Bandpass modulation

- **Bandpass modulation:** The process of converting a data signal to a sinusoidal waveform where its amplitude, phase or frequency, or a combination of them, are varied in accordance with the transmitting data.
- Bandpass signal:

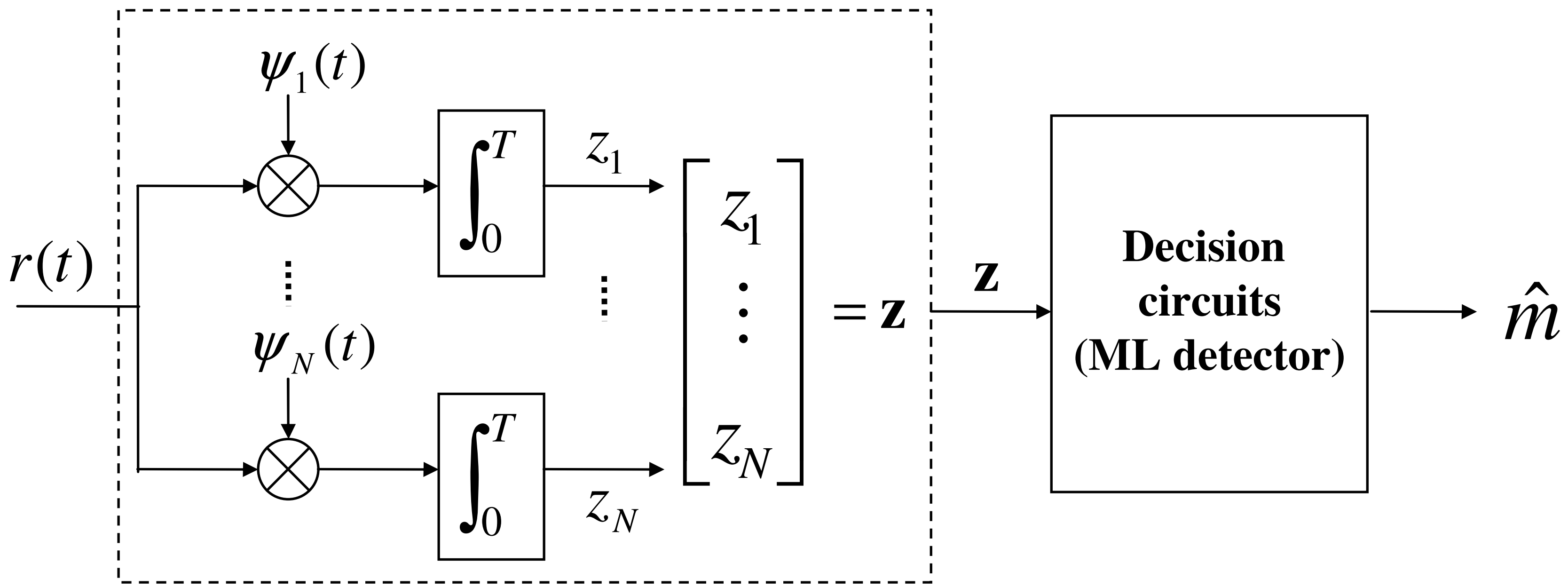
$$s_i(t) = g_T(t)\sqrt{\frac{2E_i}{T}}\cos\left(\omega_c t + (i-1)\Delta\omega t + \phi_i(t)\right) \quad 0 \le t \le T$$

where $g_T(t)$ is the baseband pulse shape with energy E_g.

- We assume here (otherwise will be stated):
  - $g_T(t)$ is a rectangular pulse shape with unit energy.
  - Gray coding is used for mapping bits to symbols.
  - E_s denotes average symbol energy given by $E_s = \frac{1}{M}\sum_{i=1}^{M} E_i$

Demodulation and detection

- **Demodulation:** The receiver signal is converted to baseband, filtered and sampled.
- **Detection:** Sampled values are used for detection using a decision rule such as the ML detection rule.

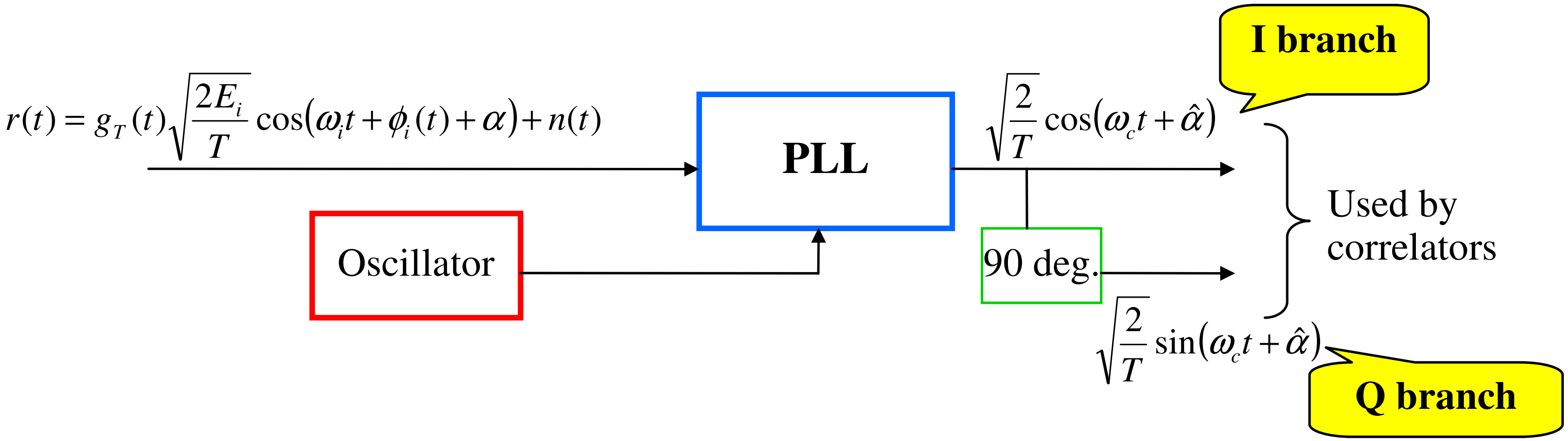


Coherent detection

- Coherent detection
  - requires carrier phase recovery at the receiver and hence, circuits to perform phase estimation.
  - Sources of carrier-phase mismatch at the receiver:
    - Propagation delay causes carrier-phase offset in the received signal.
    - The oscillators at the receiver which generate the carrier signal, are not usually phased locked to the transmitted carrier.

Coherent detection ..

- Circuits such as Phase-Locked-Loop (PLL) are implemented at the receiver for carrier phase estimation ($\alpha \approx \hat{\alpha}$).



Bandpass Modulation Schemes

- One dimensional waveforms
  - Amplitude Shift Keying (ASK)
  - M-ary Pulse Amplitude Modulation (M-PAM)
- Two dimensional waveforms
  - M-ary Phase Shift Keying (M-PSK)
  - M-ary Quadrature Amplitude Modulation (M-QAM)
- Multidimensional waveforms
  - M-ary Frequency Shift Keying (M-FSK)

One dimensional modulation, demodulation and detection

- Amplitude Shift Keying (ASK) modulation:

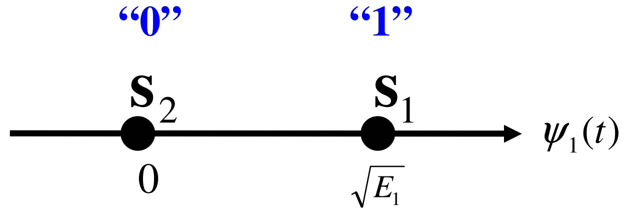
$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_c t + \phi)$$

$$s_i(t) = a_i \psi_1(t) \quad i = 1, \ldots, M$$

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t + \phi)$$

$$a_i = \sqrt{E_i}$$

On-off keying (M=2):



One dimensional mod.,…

- M-ary Pulse Amplitude modulation (M-PAM)

$$s_i(t) = a_i \sqrt{\frac{2}{T}} \cos(\omega_c t)$$

$$s_i(t) = a_i \psi_1(t) \quad i = 1, \ldots, M$$

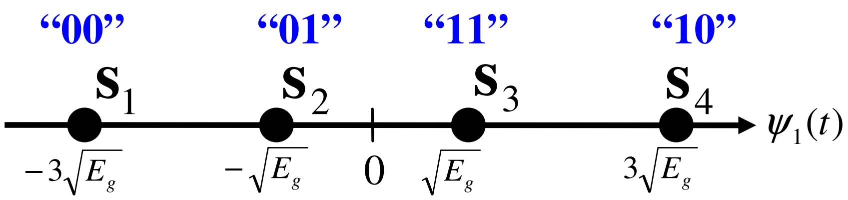
$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t)$$

$$a_i = (2i - 1 - M)\sqrt{E_g}$$

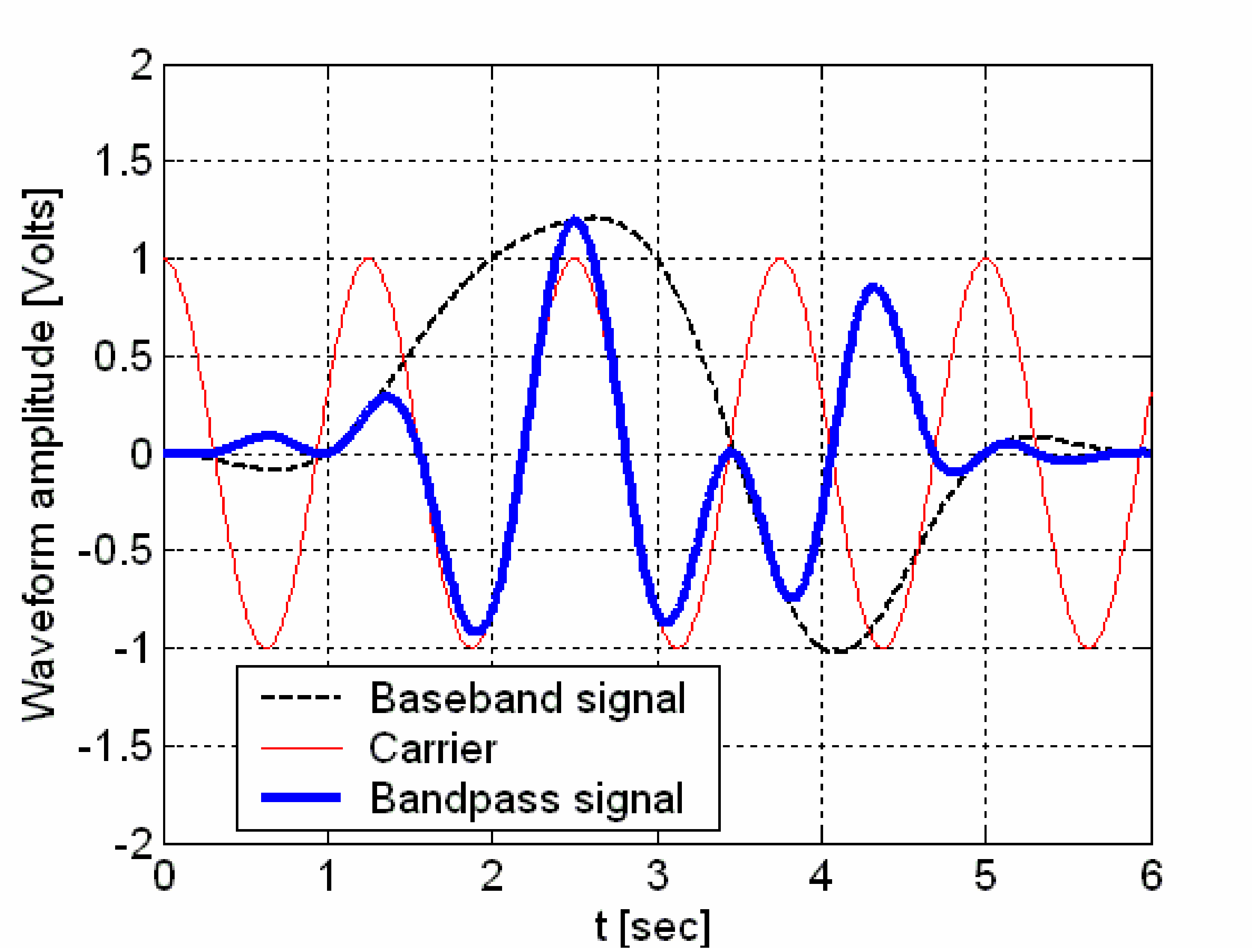
$$E_i = \|\mathbf{s}_i\|^2 = E_g (2i - 1 - M)^2$$

$$E_s = \frac{(M^2 - 1)}{3} E_g$$

4-PAM:

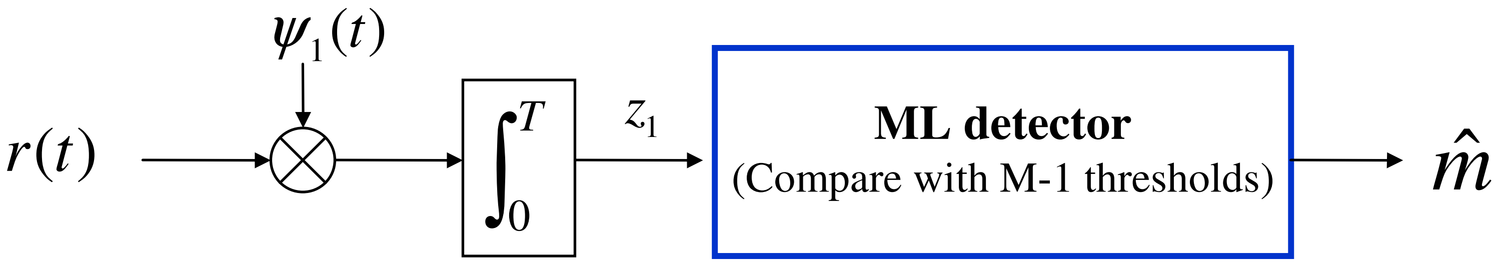


Example of bandpass modulation: Binary PAM



One dimensional mod.,...–cont'd

- Coherent detection of M-PAM



Two dimensional modulation, demodulation and detection (M-PSK)

- M-ary Phase Shift Keying (M-PSK)

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos\left(\omega_c t + \frac{2\pi i}{M} \right)$$

$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t) \quad i = 1, \dots, M$$

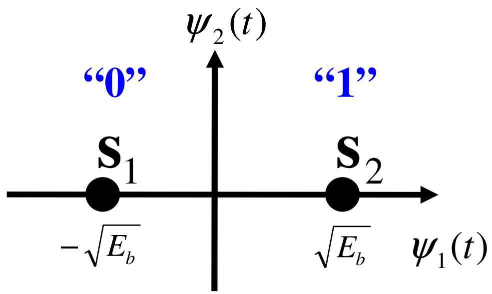
$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t) \qquad \psi_2(t) = -\sqrt{\frac{2}{T}} \sin(\omega_c t)$$

$$a_{i1} = \sqrt{E_s} \cos\left(\frac{2\pi i}{M}\right) \qquad a_{i2} = \sqrt{E_s} \sin\left(\frac{2\pi i}{M}\right)$$

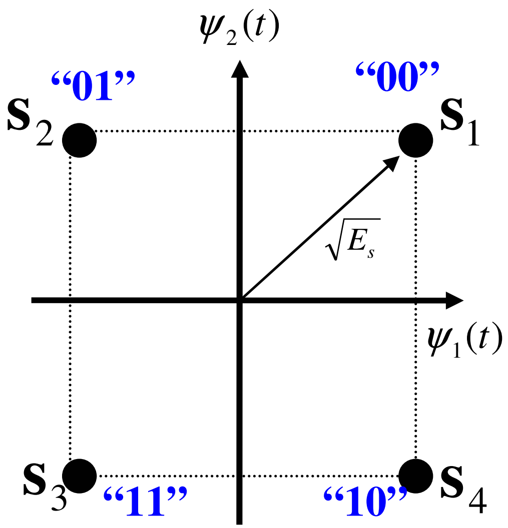
$$E_s = E_i = \|\mathbf{s}_i\|^2$$

Two dimensional mod.,… (MPSK)

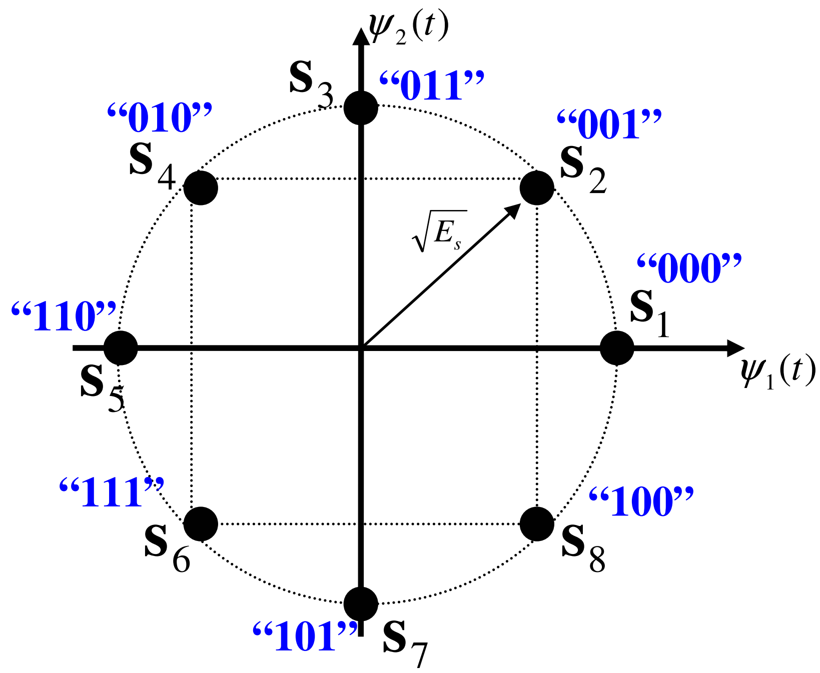
BPSK (M=2)



QPSK (M=4)

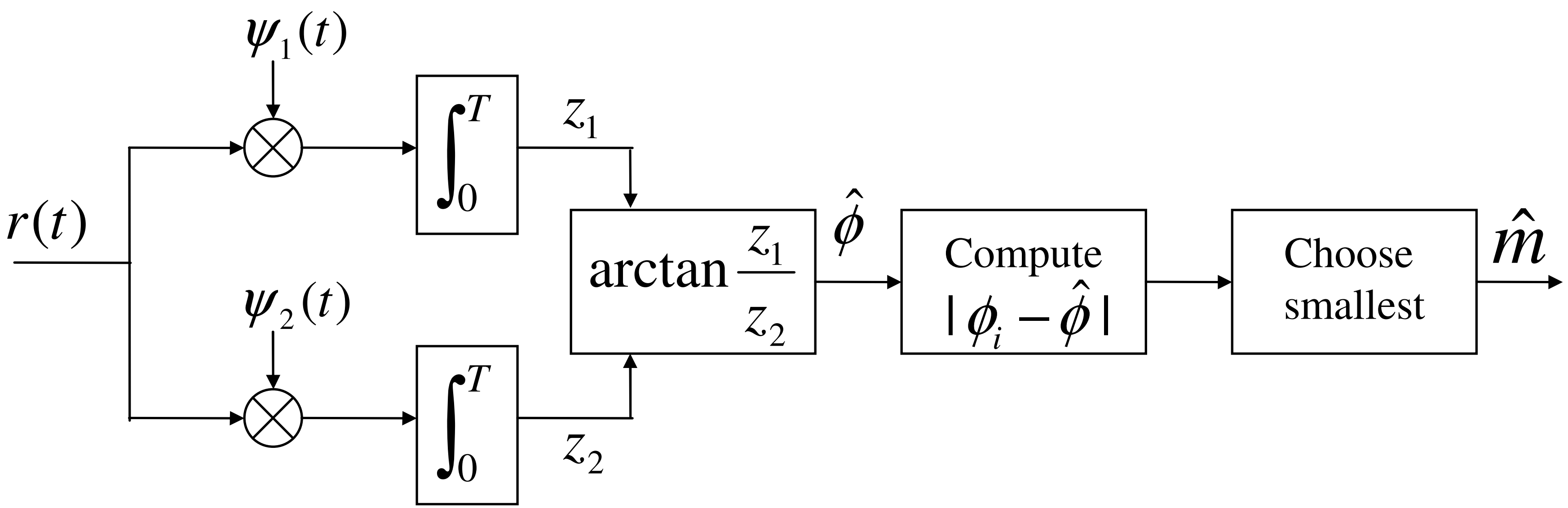


8PSK (M=8)



Two dimensional mod.,...(MPSK)

- Coherent detection of MPSK



Two dimensional mod.,… (M-QAM)

- M-ary Quadrature Amplitude Mod. (M-QAM)

$$s_i(t)=\sqrt{\frac{2E_i}{T}}\cos\left(\omega_c t+\varphi_i\right)$$

$$s_i(t)=a_{i1}\psi_1(t)+a_{i2}\psi_2(t) \quad i=1,\dots,M$$

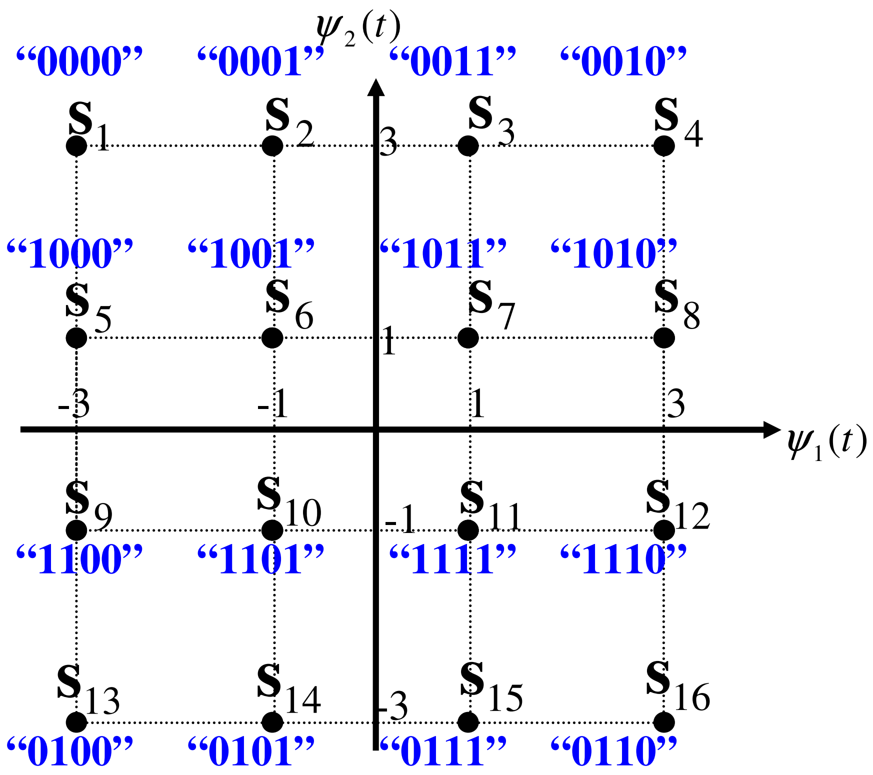
$$\psi_1(t)=\sqrt{\frac{2}{T}}\cos\left(\omega_c t\right) \quad \psi_2(t)=\sqrt{\frac{2}{T}}\sin\left(\omega_c t\right)$$

where a_{i1} and a_{i2} are PAM symbols and $E_s=\frac{2(M-1)}{3}$

$$(a_{i1},a_{i2})=\begin{bmatrix} (-\sqrt{M}+1,\sqrt{M}-1) & (-\sqrt{M}+3,\sqrt{M}-1) & \cdots & (\sqrt{M}-1,\sqrt{M}-1) \\ (-\sqrt{M}+1,\sqrt{M}-3) & (-\sqrt{M}+3,\sqrt{M}-3) & \cdots & (\sqrt{M}-1,\sqrt{M}-3) \\ \vdots & \vdots & \vdots & \vdots \\ (-\sqrt{M}+1,-\sqrt{M}+1) & (-\sqrt{M}+3,-\sqrt{M}+1) & \cdots & (\sqrt{M}-1,-\sqrt{M}+1) \end{bmatrix}$$

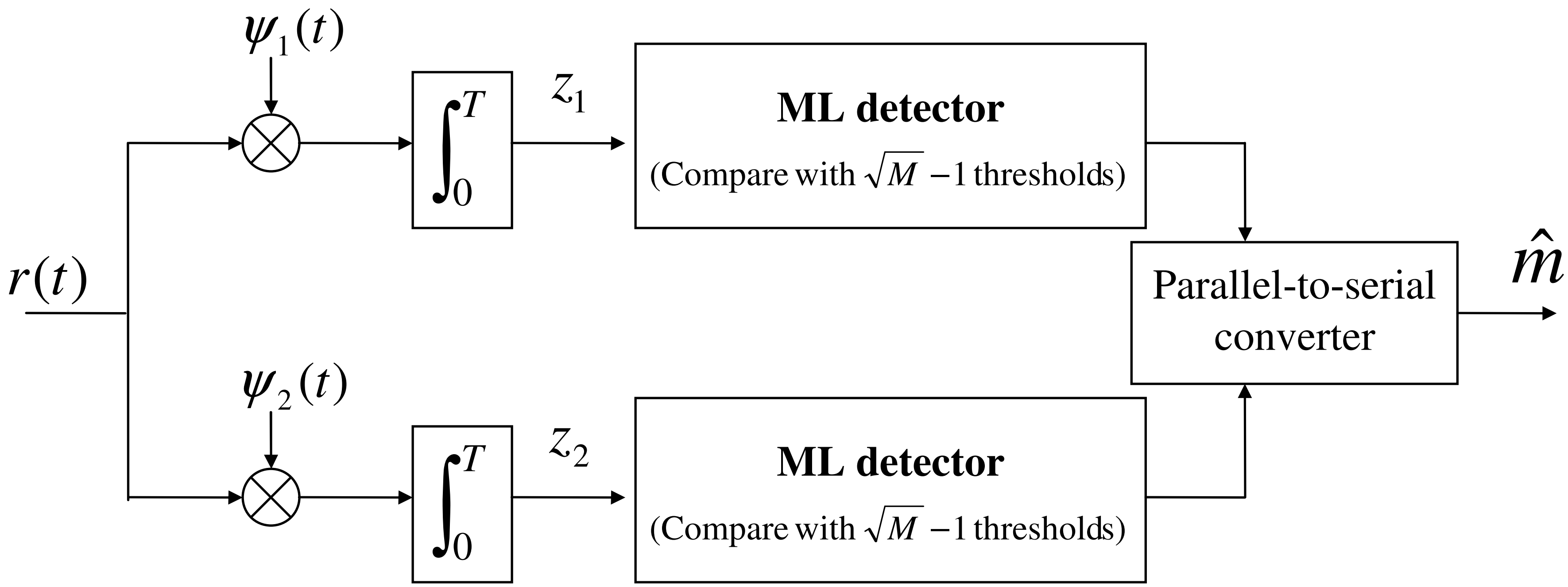
Two dimensional mod.,… (M-QAM)

16-QAM



Two dimensional mod.,… (M-QAM)

- Coherent detection of M-QAM



Multi-dimensional modulation, demodulation & detection

- M-ary Frequency Shift keying (M-FSK)

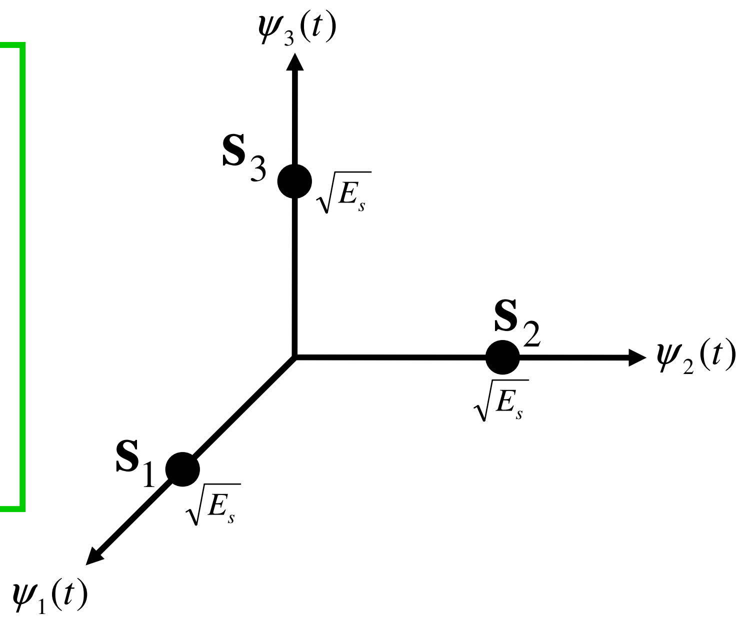
$$s_i(t) = \sqrt{\frac{2E_s}{T}}\cos(\omega_i t) = \sqrt{\frac{2E_s}{T}}\cos(\omega_c t + (i-1)\Delta\omega t)$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1}{2T}$$

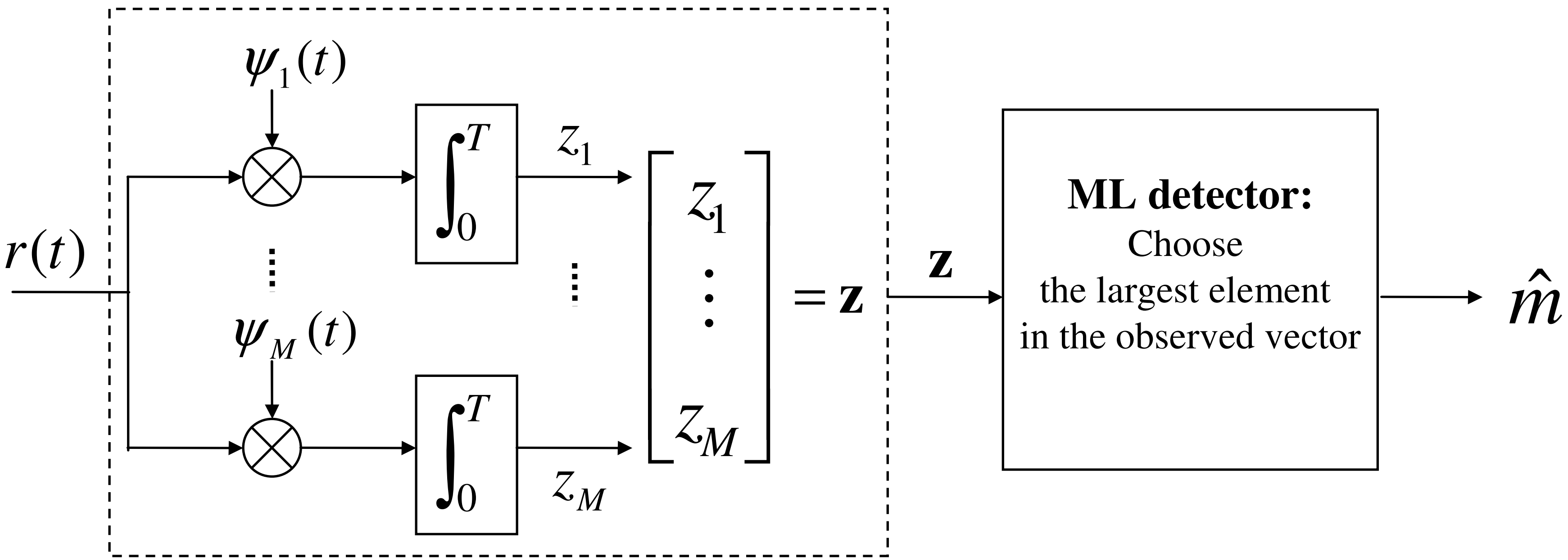
$$s_i(t) = \sum_{j=1}^{M} a_{ij}\psi_j(t) \quad i = 1,\ldots,M$$

$$\psi_i(t) = \sqrt{\frac{2}{T}}\cos(\omega_i t) \qquad a_{ij} = \begin{cases} \sqrt{E_s} & i = j \\ 0 & i \neq j \end{cases}$$

$$E_s = E_i = \|\mathbf{s}_i\|^2$$



Multi-dimensional mod.,...(M-FSK)



Non-coherent detection

- Non-coherent detection:
  - *No need for a reference in phase* with the received carrier
  - *Less complexity* compared to coherent detection at the price of *higher error rate.*

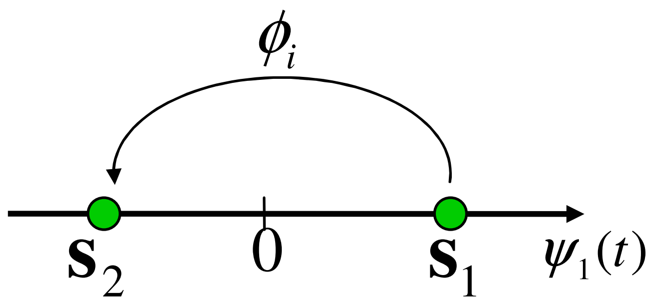
Non-coherent detection …

- Differential coherent detection
  - Differential encoding of the message
    - The symbol phase changes if the current bit is different from the previous bit.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \theta_i(t)\right), \quad 0 \le t \le T, \quad i = 1,\dots,M$$

$$\theta_k(nT) = \theta_k((n-1)T) + \phi_i(nT)$$

Symbol index: k	0	1	2	3	4	5	6	7
Data bits: m_k		1	1	0	1	0	1	1
Symbol phase: θ_k	π	π	π	0	π	0	π	π

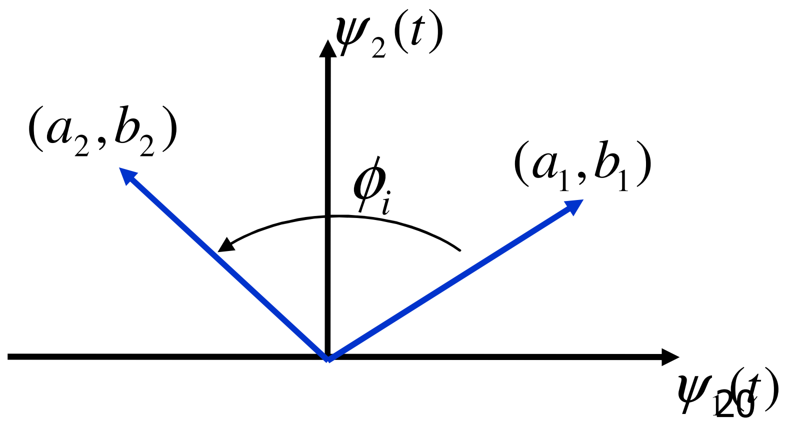


Non-coherent detection …

- Coherent detection for diff encoded mod.
  - assumes slow variation in carrier-phase mismatch during two symbol intervals.
  - correlates the received signal with basis functions
  - uses the phase difference between the current received vector and previously estimated symbol

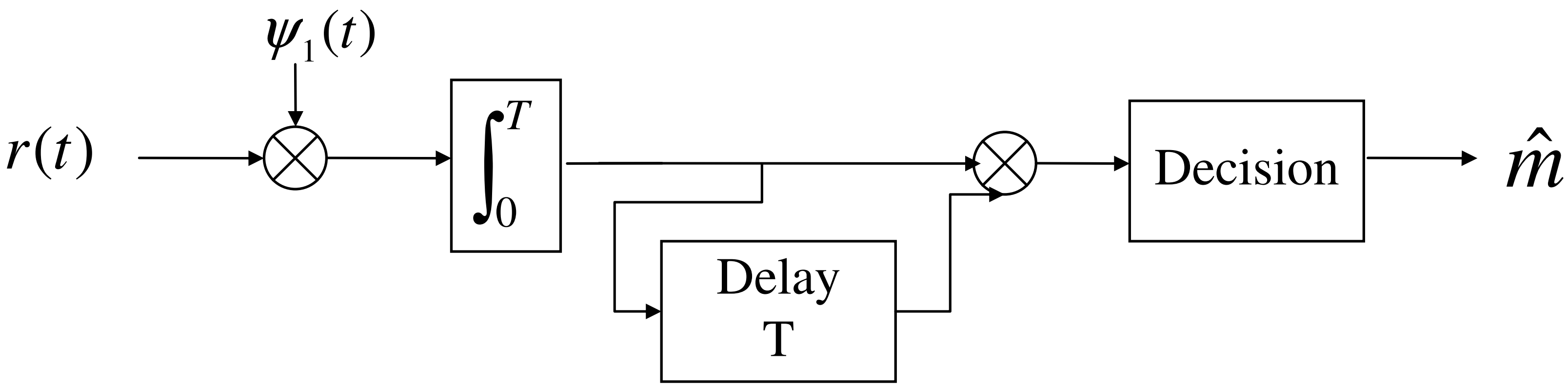
$$r(t)=\sqrt{\frac{2E}{T}}\cos\left(\omega_0 t+\theta_i(t)+\alpha\right)+n(t),\quad 0\le t\le T$$

$$\left(\theta_i(nT)+\alpha\right)-\left(\theta_j((n-1)T)+\alpha\right)=\theta_i(nT)-\theta_j((n-1)T)=\phi_i(nT)$$



Non-coherent detection …

- Optimum differentially coherent detector

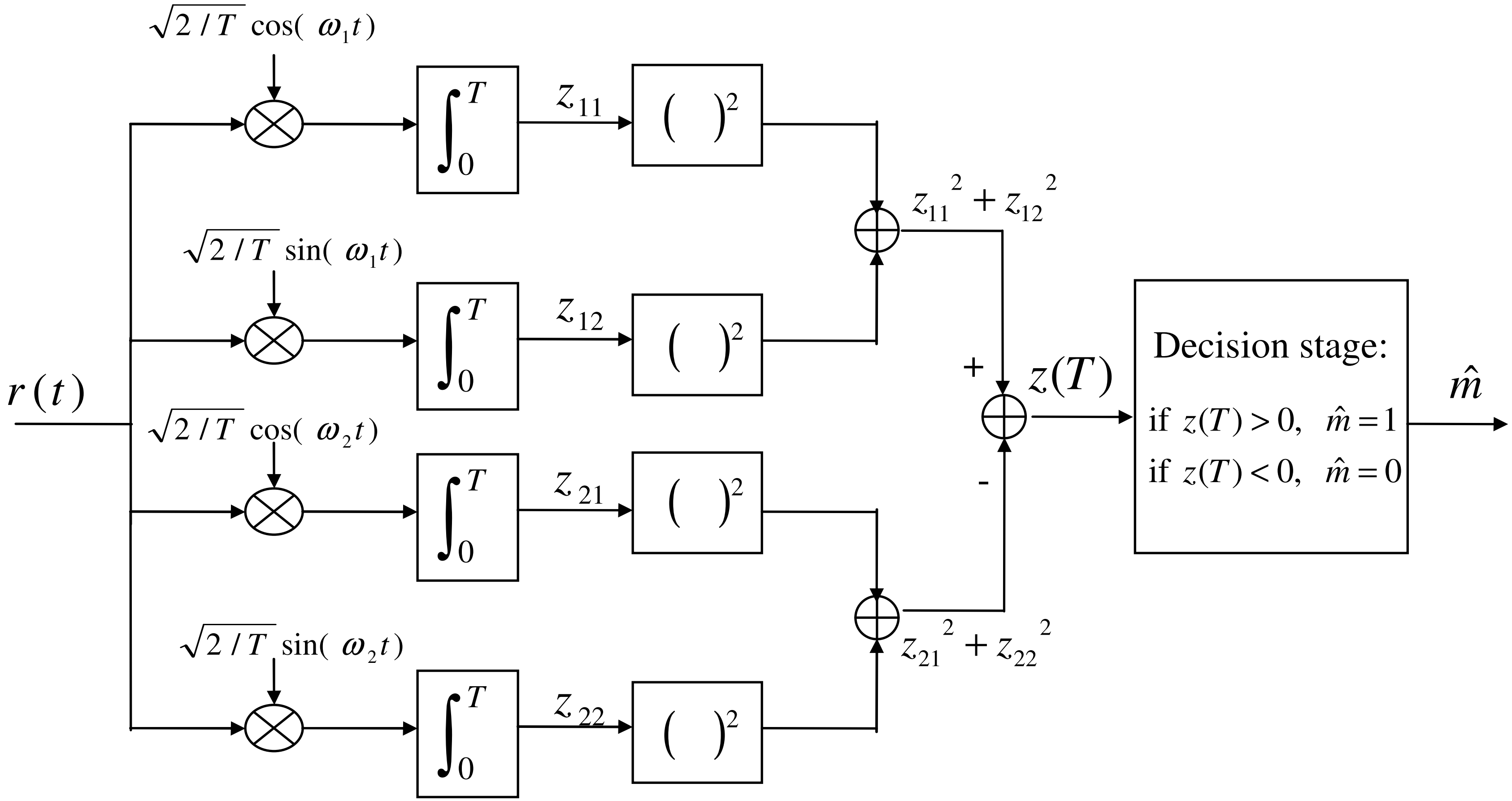


Non-coherent detection …

- Energy detection
  - Non-coherent detection for orthogonal signals (e.g. M-FSK)
    - Carrier-phase offset causes partial correlation between I and Q branches for each candidate signal.
    - The received energy corresponding to each candidate signal is used for detection.

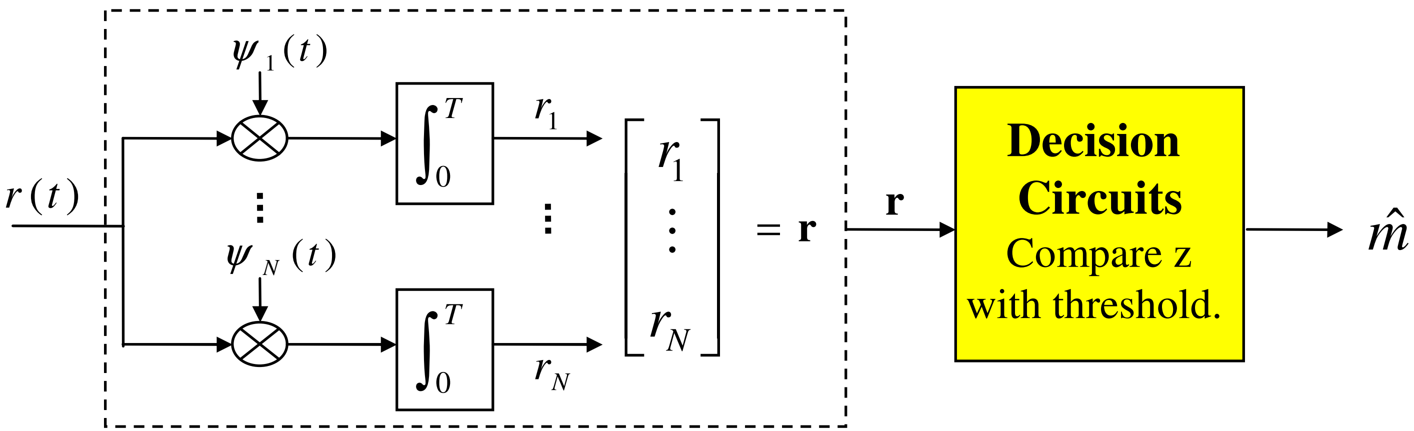
Non-coherent detection ...

- Non-coherent detection of BFSK



Error probability of bandpass modulation

- Before evaluating the error probability, it is important to remember that:
  - The type of modulation and detection (coherent or non-coherent) determines the structure of the decision circuits and hence the decision variable, denoted by z.
  - The decision variable, z, is compared with M-1 thresholds, corresponding to M decision regions for detection purposes.



Error probability …

- The matched filters output (observation vector= $\mathbf{r}$) is the detector input and the decision variable is a $z = f(\mathbf{r})$ function of $\mathbf{r}$, i.e.
  - For MPAM, MQAM and MFSK with coherent detection $z = \mathbf{r}$
  - For MPSK with coherent detection $z = \angle\mathbf{r}$
  - For non-coherent detection (M-FSK and DPSK), $z = |\mathbf{r}|$
- We know that for calculating the average probability of symbol error, we need to determine

$$\Pr(\mathbf{r} \text{ lies inside } Z_i \,|\, \mathbf{s}_i \text{ sent}) = \Pr(z \text{ satisfies condition } C_i \,|\, \mathbf{s}_i \text{ sent})$$

- *Hence, we need to know the statistics of z, which depends on the modulation scheme and the detection type.*

Error probability …

- AWGN channel model: $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$
  - The signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$ is deterministic.
  - The elements of the noise vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0/2$. The noise vector's pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

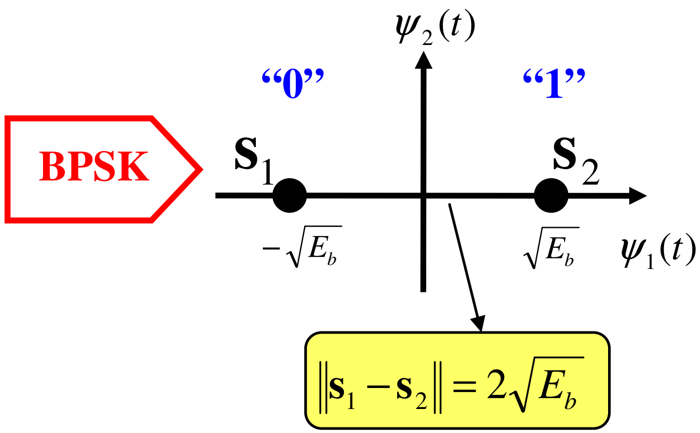
  - The elements of the observed vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{r}}(\mathbf{r} \mid \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{r} - \mathbf{s}_i\|^2}{N_0} \right)$$

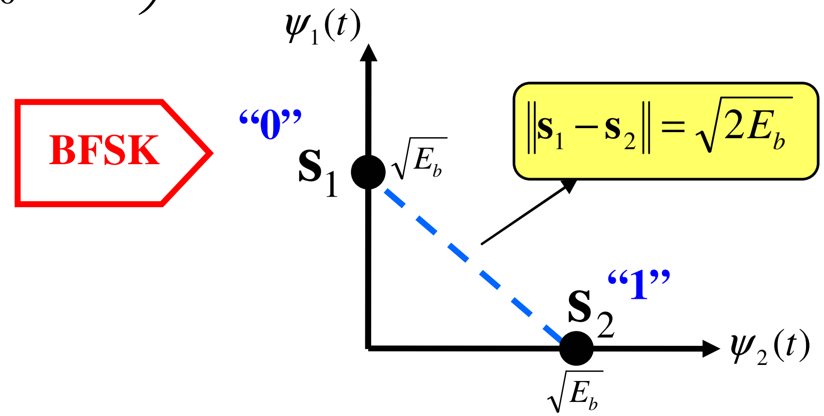
Error probability …

- BPSK and BFSK with *coherent* detection:

$$P_B = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|/2}{\sqrt{N_0/2}}\right)$$



$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

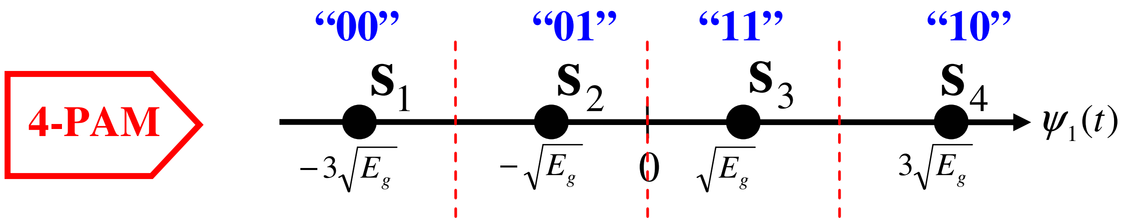


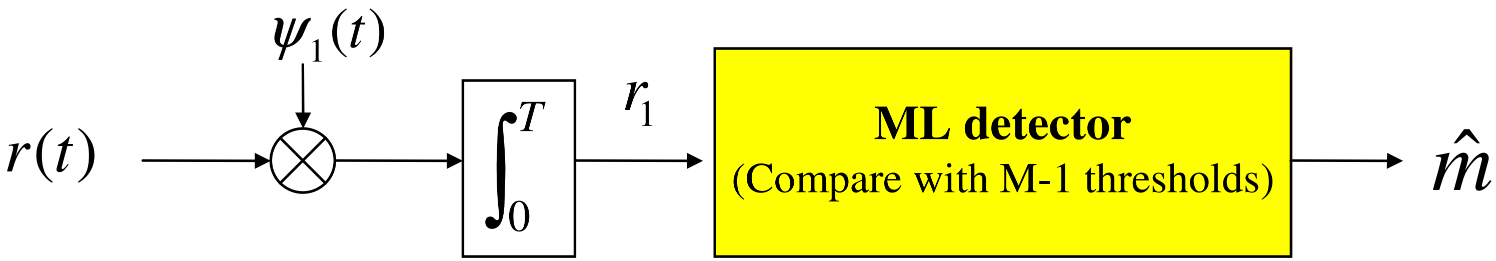
$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Error probability ….

- *Coherent* detection of M-PAM
  - Decision variable:

$$z = r_1$$





Error probability ….

- *Coherent* detection of M-PAM ….
  - Error happens if the noise, $n_1 = r_1 - \mathbf{s}_m$, exceeds in amplitude one-half of the distance between adjacent symbols. For symbols on the border, error can happen only in one direction. Hence:

$$P_e(\mathbf{s}_m) = \Pr\left(|n_1| = |r_1 - \mathbf{s}_m| > \sqrt{E_g}\right) \quad \text{for} \quad 2 < m < M-1;$$

$$P_e(\mathbf{s}_1) = \Pr\left(n_1 = r_1 - \mathbf{s}_1 > \sqrt{E_g}\right) \quad \text{and} \quad P_e(\mathbf{s}_M) = \Pr\left(n_1 = r_1 - \mathbf{s}_M < -\sqrt{E_g}\right)$$

$$P_E(M) = \frac{1}{M}\sum_{m=1}^{M} P_e(\mathbf{s}_m) = \frac{M-2}{M}\Pr\left(|n_1| > \sqrt{E_g}\right) + \frac{1}{\mathrm{M}}\Pr\left(n_1 > \sqrt{E_g}\right) + \frac{1}{\mathrm{M}}\Pr\left(n_1 < -\sqrt{E_g}\right)$$

$$= \frac{2(M-1)}{M}\Pr\left(n_1 > \sqrt{E_g}\right) = \frac{2(M-1)}{M}\int_{\sqrt{E_g}}^{\infty} p_{n_1}(n)\,dn = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

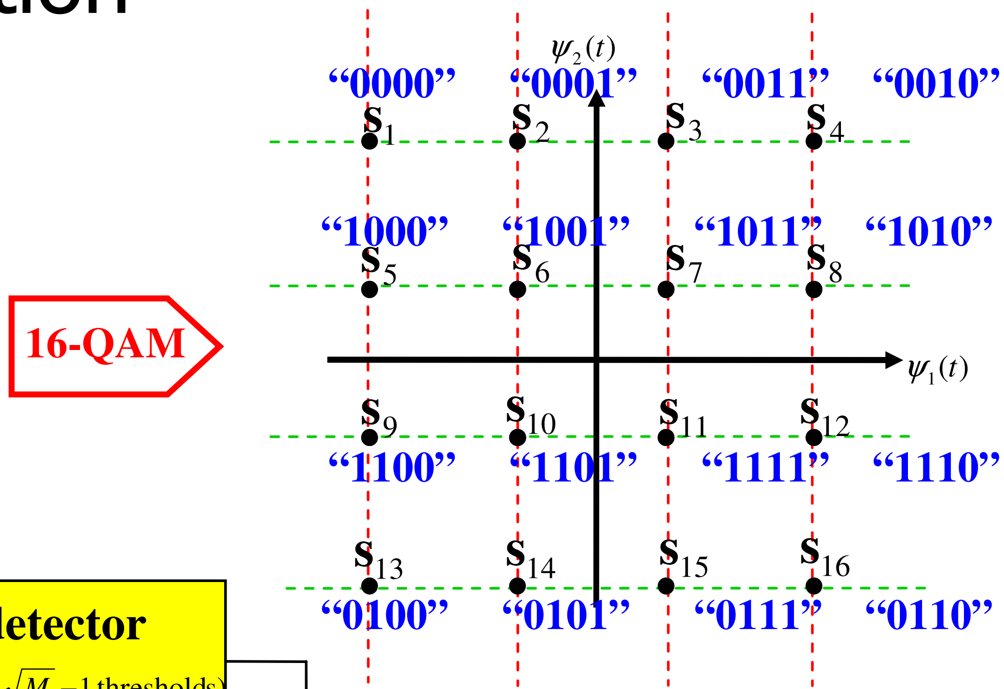
$$E_s = (\log_2 M)E_b = \frac{(M^2-1)}{3}E_g$$

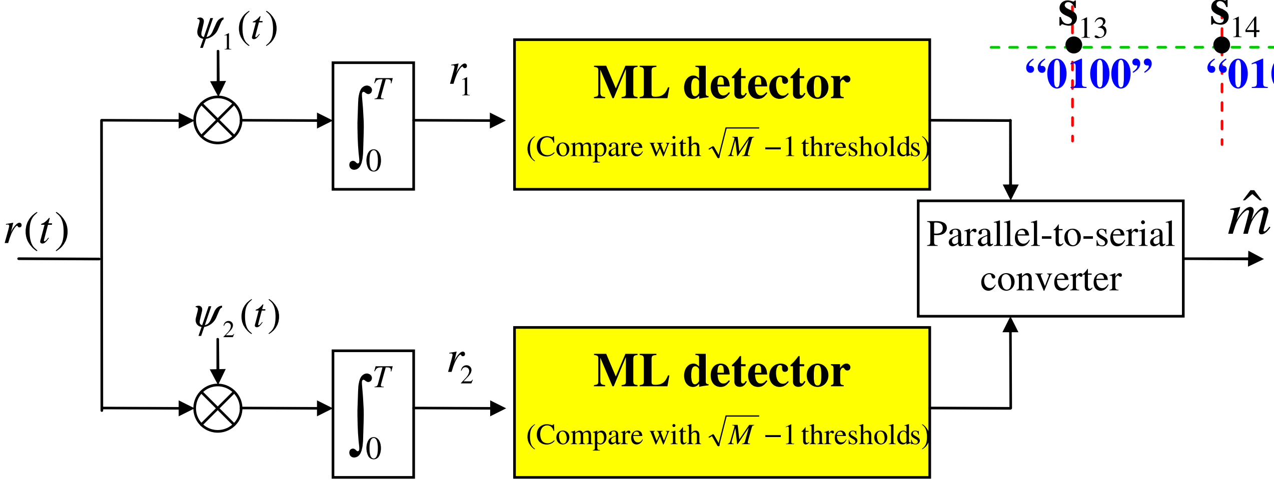
Gaussian pdf with zero mean and variance $N_0/2$

$$P_E(M) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)$$

Error probability …

- *Coherent* detection of M-QAM





Error probability …

Coherent detection of M-QAM …

- M-QAM can be viewed as the combination of two $\sqrt{M}-\mathrm{PAM}$ modulations on I and Q branches, respectively.
- No error occurs if no error is detected on either the I or the Q branch.
- Considering the symmetry of the signal space and the orthogonality of the I and Q branches:

$$P_E(M) = 1 - P_C(M) = 1 - \Pr(\text{no error detected on I and Q branches})$$

$$\Pr(\text{no error detected on I and Q branches}) = \Pr(\text{no error on I})\Pr(\text{no error on Q})$$

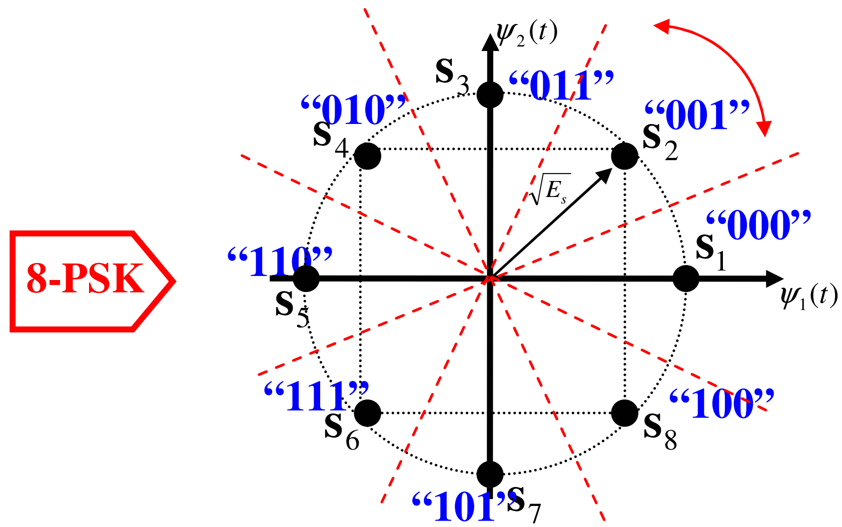
$$= \Pr(\text{no error on I})^2 = \left(1 - P_E\left(\sqrt{M}\right)\right)^2$$

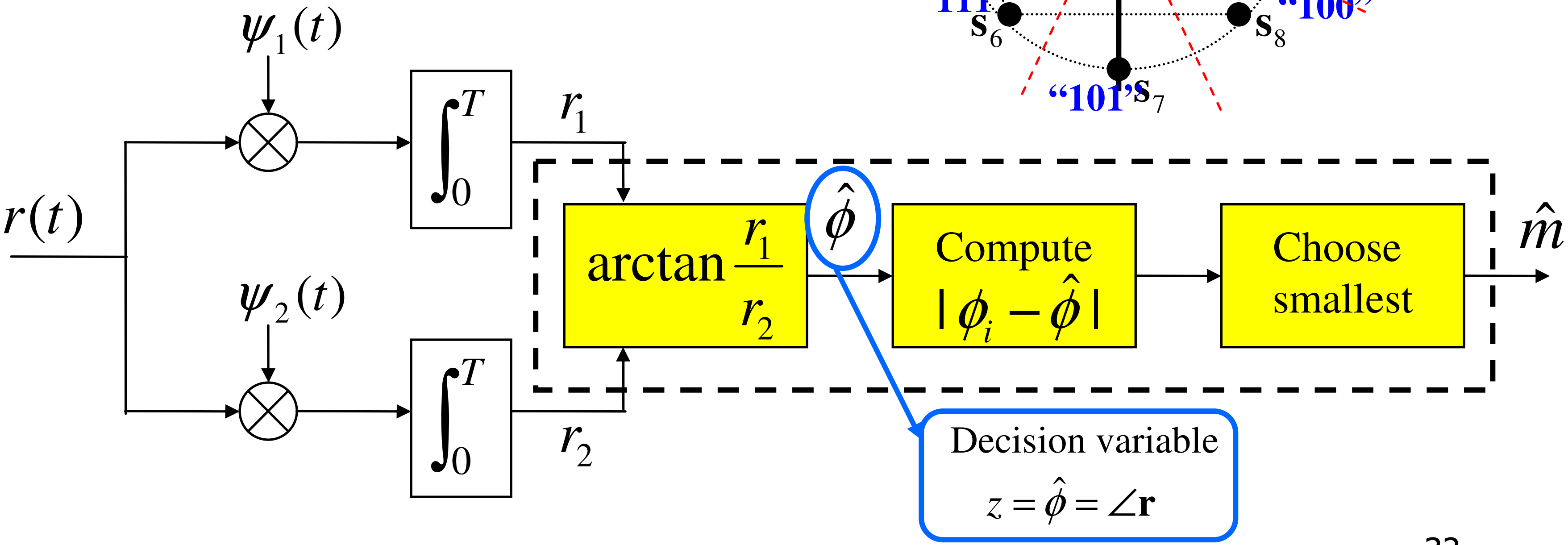
Average probability of symbol error for $\sqrt{M}-\mathrm{PAM}$

$$P_E(M) = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)$$

Error probability …

- *Coherent* detection of MPSK





Error probability …

- *Coherent* detection of MPSK …
  - The detector compares the phase of observation vector to M-1 thresholds.
  - Due to the circular symmetry of the signal space, we have:

$$P_E(M) = 1 - P_C(M) = 1 - \frac{1}{M}\sum_{m=1}^{M} P_c(\mathbf{s}_m) = 1 - P_c(\mathbf{s}_1) = 1 - \int_{-\pi/M}^{\pi/M} p_{\hat{\phi}}(\phi)\,d\phi$$

where

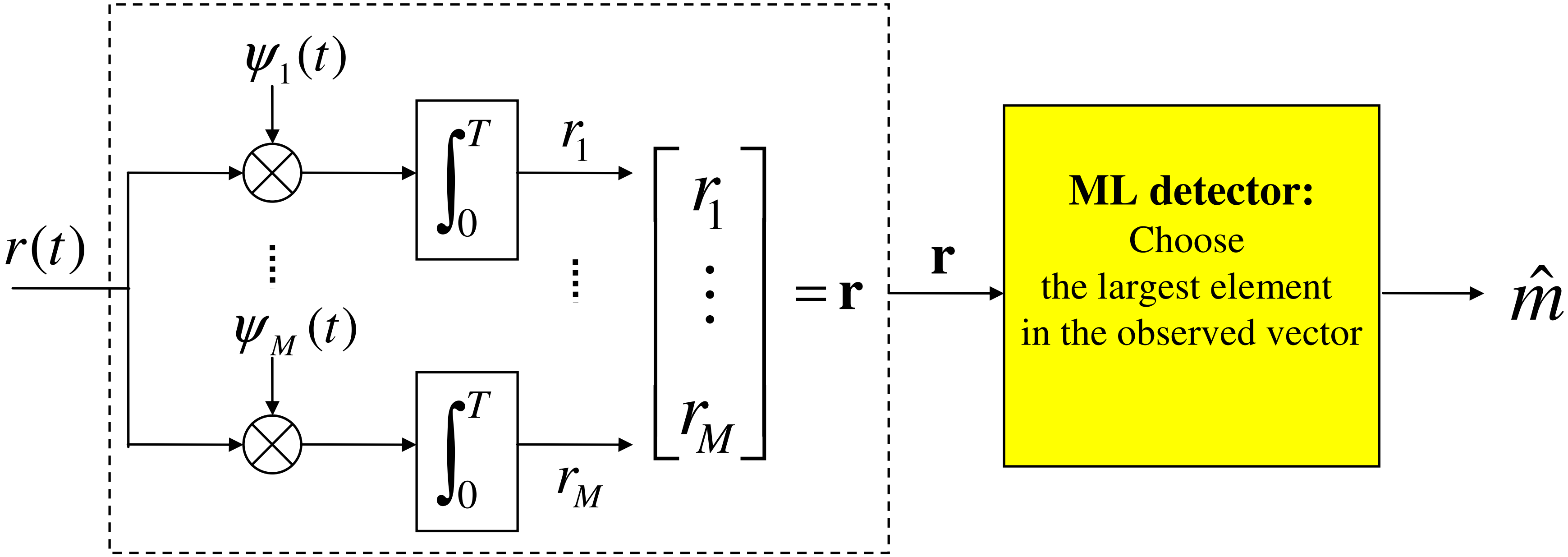
$$p_{\hat{\phi}}(\phi) \approx \sqrt{\frac{2}{\pi}\frac{E_s}{N_0}}\cos(\phi)\exp\left(-\frac{E_s}{N_0}\sin^2\phi\right); \quad |\phi| \leq \frac{\pi}{2}$$

  - It can be shown that

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) \quad \text{or} \quad P_E(M) \approx 2Q\left(\sqrt{\frac{2(\log_2 M)E_b}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

Error probability …

- *Coherent* detection of M-FSK



Error probability …

- *Coherent* detection of M-FSK …

- The dimension of the signal space is *M*. An upper bound for the average symbol error probability can be obtained by using the union bound. Hence:

$$P_E(M) \le (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

or, equivalently

$$P_E(M) \le (M-1)Q\left(\sqrt{\frac{(\log_2 M)E_b}{N_0}}\right)$$

Bit error probability versus symbol error probability

- Number of bits per symbol $k = \log_2 M$
- For orthogonal M-ary signaling (M-FSK)

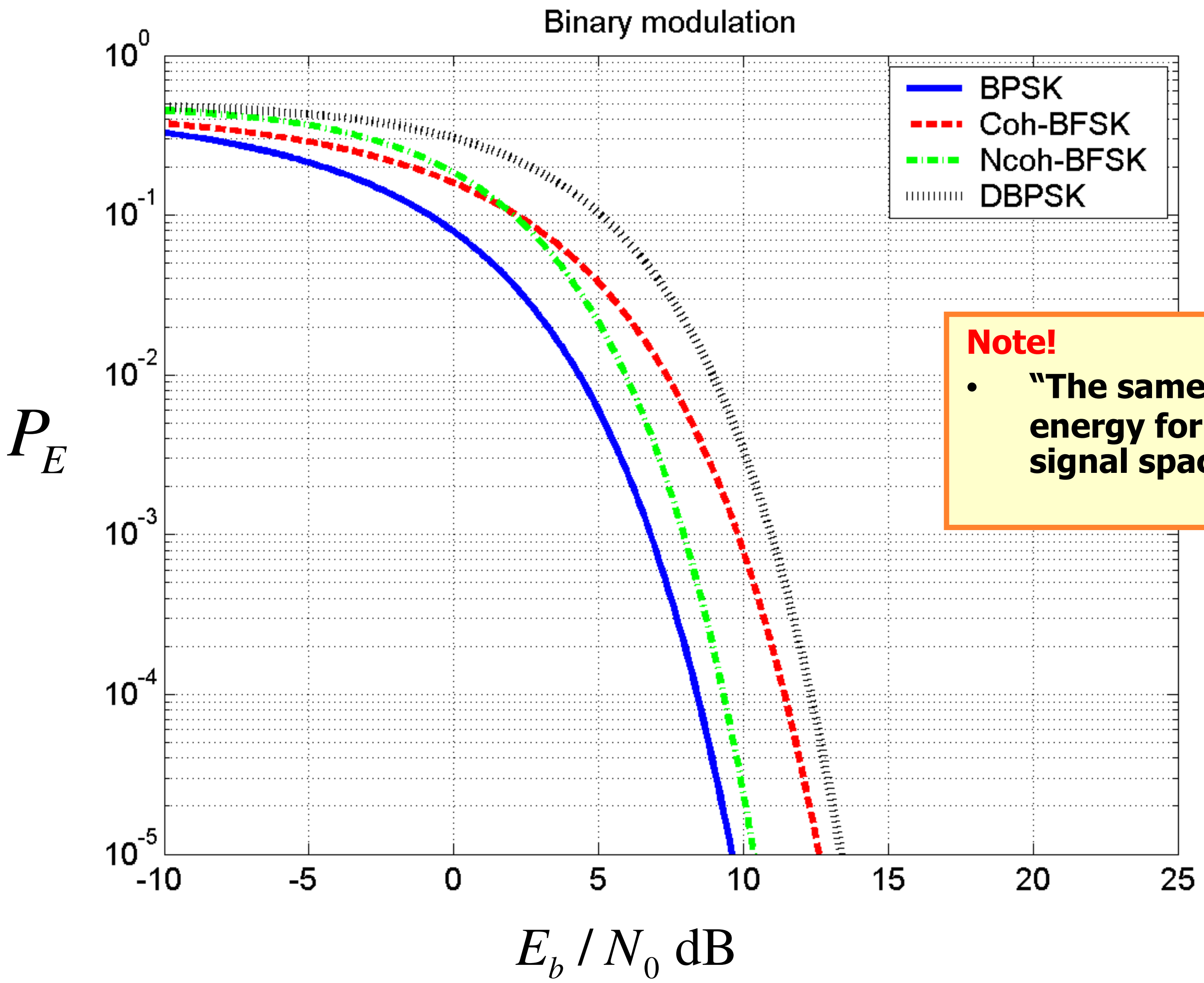
$$\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1}$$

$$\lim_{k \to \infty} \frac{P_B}{P_E} = \frac{1}{2}$$

- For M-PSK, M-PAM and M-QAM

$$P_B \approx \frac{P_E}{k} \quad \text{for} \quad P_E << 1$$

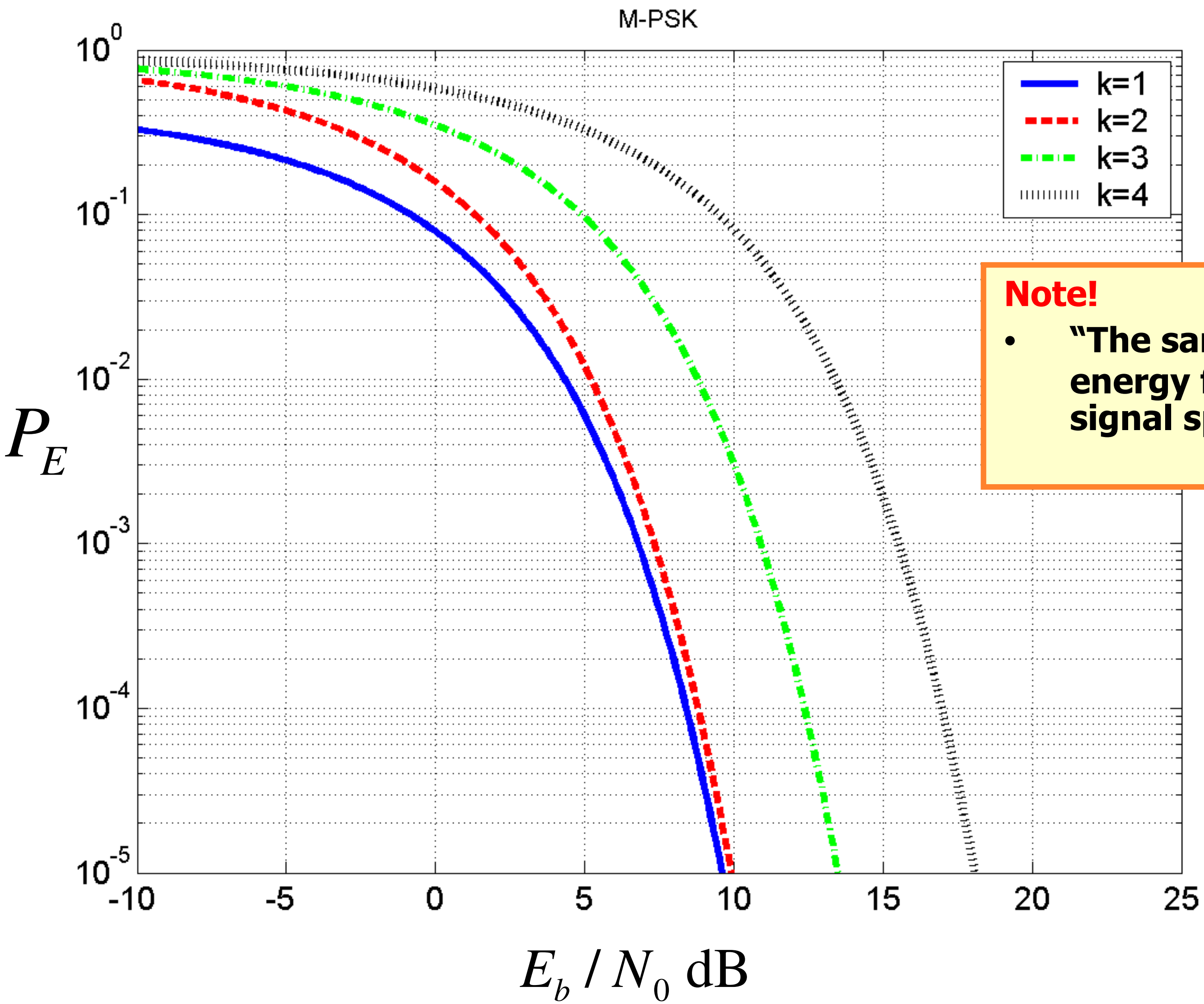
Probability of symbol error for binary modulation



Note!

- **"The same average symbol energy for different sizes of signal space"**

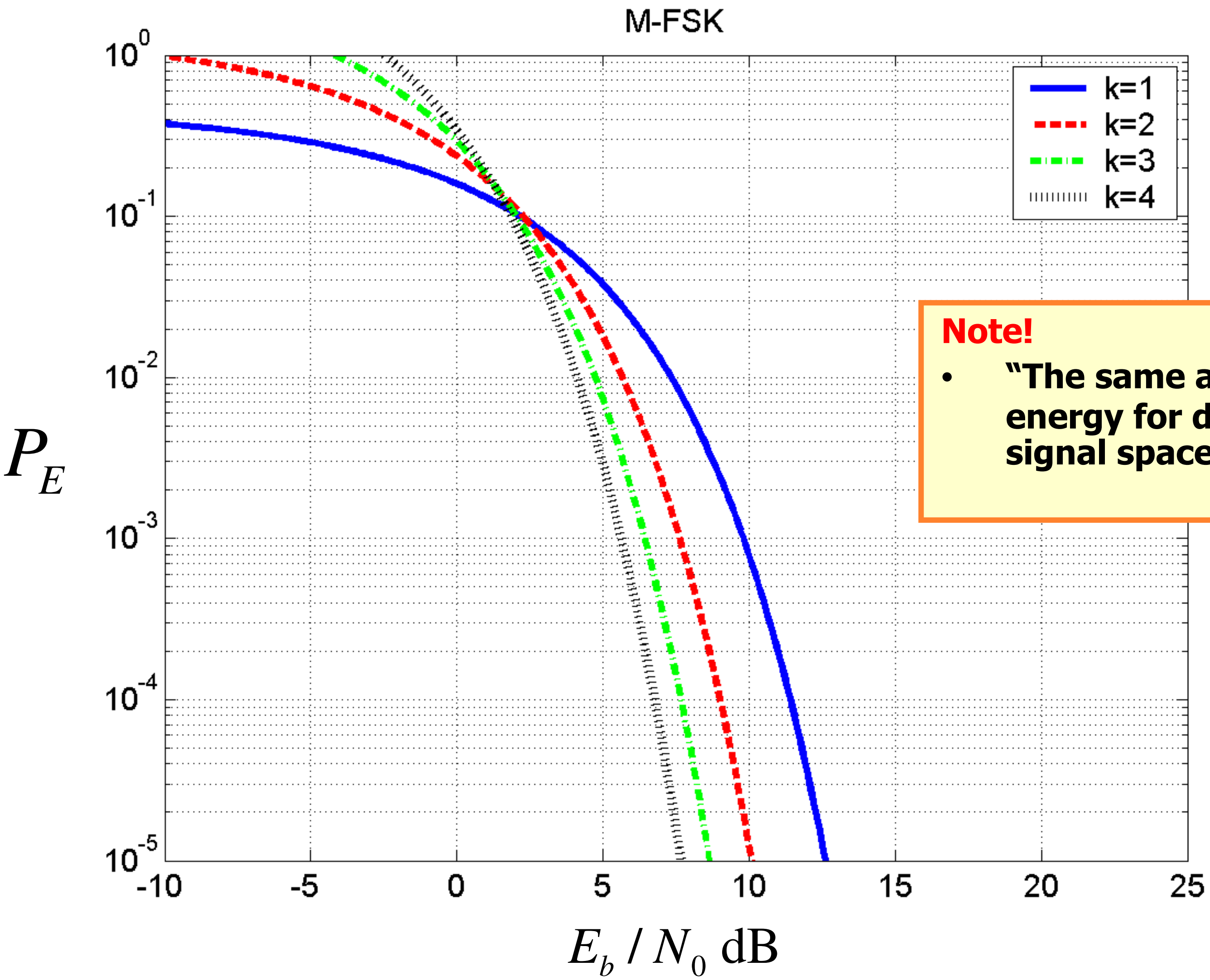
Probability of symbol error for M-PSK



Note!

- **"The same average symbol energy for different sizes of signal space"**

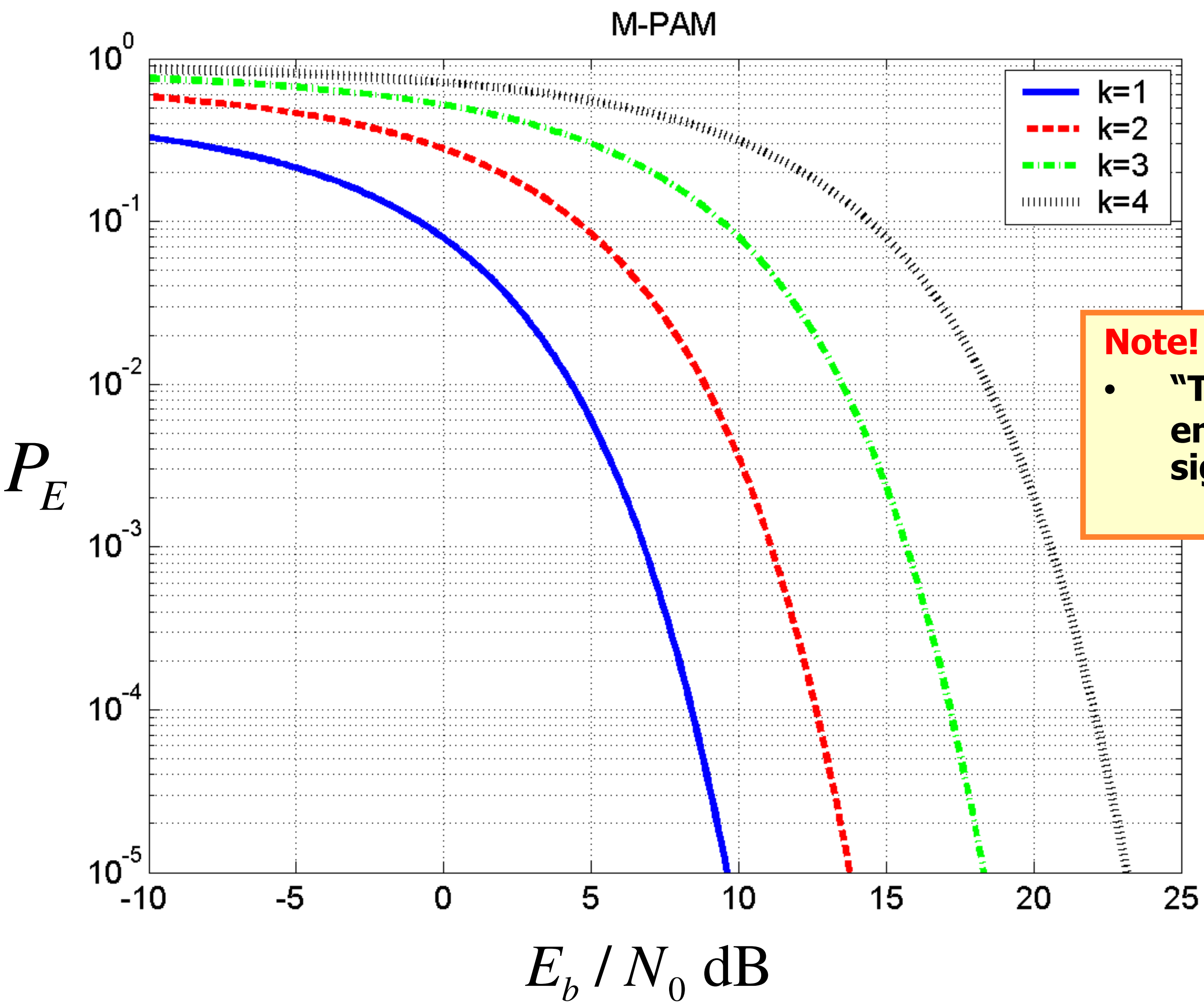
Probability of symbol error for M-FSK



Note!

- **"The same average symbol energy for different sizes of signal space"**

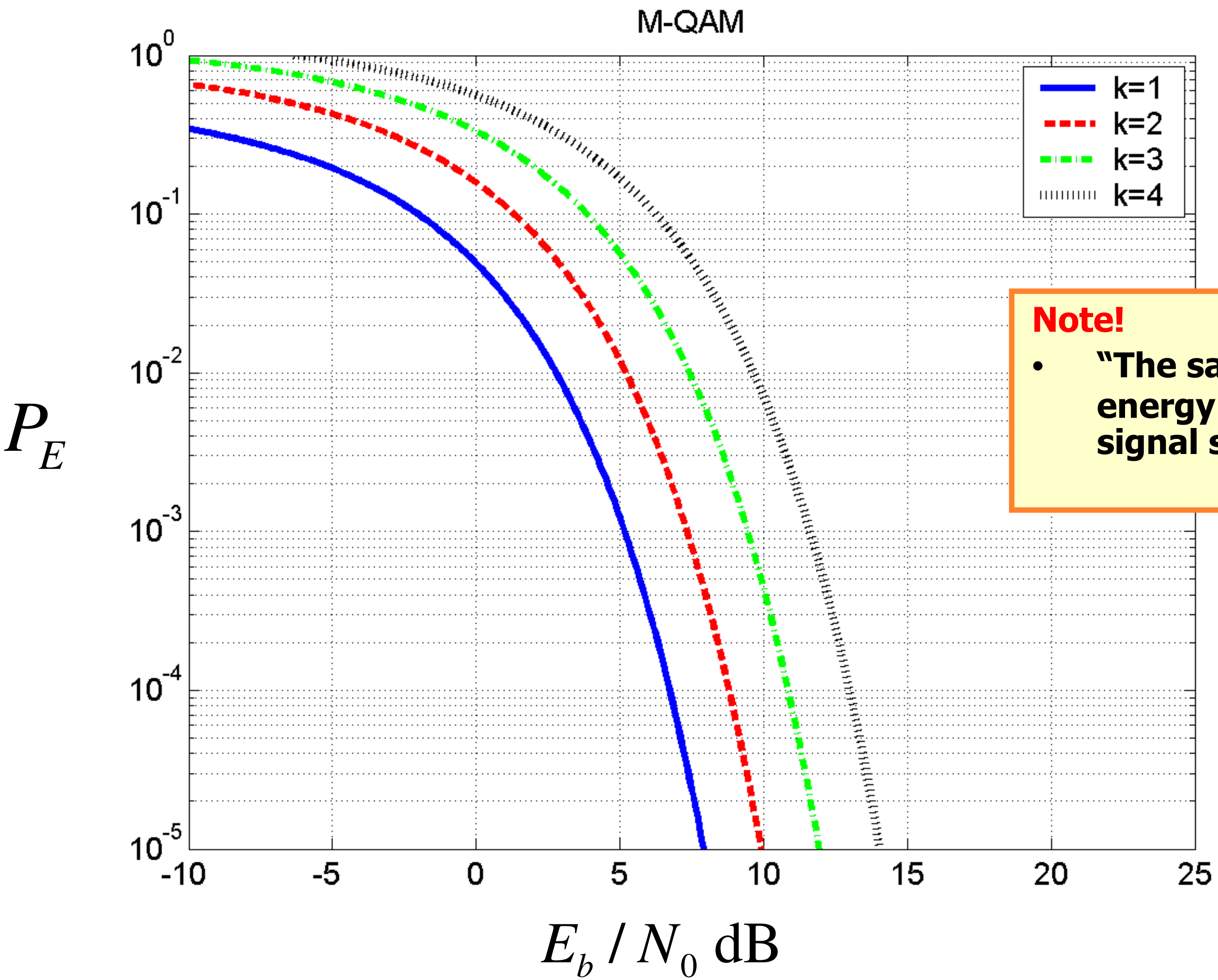
Probability of symbol error for M-PAM



Note!

- **"The same average symbol energy for different sizes of signal space"**

Probability of symbol error for M-QAM



Note!

- **"The same average symbol energy for different sizes of signal space"**

Example of samples of matched filter output for some bandpass modulation schemes