EC 7xx Advanced Digital Communications
Spring 2008

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Overview, Probabilities, Random variables, Random process

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Syllabus

- Tentatively

Week 1	Overview, Probabilities, Random variables, Random process
Week 2	
Week 3	
Week 4	
Week 5	
Week 6	
Week 7	
Week 8	
Week 9	
Week 10	
Week 11	
Week 12	
Week 13	
Week 14	
Week 15	

Grades

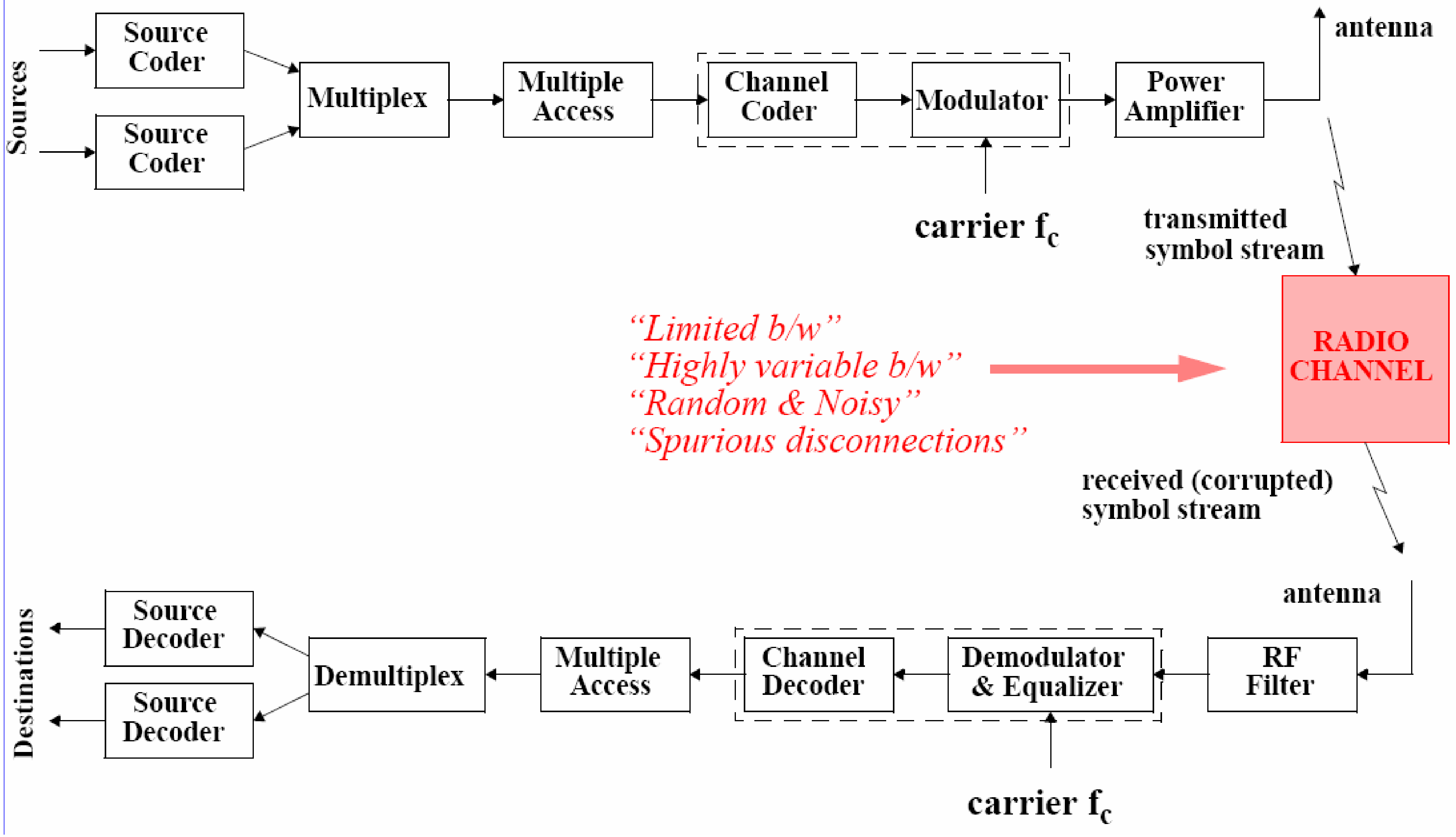
Load	Percentage	Date
7^{th} Week Exam	35%	27 April 2008
Final Exam	40%	
Participation	10%	
Report	15%	10^{th} week and up

Presentation and Report

- Roadmap

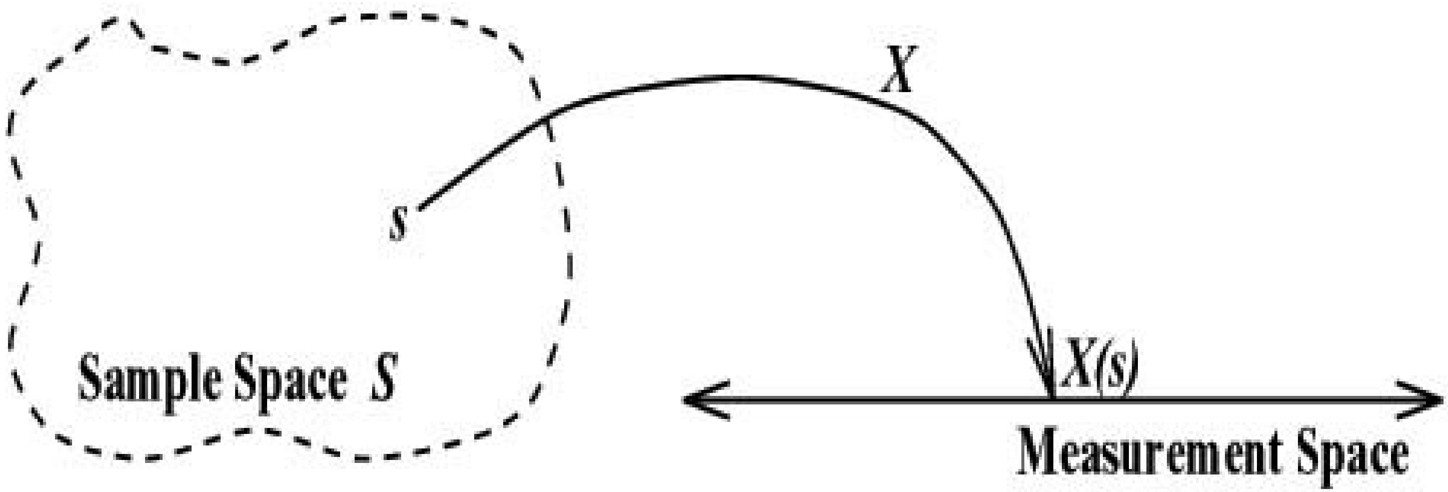
Week 4	Point Distribution
Week 8	Progress report 5%
Week 13	Presentation starts 10%
Week 15	Report 10%

Simplified View of a Digital Radio Link



Random Variable as a Measurement

- Thus a random variable can be thought of as a <u>measurement</u> (yielding a real number) on an experiment
  - Maps "*events*" to "*real numbers*"
  - We can then talk about the pdf, define the mean/variance and other moments



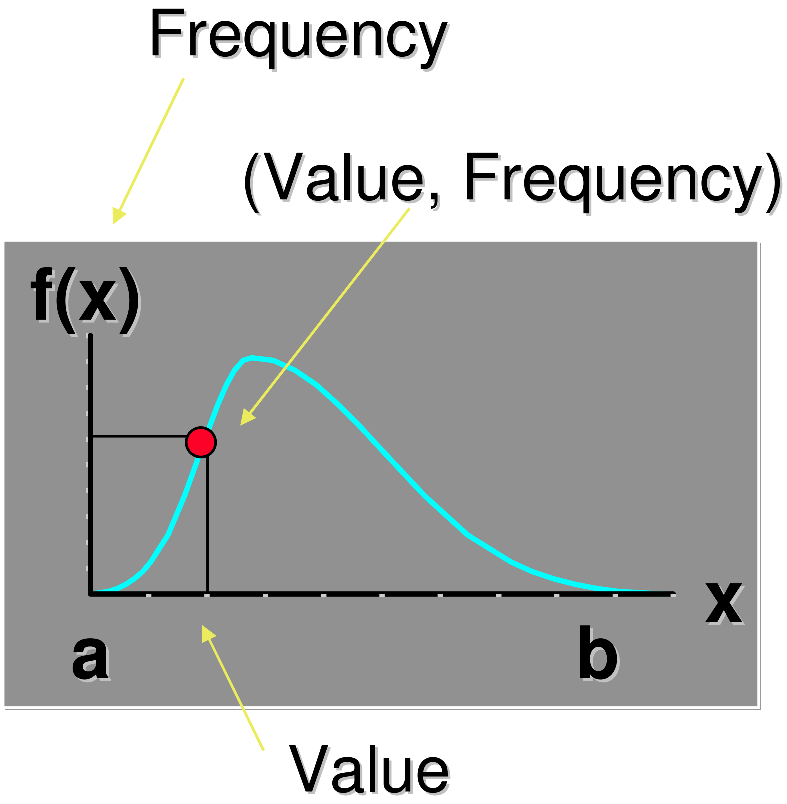
Continuous Probability Density Function

- 1. Mathematical Formula
- 2. Shows All Values, *x*, & Frequencies, f(*x*)
  - f(*X*) Is ***Not*** Probability
- 3. Properties

$$\int_{\text{All } X} f(x)dx = 1$$

(Area Under Curve)

$$f(x) \geq 0, \; a \leq x \leq b$$



Cumulative Distribution Function

- The cumulative distribution function (CDF) for a random variable ***X*** is

$$F_X(x) = P(X \le x) = P(\{s \in S \mid X(s) \le x\})$$

- Note that $F_X(x)$ is non-decreasing in ***x***, i.e.

$$x_1 \le x_2 \Rightarrow F_x(x_1) \le F_x(x_2)$$

- Also $\lim\limits_{x \to -\infty} F_x(x) = 0$ and $\lim\limits_{x \to \infty} F_x(x) = 1$

Expectation of a Random Variable: E[X]

- The expectation (average) of a (discrete-valued) random variable ***X*** is

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad E(X) = \sum_{x=-\infty}^{\infty} x P_X(x)$$

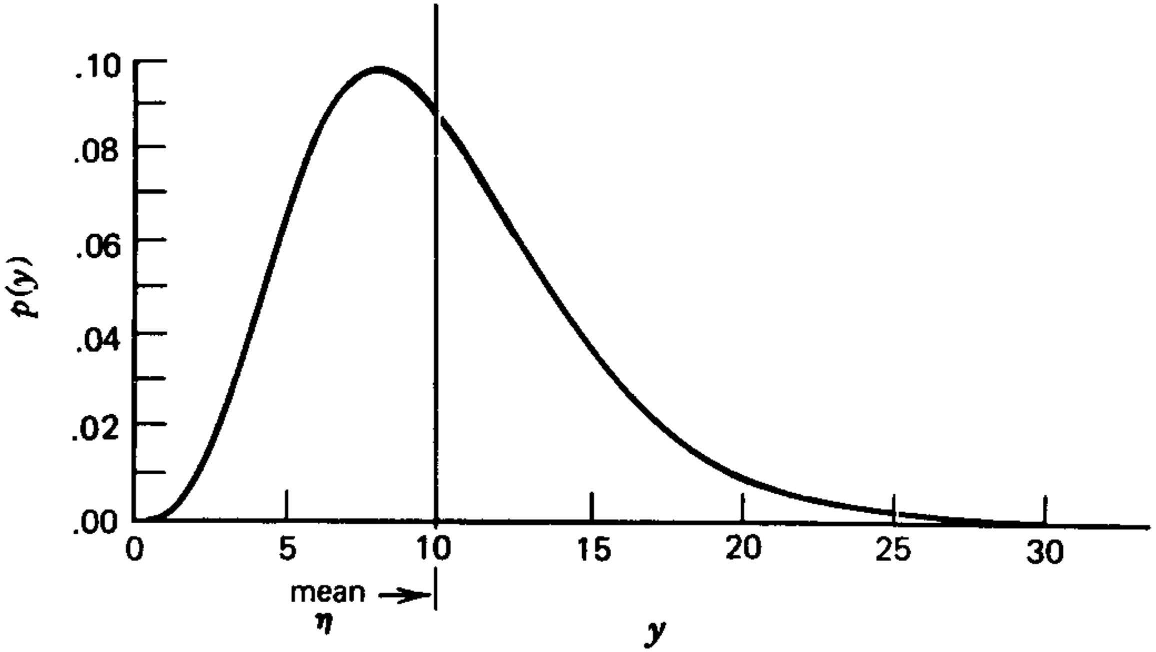


FIGURE 2.7. The mean $\eta = E(y)$ as the center of gravity of a distribution.

Standard Deviation, Coeff. Of Variation, SIQR

- **<u>Variance</u>**: second moment around the mean:
  - $\sigma^2 = E[(X-\mu)^2]$
- **<u>Standard deviation</u>** = σ

$$\mathrm{stdv}(x) = \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\mu'_2 - \mu^2},$$

Covariance and Correlation: Measures of Dependence

- **<u>Covariance</u>:** $\langle (x_i - \mu_i)(x_j - \mu_j) \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle ,$
  - For i = j, covariance = variance!
  - Independence => covariance = 0 (not vice-versa!)

- **<u>Correlation (coefficient)</u>** is a normalized (or scaleless) form of covariance:

$$\mathrm{cor}(x_i, x_j) \equiv \frac{\mathrm{cov}(x_i, x_j)}{\sigma_i \sigma_j},$$

  - Between –1 and +1.
    - Zero => no correlation (uncorrelated).
    - Note: uncorrelated <u>DOES NOT</u> mean independent!

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Random Vectors & Sum of R.V.s

- Random Vector = $[X_1, \ldots, X_n]$, where Xi = r.v.
- Covariance Matrix:
  - **K** is an nxn matrix…
  - $K_{ij} = Cov[X_i, X_j]$
  - $K_{ii} = Cov[X_i, X_i] = Var[X_i]$

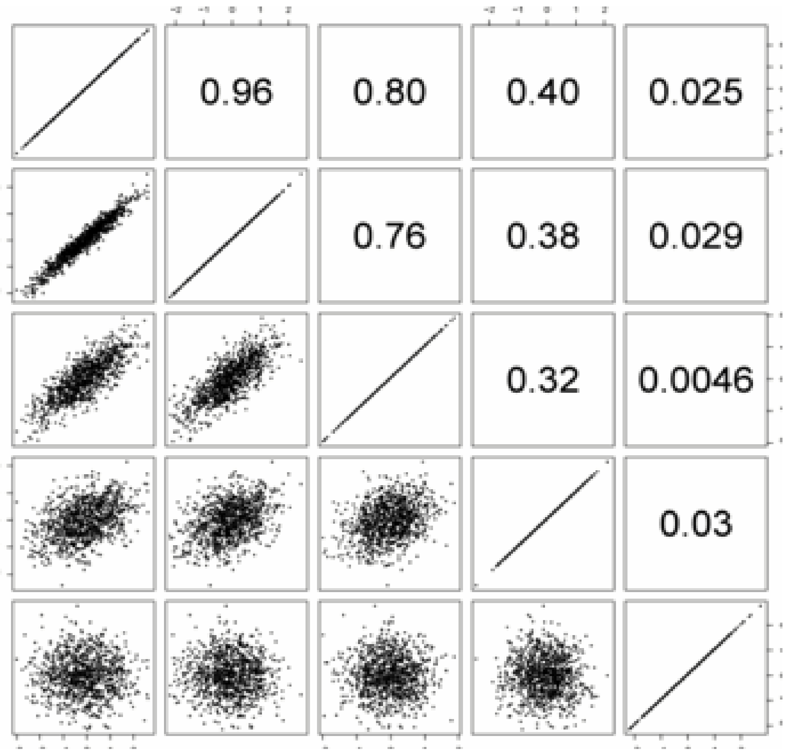
- Sum of *independent* R.v.s
  - $Z = X + Y$
  - PDF of Z is the *convolution* of PDFs of X and Y

    $p_Z(z) = p_X(x) * p_Y(y)$. Can use transforms!

Correlation

- indicates the strength and direction of a linear relationship between two random variables

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y}$$



Important (Discrete) Random Variable: Bernoulli

- The simplest possible measurement on an experiment:
  - **Success** (***X* = *1***) or **failure** ***(X* = *0)***.

- Usual notation:

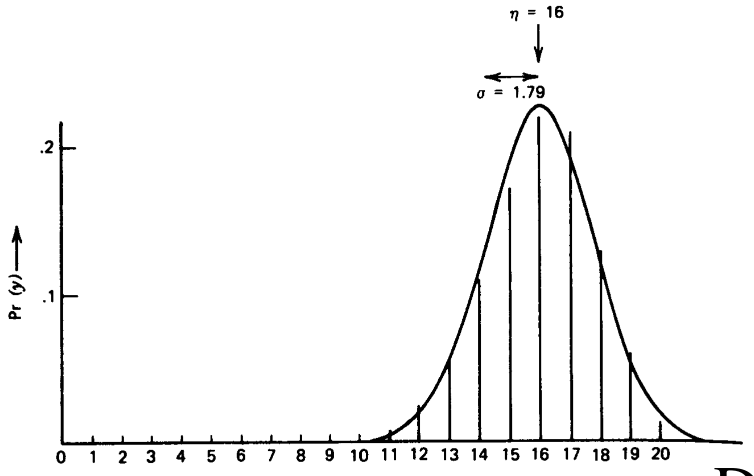
$$P_X(1) = P(X = 1) = p \qquad P_X(0) = P(X = 0) = 1 - p$$

  - ***E(X)*=**

Binomial can be skewed or normal

$$p(Y = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

$$\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}.$$



(c) Binomial distribution with mean p = 0.8 and n = 20.

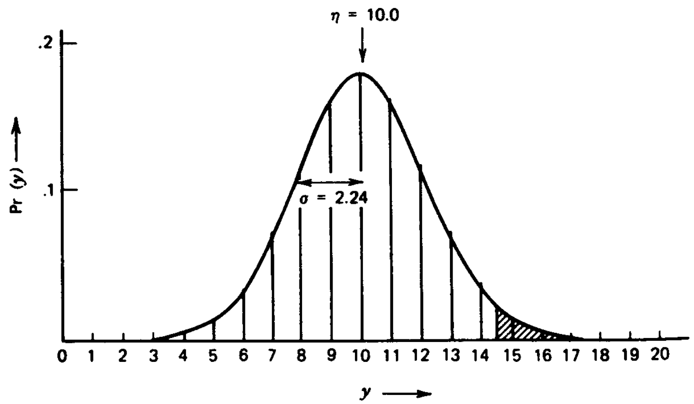
Depends upon p and n !

Mean

$$\mu = E(x) = np$$

Standard Deviation

$$\sigma = \sqrt{np(1-p)}$$



(d) Binomial distribution with mean p = 0.5 and n = 20.

FIGURE 5.4. (continued)

Important Random Variable: <u>Poisson</u>

- A Poisson random variable ***X*** is defined by its PMF: (limit of binomial)

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

Where $\lambda > 0$ is a constant

E(X) **=** λ

- Poisson random variables are good for <u>*counting frequency of occurrence*</u>: like the number of customers that arrive to a bank in one hour, or the number of packets that arrive to a router in one second.

Important Continuous Random Variable: Exponential

- Used to represent time, e.g. until the next arrival
- Has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

for some $\lambda > 0$

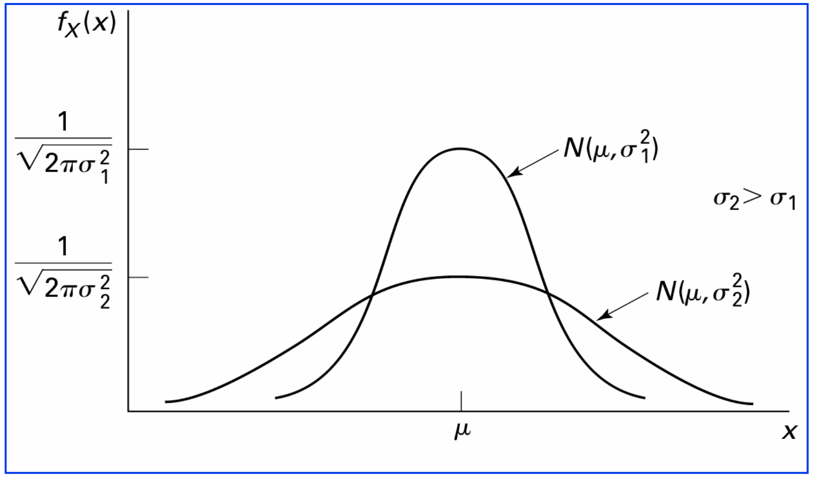
- Properties:

$$\int_0^\infty f_X(x)dx = 1 \quad \text{and} \quad E(X) = \frac{1}{\lambda}$$

  - Need to use integration by Parts!

Gaussian/Normal

- **<u>Normal Distribution:</u>** Completely characterized by mean (μ) and variance (σ^2)

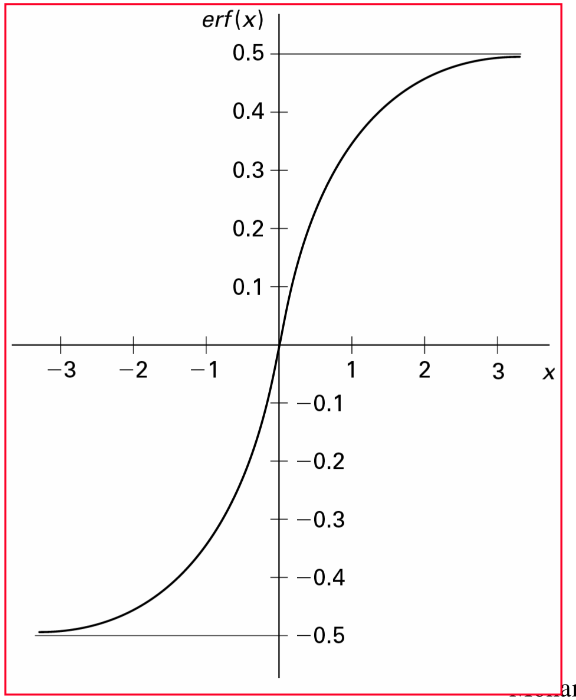


- **<u>Q-function:</u>** one-sided tail of normal pdf

$$Q(z) \triangleq p(x > z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

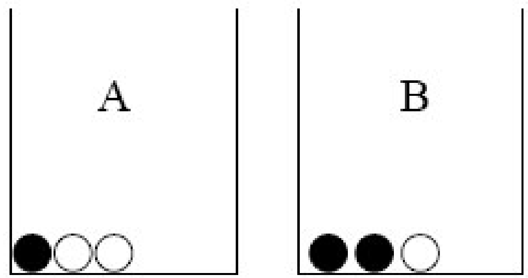
- **<u>erfc():</u>** two-sided tail.
- So:

$$Q(z) = \frac{1}{2}\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right)$$



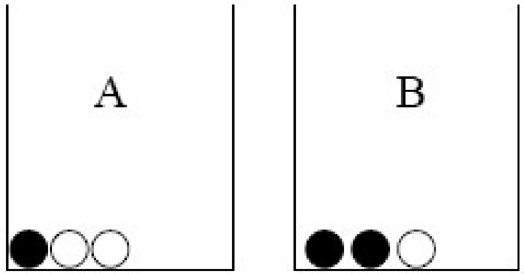
Maximum Likelihood (ML) Detection: Concepts

Likelihood Principle



- Experiment:
  - Pick Urn A or Urn B at random
  - Select a ball from that Urn.
- The ball is black.
- What is the probability that the selected Urn is A?

Likelihood Principle (Contd)



- Write out what you know!
- **P(Black | UrnA) = 1/3**
- **P(Black | UrnB) = 2/3**
- **P(Urn A) = P(Urn B) = 1/2**
- We want **P(Urn A | Black).**
- Gut feeling: Urn B is more likely than Urn A (given that the ball is black). But by how much?
- This is an inverse probability problem.
  - Make sure you understand the inverse nature of the conditional probabilities!
- Solution technique: Use Bayes Theorem.

Likelihood Principle (Contd)

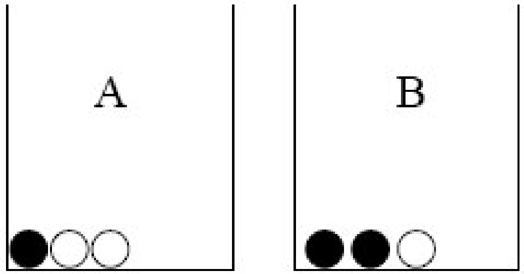
- **<u>Bayes manipulations:</u>**
- **P(Urn A | Black) =**
  - **P(Urn A and Black) /P(Black)**
- Decompose the numerator and denomenator in terms of the probabilities we know.

- **P(Urn A and Black) = P(Black | UrnA)*P(Urn A)**
- **P(Black) = P(Black| Urn A)*P(Urn A) + P(Black| UrnB)*P(UrnB)**

- We know all these values Plug in and crank.
- **P(Urn A and Black) = 1/3 * 1/2**
- **P(Black) = 1/3 * 1/2 + 2/3 * 1/2 = 1/2**
- **P(Urn A and Black) /P(Black) = 1/3 = 0.333**
- Notice that it matches our gut feeling that Urn A is less likely, once we have seen black.

- *<u>The information that the ball is black has CHANGED !</u>*
  - From P(Urn A) = 0.5 to P(Urn A | Black) = 0.333

Likelihood Principle



- Way of thinking…
- Hypotheses: Urn A or Urn B ?
- Observation: "Black"
- Prior probabilities: P(Urn A) and P(Urn B)
- Likelihood of Black given choice of Urn: {aka *forward probability*}
  - P(Black | Urn A) and P(Black | Urn B)
- Posterior Probability: of each hypothesis given evidence
  - P(Urn A | Black) {aka *inverse probability*}
- Likelihood Principle (informal): All inferences depend ONLY on
  - The likelihoods P(Black | Urn A) and P(Black | Urn B), and
  - The priors P(Urn A) and P(Urn B)
- Result is a probability (or distribution) model over the space of possible hypotheses.

Maximum Likelihood (intuition)

- Recall:
- **P(Urn A | Black) = P(Urn A and Black) /P(Black) =**

  P(Black | UrnA)*P(Urn A) / P(Black)

- **P(Urn? | Black)** is maximized when **P(Black | Urn?)** is maximized.
  - Maximization over the hypotheses space (Urn A or Urn B)

- **P(Black | Urn?)** = "likelihood"
- => "*Maximum Likelihood*" approach to maximizing posterior probability

Maximum Likelihood (ML): mechanics

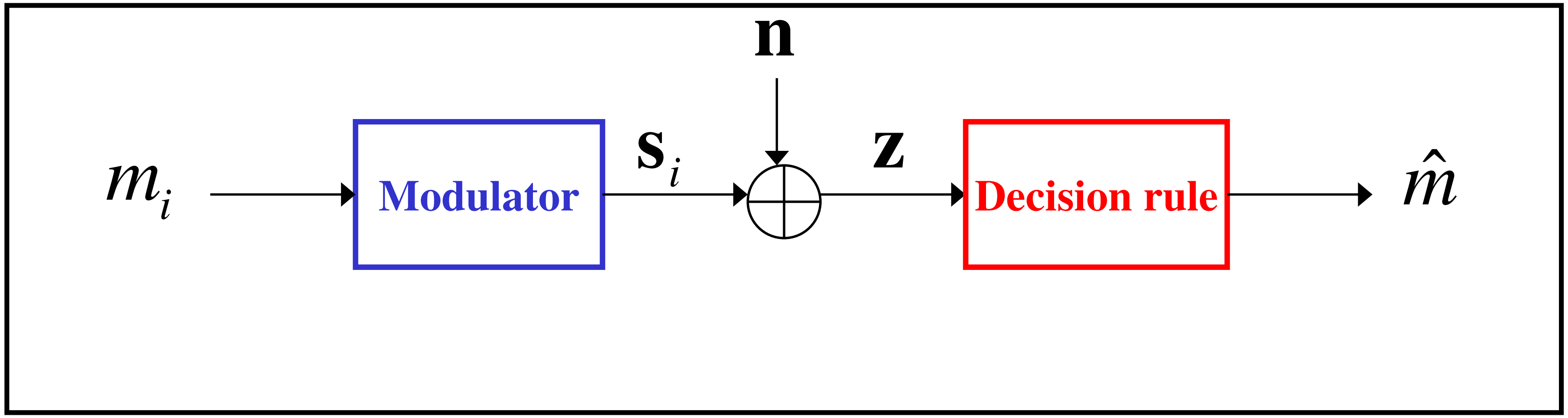
- **Independent Observations** (like Black): $\mathbf{X_1, \ldots, X_n}$
- **Hypothesis θ**
- **Likelihood Function**: $L(\theta) = P(X_1, \ldots, X_n \mid \theta) = \Pi_i P(X_i \mid \theta)$
  - {Independence => multiply individual likelihoods}
- **Log Likelihood** $\mathbf{LL(\theta) = \Sigma_i \log P(X_i \mid \theta)}$
- **Maximum likelihood:** by taking derivative and setting to zero and solving for θ

$$\hat{\theta}_{\mathrm{ML}}(x) = \arg\max_{\theta} \mathrm{P}(x|\theta)$$

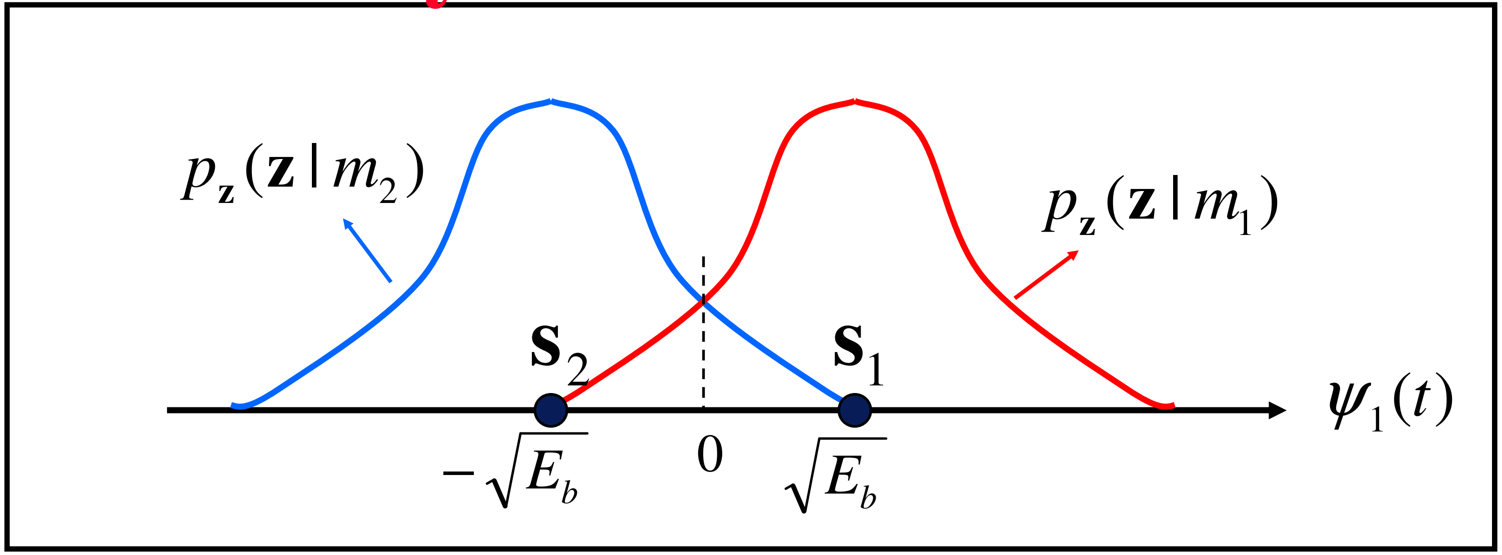
- **Maximum A Posteriori (MAP):** if non-uniform prior probabilities/distributions
  - Optimization function

Not Just Urns and Balls:
Detection of signal in AWGN

- Detection problem:
  - Given the observation vector $\mathbf{z}$, perform a mapping from $\mathbf{z}$ to an estimate $\hat{m}$ of the transmitted symbol, m_i, such that the average probability of error in the decision is minimized.



Binary PAM + AWGN Noise



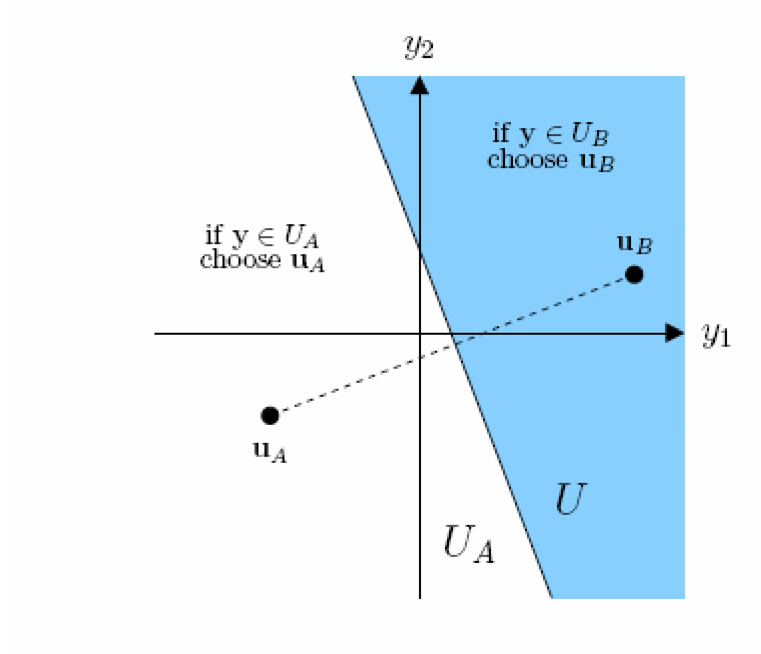
Signal s1 or s2 is sent. **z** is received

Additive white gaussian noise (AWGN) => the likelihoods are

$p_{\mathbf{z}}(\mathbf{z} \mid m_1)$ $p_{\mathbf{z}}(\mathbf{z} \mid m_2)$ bell-shaped pdfs around s1 and s2

MLE => at any point on the x-axis, see which curve (blue or red) has a higher (maximum) value and select the corresponding signal (s1 or s2) : simplifies into a "*nearest-neighbor*" rule

AWGN Nearest Neighbor Detection



- Projection onto the signal directions (subspace) is called *matched filtering* to get the "*sufficient statistic*"
- Error probability is the tail of the normal distribution (Q-function), based upon the mid-point between the two signals

$$Q\left(\frac{\|\mathbf{u}_A - \mathbf{u}_B\|}{2\sqrt{N_0/2}}\right),$$

Questions?