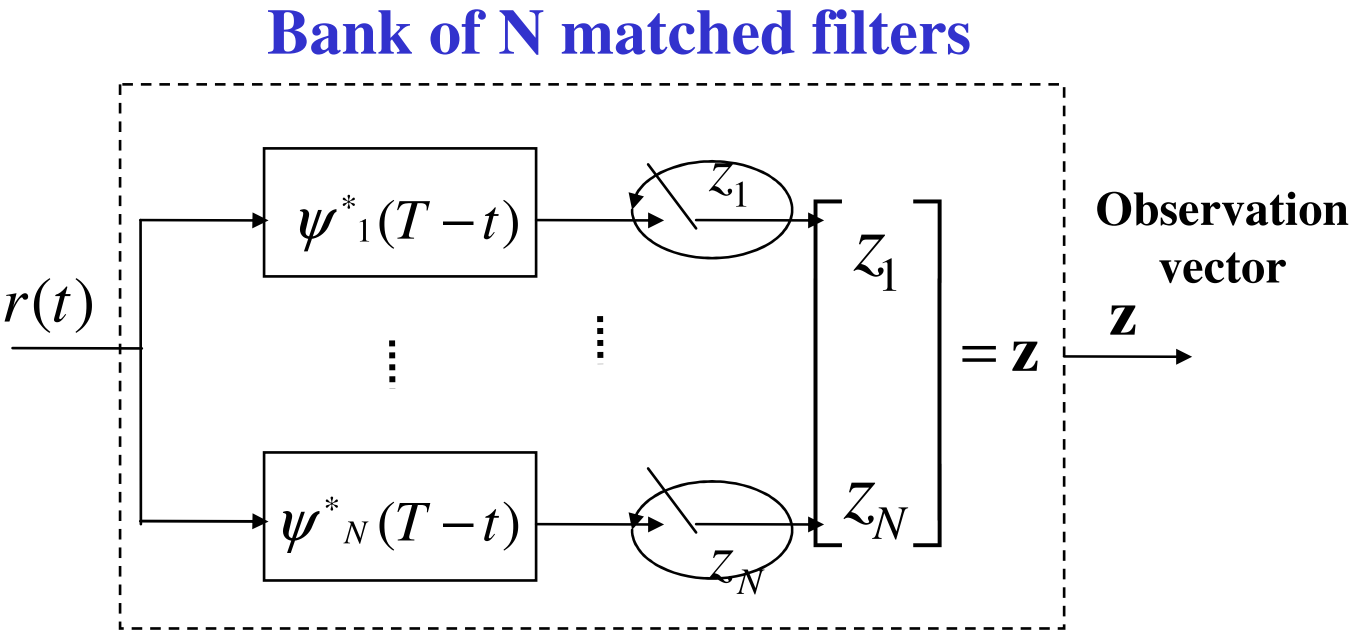
$s_1(t) = A\psi_1(t)$

$s_2(t) = -A\psi_1(t)$

$\mathbf{s}_1 = (A) \quad \mathbf{s}_2 = (-A)$



Implementation of the matched filter receiver



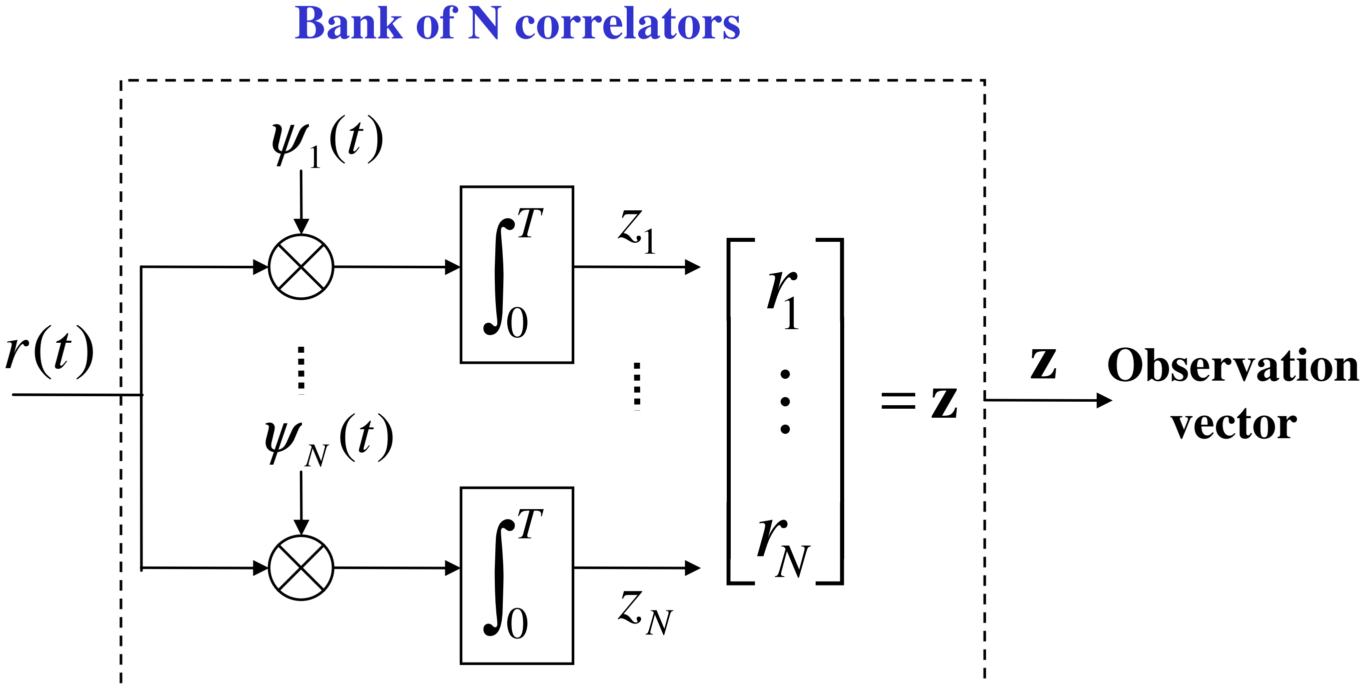
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

$$N \le M$$

$$z_j = r(t) * \psi_j(T-t) \qquad j = 1,\ldots,N$$

Implementation of the correlator receiver



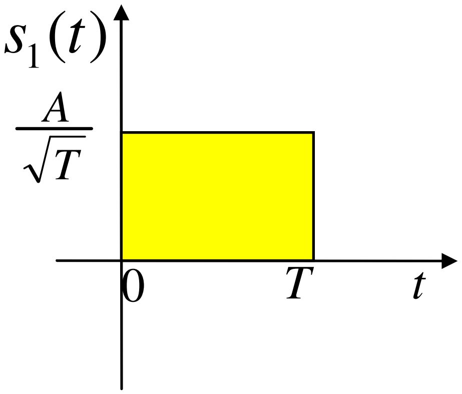
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

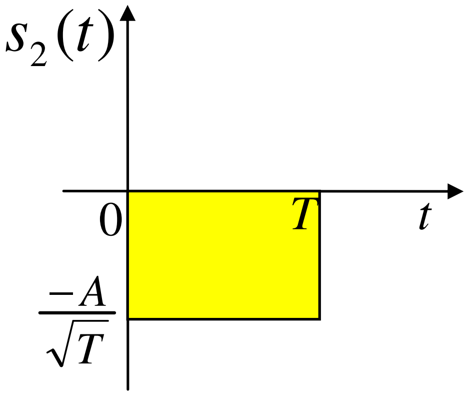
$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

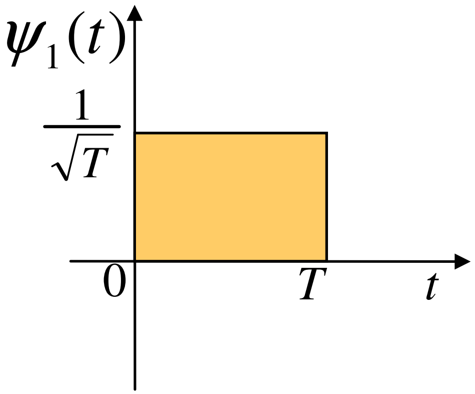
$$z_j = \int_0^T r(t)\psi_j(t)dt \qquad j = 1,\ldots,N$$

$$\boxed{N \le M}$$

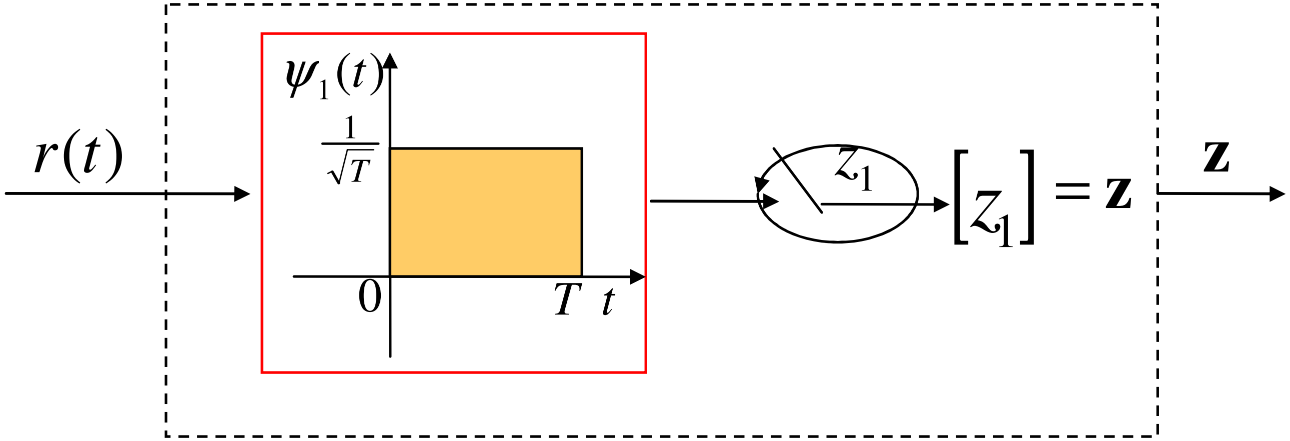
Example of matched filter receivers using basic functions







1 matched filter



- ❑ Number of matched filters (or correlators) is reduced by 1 compared to using matched filters (correlators) to the transmitted signal.

- ❑ Reduced number of filters (or correlators)

White noise in the orthonormal signal space

- AWGN, *n(t)*, can be expressed as

$$n(t) = \underbrace{\hat{n}(t)} + \underbrace{\tilde{n}(t)}$$

Noise projected on the signal space which impacts the detection process.

Noise outside on the signal space

$$\begin{cases} \hat{n}(t) = \sum_{j=1}^{N} n_j \psi_j(t) \\ n_j = < n(t), \psi_j(t) > \qquad j = 1, \dots, N \\ < \tilde{n}(t), \psi_j(t) > = 0 \qquad j = 1, \dots, N \end{cases}$$

Vector representation of $\hat{n}(t)$

$$\mathbf{n} = (n_1, n_2, \dots, n_N)$$

$\{n_j\}_{j=1}^{N}$ independent zero-mean Gaussain random variables with variance $\text{var}(n_j) = N_0 / 2$

Statistics of the observation Vector

- AWGN channel model: $\mathbf{z} = \mathbf{s}_i + \mathbf{n}$
  - Signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, ..., a_{iN})$ is deterministic.
  - Elements of noise vector $\mathbf{n} = (n_1, n_2, ..., n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0 / 2$. The noise vector pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

  - The elements of observed vector $\mathbf{z} = (z_1, z_2, ..., z_N)$ are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{z}}(\mathbf{z} \,|\, \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{z} - \mathbf{s}_i\|^2}{N_0} \right)$$

Detection

- Optimum decision rule (maximum a posteriori probability):

Set $\hat{m} = m_i$ if

$\Pr(m_i \text{ sent} \mid \mathbf{z}) \geq \Pr(m_k \text{ sent} \mid \mathbf{z})$, for all $k \neq i$

where $k = 1, \ldots, M$.

- Applying Bayes' rule gives:

Set $\hat{m} = m_i$ if

$$p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}, \text{ is maximum for all } k = i$$

Detection …

- Partition the signal space into *M* decision regions, such that $Z_1,\ldots,Z_M$

Vector **z** lies inside region Z_i if

$$\ln[p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}], \text{ is maximum for all } k = i.$$

That means

$$\hat{m} = m_i$$

Detection (ML rule)

- For equal probable symbols, the optimum decision rule (maximum posteriori probability) is simplified to:

$$\boxed{\begin{array}{l}\text{Set } \hat{m} = m_i \text{ if} \\ p_{\mathbf{z}}(\mathbf{z} \mid m_k), \text{ is maximum for all } k = i\end{array}}$$

or equivalently:

$$\boxed{\begin{array}{l}\text{Set } \hat{m} = m_i \text{ if} \\ \ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)], \text{ is maximum for all } k = i\end{array}}$$

which is known as *maximum likelihood*.

Detection (ML)…

- Partition the signal space into *M* decision regions,

  . $Z_1, \ldots, Z_M$

- Restate the maximum likelihood decision rule as follows:

Vector $\mathbf{z}$ lies inside region Z_i if

$\ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)]$, is maximum for all $k = i$

That means

$\hat{m} = m_i$

Detection rule (ML)…

- It can be simplified to:

Vector $\mathbf{z}$ lies inside region Z_i if

$\left\|\mathbf{z}-\mathbf{s}_k\right\|$, is minimum for all $k=i$

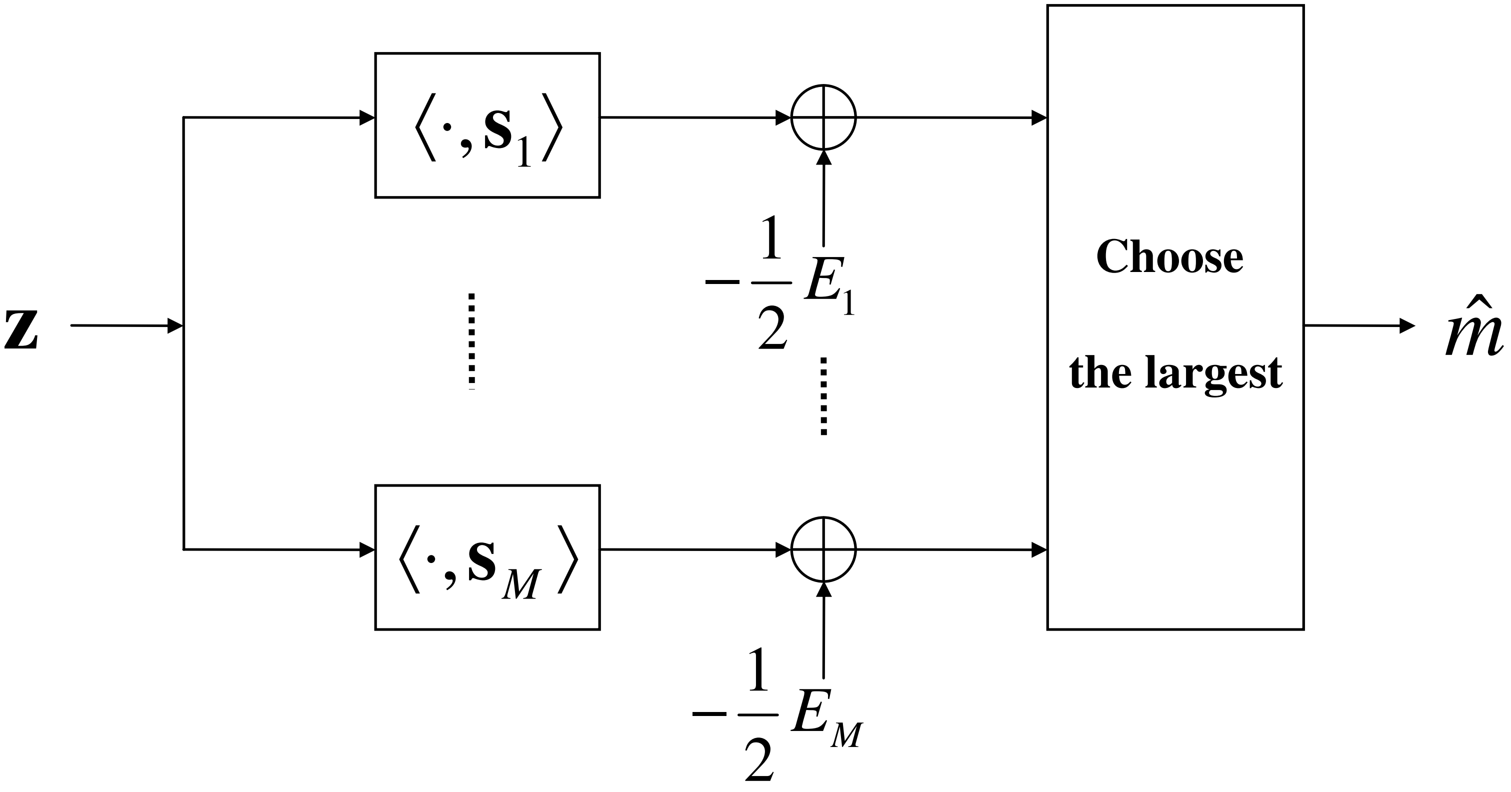
or equivalently:

Vector $\mathbf{r}$ lies inside region Z_i if

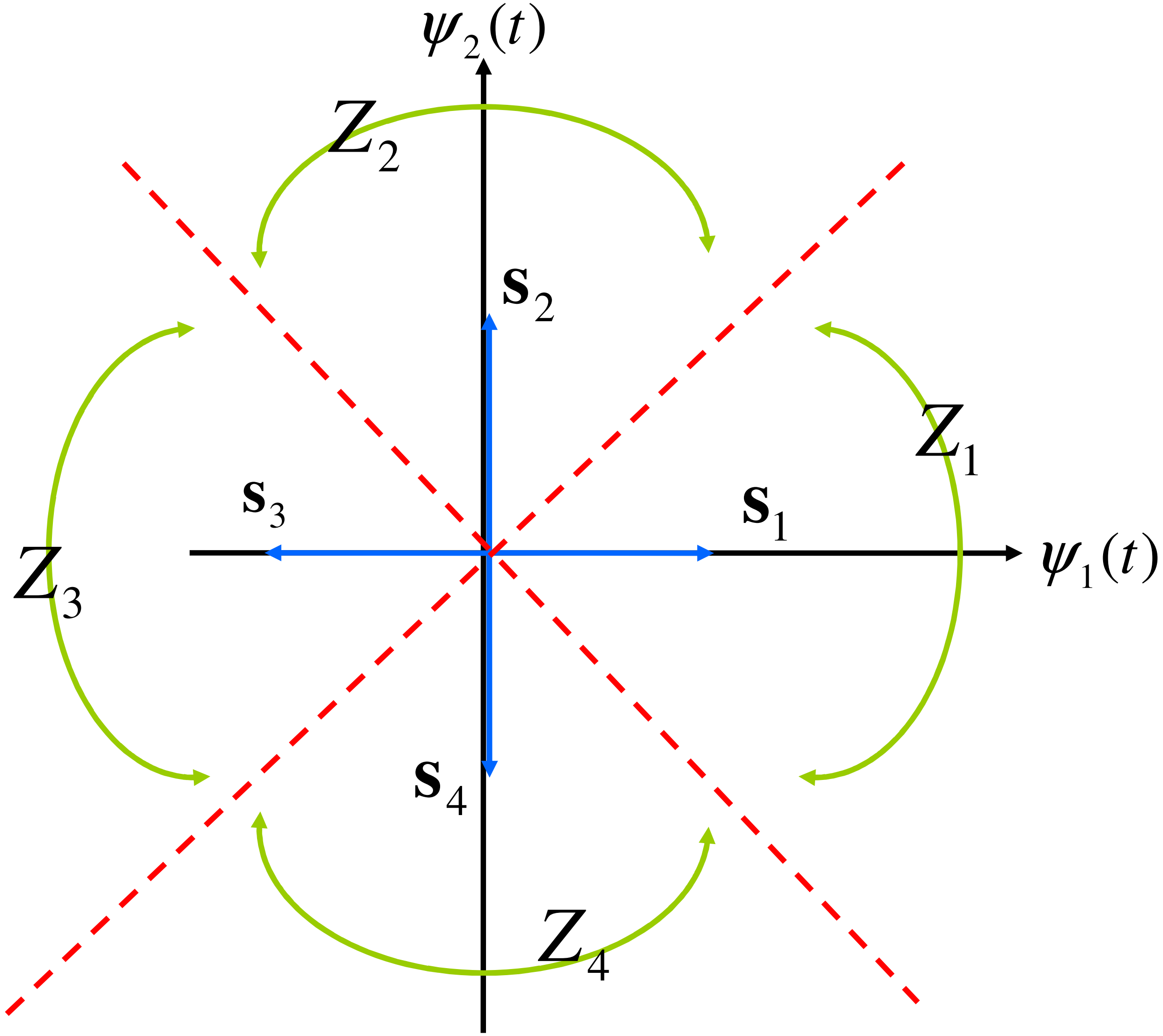
$$\sum_{j=1}^{N} z_j a_{kj} - \frac{1}{2}E_k, \text{ is maximum for all } k=i$$

where E_k is the energy of $s_k(t)$.

Maximum likelihood detector block diagram



Schematic example of the ML decision regions



Average probability of symbol error

- Erroneous decision: For the transmitted symbol m_i or equivalently signal vector $\mathbf{s}_i$, an error in decision occurs if the observation vector $\mathbf{z}$ does not fall inside region Z_i
  - Probability of erroneous decision for a transmitted symbol

or equivalently

$$\boxed{P_e(m_i) = \Pr(\hat{m} \neq m_i \text{ and } m_i \text{ sent})}$$

$$\boxed{\Pr(\hat{m} \neq m_i) = \Pr(m_i \text{ sent}) \Pr(\mathbf{z} \text{ does not lie inside } Z_i | m_i \text{ sent})}$$

  - Probability of correct decision for a transmitted symbol

$$\boxed{\Pr(\hat{m} = m_i) = \Pr(m_i \text{ sent}) \Pr(\mathbf{z} \text{ lies inside } Z_i | m_i \text{ sent})}$$

$$P_c(m_i) = \Pr(\mathbf{z} \text{ lies inside } Z_i | m_i \text{ sent}) = \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} | m_i) d\mathbf{z}$$

$$P_e(m_i) = 1 - P_c(m_i)$$

Av. prob. of symbol error …

- ☐ Average probability of symbol error :

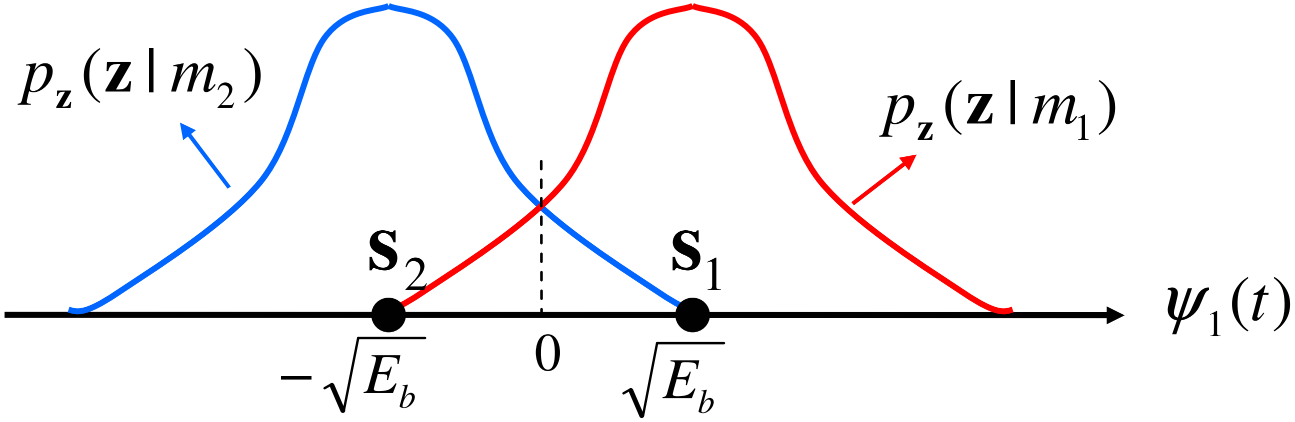
$$P_E(M) = \sum_{i=1}^{M} \Pr(\hat{m} \neq m_i)$$

- ☐ For equally probable symbols:

$$P_E(M) = \frac{1}{M}\sum_{i=1}^{M} P_e(m_i) = 1 - \frac{1}{M}\sum_{i=1}^{M} P_c(m_i)$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M} \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \mid m_i)d\mathbf{z}$$

Example for binary PAM



$$\boxed{P_e(m_1) = P_e(m_2) = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|/2}{\sqrt{N_0/2}}\right)}$$

$$\boxed{P_B = P_E(2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Union bound

Union bound

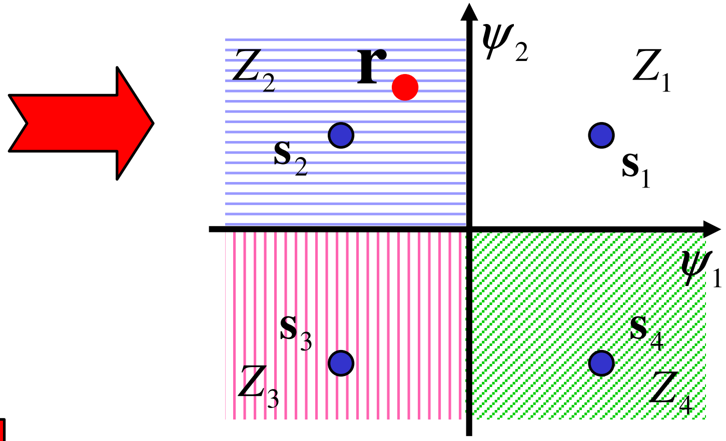
The probability of a finite union of events is upper bounded by the sum of the probabilities of the individual events.

- Let A_{ki} denote that the observation vector $\mathbf{z}$ is closer to the symbol vector $\mathbf{s}_k$ than $\mathbf{s}_i$, when $\mathbf{s}_i$ is transmitted.
- $\Pr(A_{ki}) = P_2(\mathbf{s}_k, \mathbf{s}_i)$ depends only on $\mathbf{s}_i$ and $\mathbf{s}_k$.
- Applying Union bounds yields

$$\boxed{P_e(m_i) \le \sum_{\substack{k=1\\k\ne i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i)} \Longrightarrow \boxed{P_E(M) \le \frac{1}{M}\sum_{i=1}^{M}\sum_{\substack{k=1\\k\ne i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i)}$$

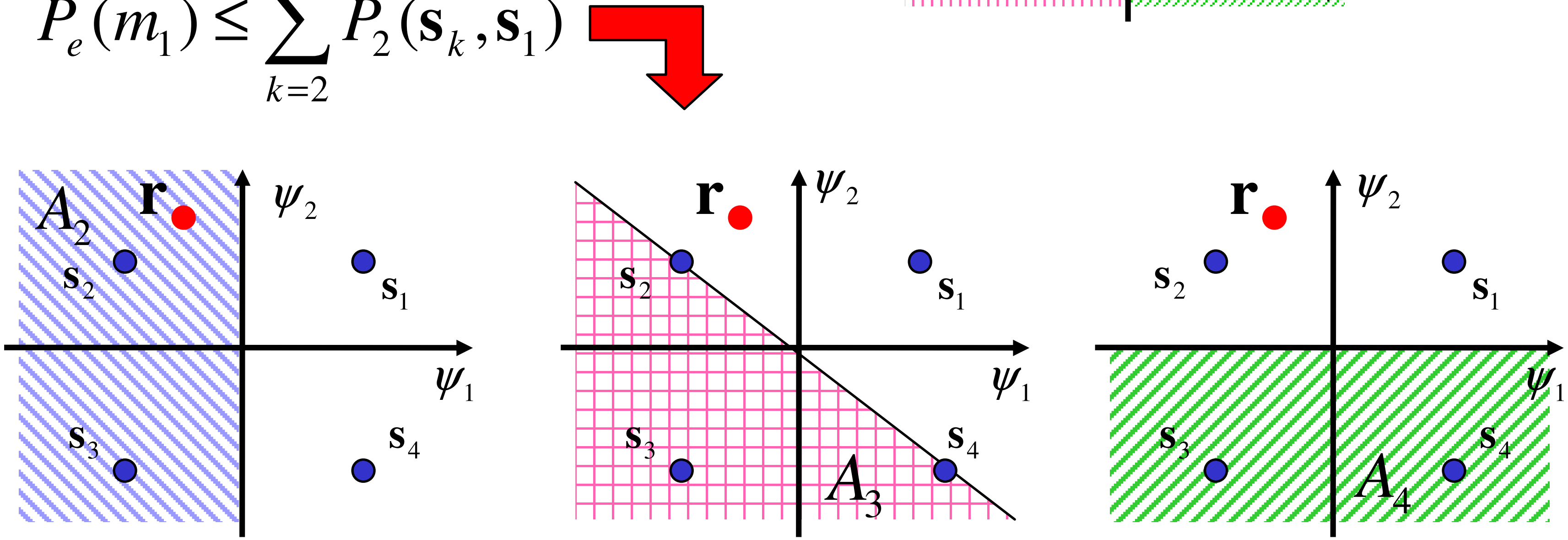
Example of union bound

$$P_e(m_1) = \int_{Z_2 \cup Z_3 \cup Z_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$



Union bound:

$$P_e(m_1) \le \sum_{k=2}^{4} P_2(\mathbf{s}_k, \mathbf{s}_1)$$



$$P_2(\mathbf{s}_2, \mathbf{s}_1) = \int_{A_2} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$

$$P_2(\mathbf{s}_3, \mathbf{s}_1) = \int_{A_3} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$

$$P_2(\mathbf{s}_4, \mathbf{s}_1) = \int_{A_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$

Upper bound based on minimum distance

$$P_2(\mathbf{s}_k, \mathbf{s}_i) = \Pr(\mathbf{z} \text{ is closer to } \mathbf{s}_k \text{ than } \mathbf{s}_i, \text{ when } \mathbf{s}_i \text{ is sent})$$

$$= \int_{d_{ik}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{u^2}{N_0}) du = Q\left(\frac{d_{ik}/2}{\sqrt{N_0/2}}\right)$$

$$d_{ik} = \left\| \mathbf{s}_i - \mathbf{s}_k \right\|$$

$$P_E(M) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \leq (M-1) Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right)$$

Minimum distance in the signal space: $d_{\min} = \min_{\substack{i,k \\ i \neq k}} d_{ik}$

Eb/No figure of merit in digital communications

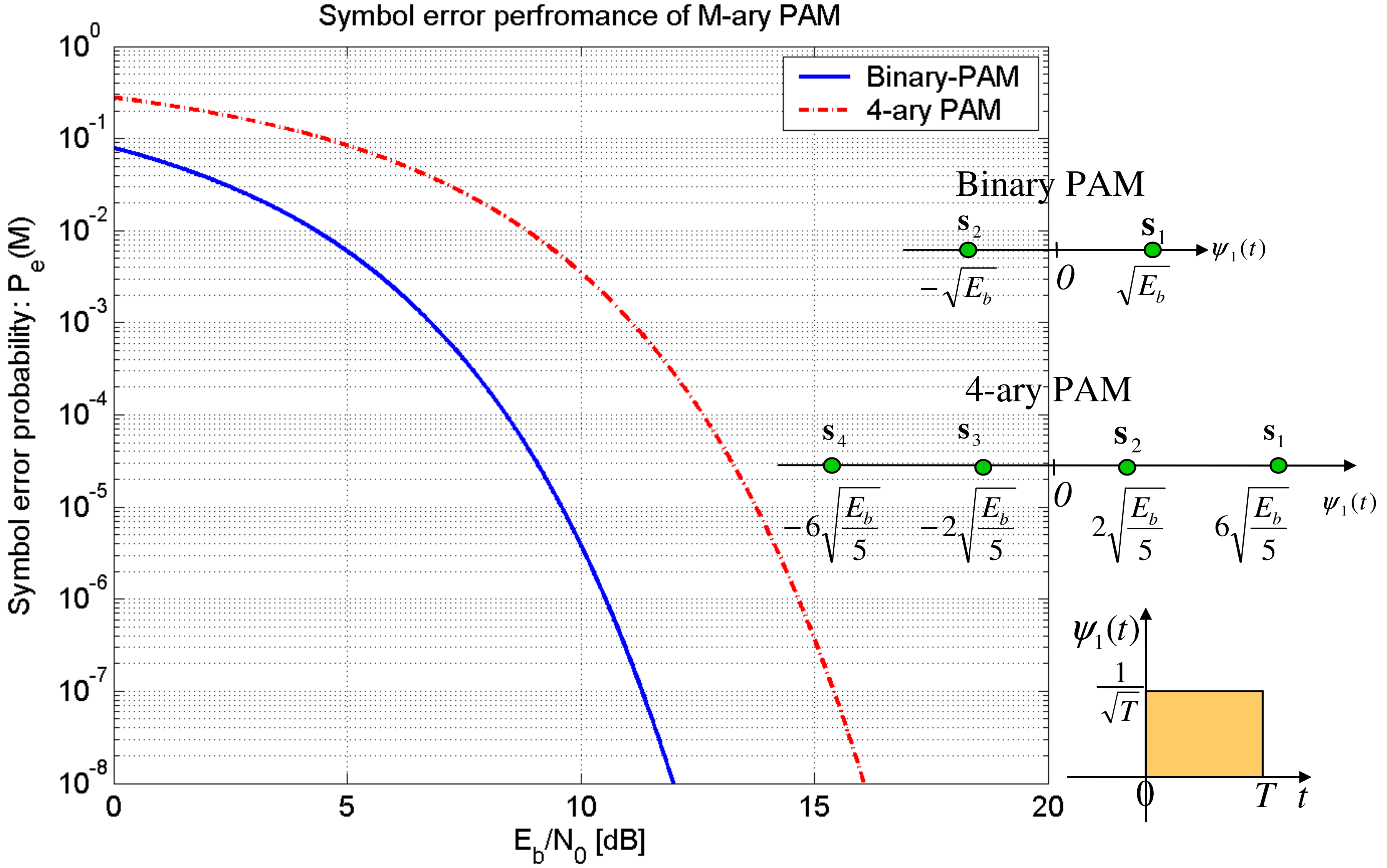
- SNR or S/N is the average signal power to the average noise power. SNR should be modified in terms of bit-energy in DCS, because:
  - Signals are transmitted within a symbol duration and hence, are energy signal (zero power).
  - A merit at bit-level facilitates comparison of different DCSs transmitting different number of bits per symbol.

$$\boxed{\frac{E_b}{N_0} = \frac{ST_b}{N/W} = \frac{S}{N}\frac{W}{R_b}}$$

R_b : Bit rate

W : Bandwidth

Example of Symbol error prob. For PAM signals



Questions?