EC 721 Advanced Digital Communications Spring 2008

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Shannon theorem, LP Representation

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Shannon limit

- Channel capacity: The maximum data rate at which the error-free communication over the channel is performed.
- Channel capacity on AWGV channel (Shannon-Hartley capacity theorem):

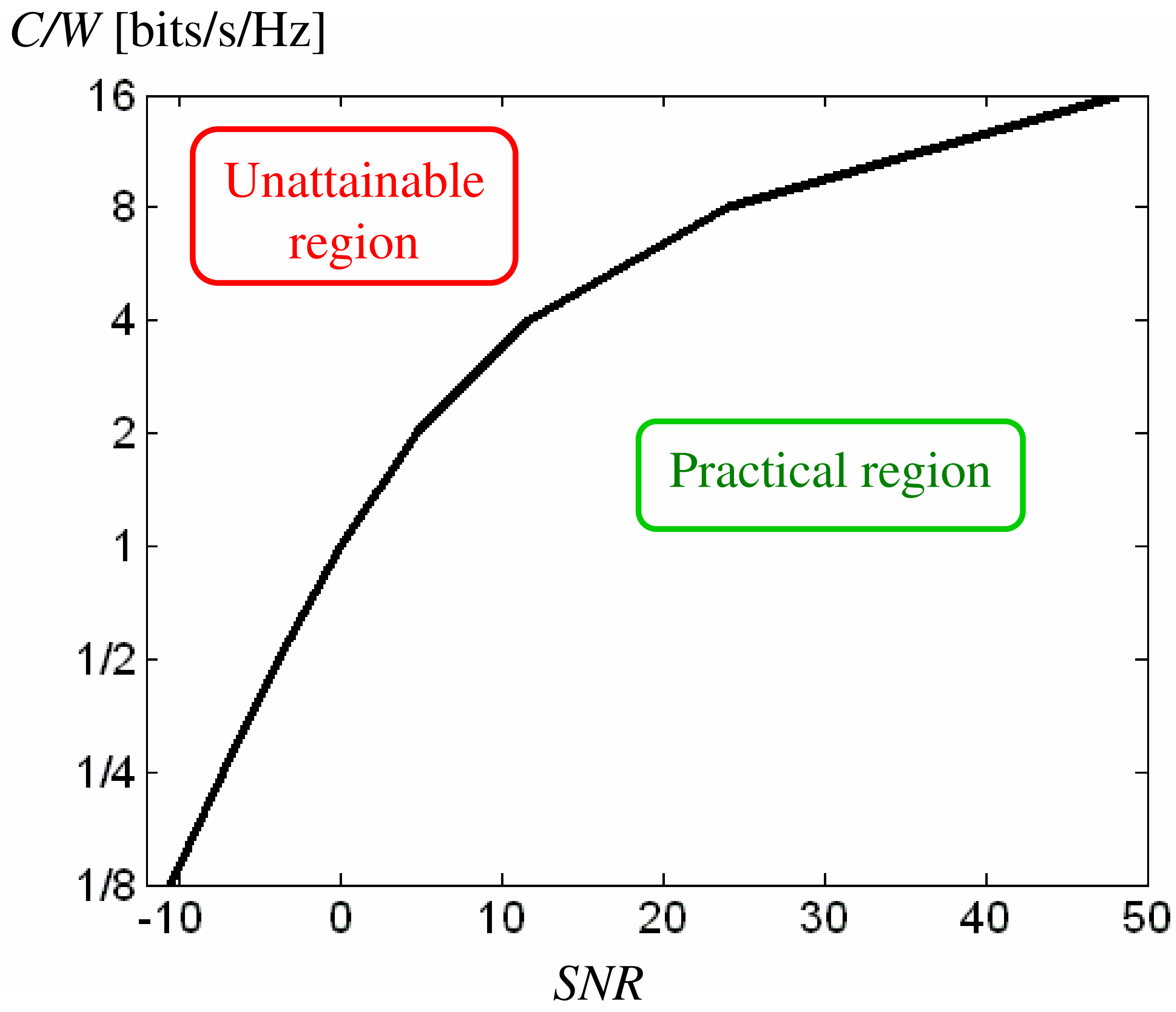
$$C = W \log_2\left(1 + \frac{S}{N}\right) \quad \text{[bits/s]}$$

W [Hz]: Bandwidth

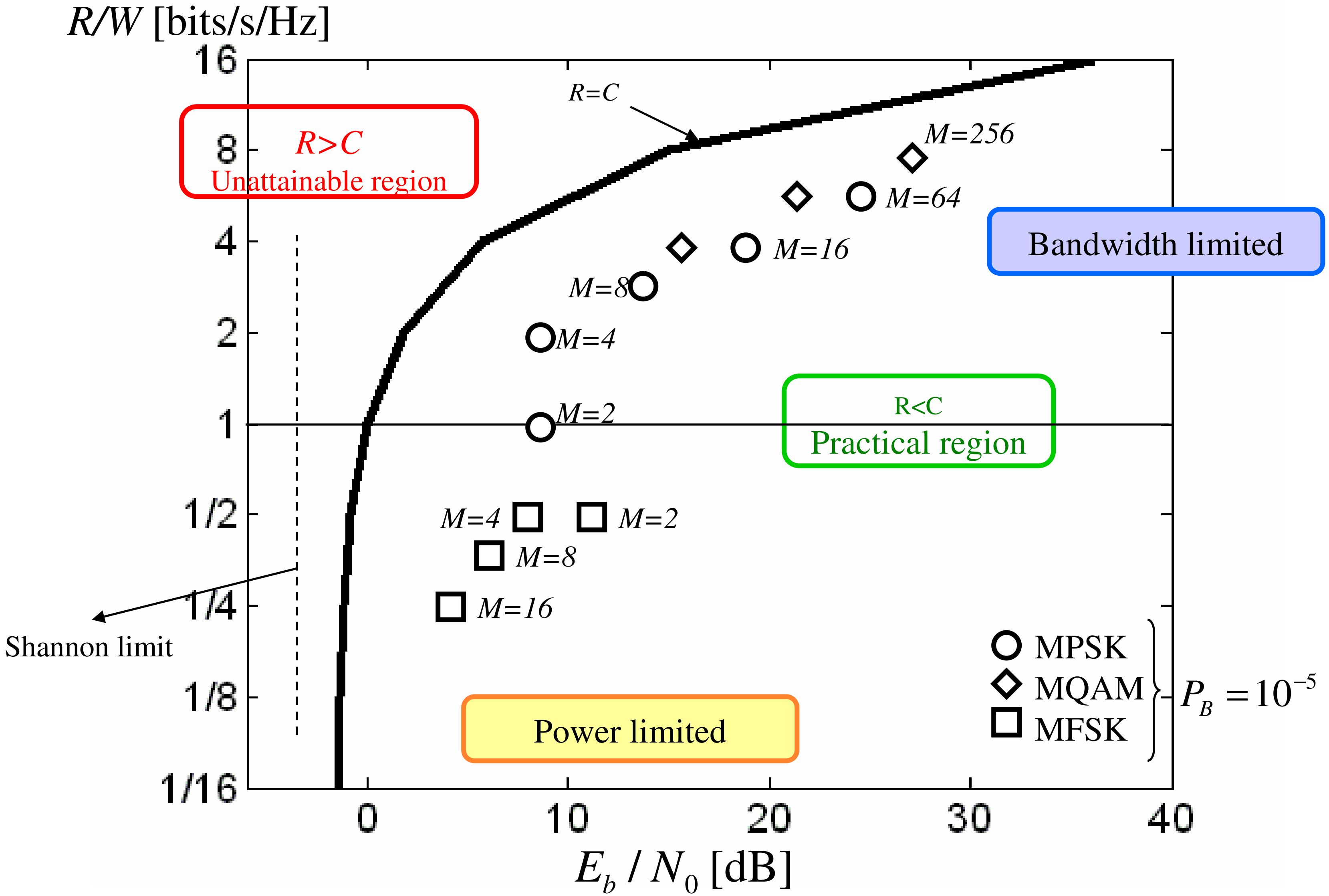
$S = E_b C$ [Watt]: Average received signal power

$N = N_0 W$ [Watt]: Average noise power

Shannon limit …



Bandwidth efficiency plane



Power and bandwidth limited systems

- Two major communication resources:
  - Transmit power and channel bandwidth
- In many communication systems, one of these resources is more precious than the other. Hence, systems can be classified as:
  - **Power-limited systems:**
    - save power at the expense of bandwidth (for example by using coding schemes)
  - **Bandwidth-limited systems:**
    - save bandwidth at the expense of power (for example by using spectrally efficient modulation schemes)

M-ary signaling

- Bandwidth efficiency:

$$\frac{R_b}{W} = \frac{\log_2 M}{WT_s} = \frac{1}{WT_b} \quad \text{[bits/s/Hz]}$$

  - Assuming Nyquist (ideal rectangular) filtering at baseband, the required passband bandwidth is:

$$W = 1/T_s = R_s \quad \text{[Hz]}$$

- M-PSK and M-QAM (bandwidth-limited systems)

$$R_b / W = \log_2 M \quad \text{[bits/s/Hz]}$$

  - Bandwidth efficiency increases as M increases.

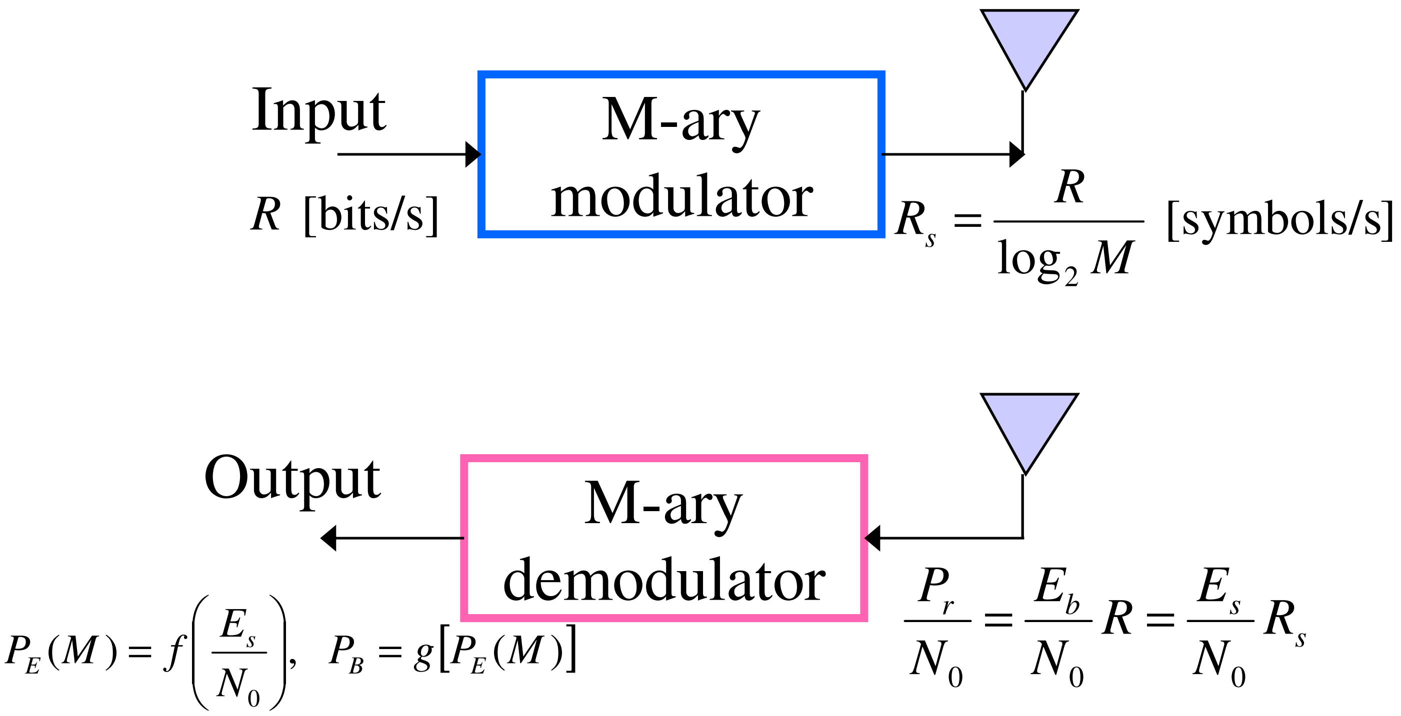
- MFSK (power-limited systems)

$$R_b / W = \log_2 M / M \quad \text{[bits/s/Hz]}$$

  - Bandwidth efficiency decreases as M increases.

Design example of uncoded systems

- Design goals:
  1. The bit error probability at the modulator output must meet the system error requirement.
  2. The transmission bandwidth must not exceed the available channel bandwidth.



Design example of uncoded systems …

- Choose a modulation scheme that meets the following system requirements:

  An AWGN channel with $W_C = 4000$ [Hz]

  $$\frac{P_r}{N_0} = 53 \text{ [dB.Hz]} \qquad R_b = 9600 \text{ [bits/s]} \qquad P_B \le 10^{-5}$$

⇨ $R_b > W_C \Rightarrow \text{Band-limited channel} \Rightarrow \text{MPSK modulation}$

⇨ $M = 8 \Rightarrow R_s = R_b / \log_2 M = 9600/3 = 3200 \text{ [sym/s]} < W_C = 4000 \text{ [Hz]}$

⇨ $\frac{E_s}{N_0} = (\log_2 M)\frac{E_b}{N_0} = (\log_2 M)\frac{P_r}{N_0}\frac{1}{R_b} = 62.67$

⇨ $P_E(M = 8) \approx 2Q\left[\sqrt{2E_s / N_0}\sin(\pi / M)\right] = 2.2\times 10^{-5}$

⇨ $P_B \approx \frac{P_E(M)}{\log_2 M} = 7.3\times 10^{-6} < 10^{-5}$

Design example of uncoded systems …

- Choose a modulation scheme that meets the following system requirements:

$$\text{An AWGN channel with } W_C = 45 \text{ [kHz]}$$

$$\frac{P_r}{N_0} = 48 \text{ [dB.Hz]} \qquad R_b = 9600 \text{ [bits/s]} \qquad P_B \le 10^{-5}$$

$$\Rightarrow \frac{E_b}{N_0} = \frac{P_r}{N_0}\frac{1}{R_b} = 6.61 = 8.2 \text{ [dB]}$$

$$\Rightarrow R_b < W_C \text{ and relatively small } E_b / N_0 \Rightarrow \text{power-limited channel} \Rightarrow \text{MFSK}$$

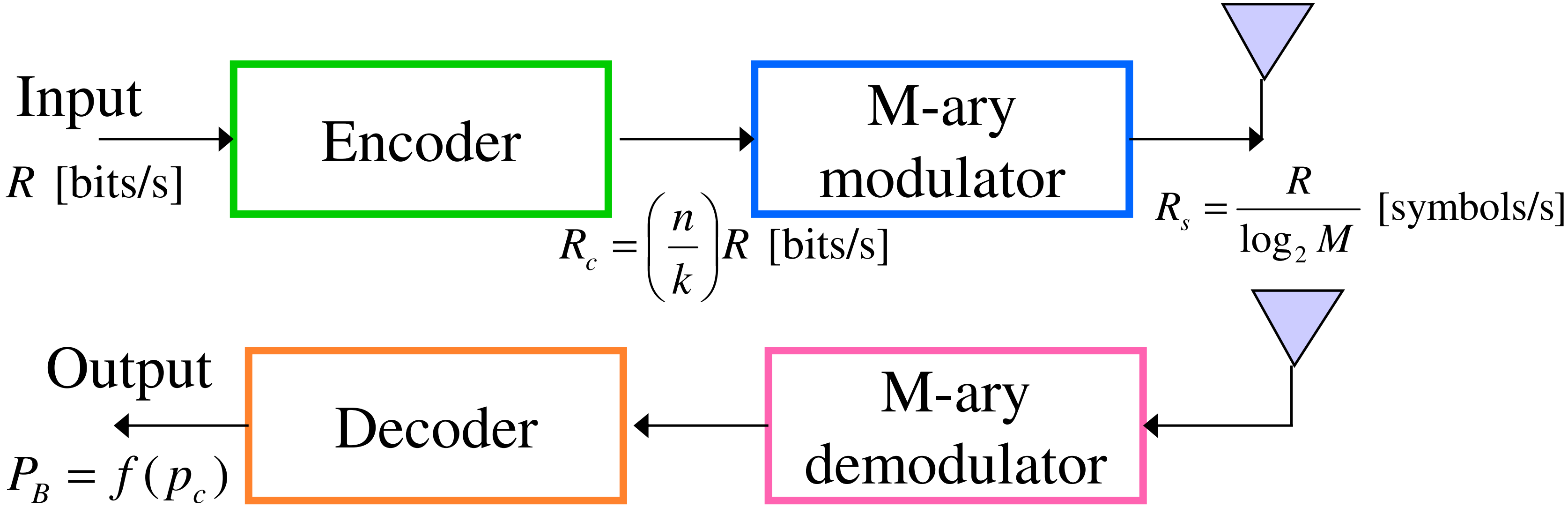
$$\Rightarrow M = 16 \Rightarrow W = MR_s = MR_b / (\log_2 M) = 16 \times 9600 / 4 = \underline{38.4 \text{ [ksym/s]} < W_C = 45 \text{ [kHz]}}$$

$$\Rightarrow \frac{E_s}{N_0} = (\log_2 M)\frac{E_b}{N_0} = (\log_2 M)\frac{P_r}{N_0}\frac{1}{R_b} = 26.44$$

$$\Rightarrow P_E(M = 16) \le \frac{M-1}{2}\exp\left(-\frac{E_s}{2N_0}\right) = 1.4 \times 10^{-5} \Rightarrow P_B \approx \frac{2^{k-1}}{2^k - 1} P_E(M) = \underline{7.3 \times 10^{-6} < 10^{-5}}$$

Design example of coded systems

- Design goals:
  1. The bit error probability at the decoder output must meet the system error requirement.
  2. The rate of the code must not expand the required transmission bandwidth beyond the available channel bandwidth.
  3. The code should be as simple as possible. Generally, the shorter the code, the simpler will be its implementation.



$$P_E(M) = f\left(\frac{E_s}{N_0}\right), \quad p_c = g\left[P_E(M)\right] \qquad \frac{P_r}{N_0} = \frac{E_b}{N_0} R = \frac{E_c}{N_0} R_c = \frac{E_s}{N_0} R_s$$

Design example of coded systems …

- Choose a modulation/coding scheme that meets the following system requirements:

$$\text{An AWGN channel with } W_C = 4000 \text{ [Hz]}$$

$$\frac{P_r}{N_0} = 53 \text{ [dB.Hz]} \qquad R_b = 9600 \text{ [bits/s]} \qquad P_B \leq 10^{-9}$$

⇨ $R_b > W_C \Rightarrow \text{Band-limited channel} \Rightarrow \text{MPSK modulation}$

⇨ $M = 8 \Rightarrow R_s = R_b / \log_2 M = 9600/3 = 3200 < 4000$

⇨ $P_B \approx \dfrac{P_E(M)}{\log_2 M} = 7.3 \times 10^{-6} > 10^{-9} \Rightarrow \text{Not low enough: power-limited system}$

- The requirements are similar to the bandwidth-limited uncoded system, except the target bit error probability is much lower.

Design example of coded systems

- Using 8-PSK, satisfies the bandwidth constraint, but not the bit error probability constraint. Much higher power is required for uncoded 8-PSK.

$$P_B \leq 10^{-9} \Rightarrow \left(\frac{E_b}{N_0} \right)_{uncoded} \geq 16 \ \ \text{dB}$$

- The solution is to use channel coding (block codes or convolutional codes) to save the power at the expense of bandwidth while meeting the target bit error probability.

Design example of coded systems

- For simplicity, we use BCH codes.
- The required coding gain is:

$$G(\text{dB}) = \left(\frac{E_b}{N_0}\right)_{uncoded}(\text{dB}) - \left(\frac{E_c}{N_0}\right)_{coded}(\text{dB}) = 16 - 13.2 = 2.8 \ \text{dB}$$

- The maximum allowable bandwidth expansion due to coding is:

$$R_s = \frac{R}{\log_2 M} = \left(\frac{n}{k}\right)\frac{R_b}{\log_2 M} \le W_C \Rightarrow \left(\frac{n}{k}\right)\frac{9600}{3} \le 4000 \Rightarrow \frac{n}{k} \le 1.25$$

- The current bandwidth of uncoded 8-PSK can be expanded still by 25% to remain below the channel bandwidth.

- Among the BCH codes, we choose the one which provides the required coding gain and bandwidth expansion with minimum amount of redundancy.

Design example of coded systems …

- Bandwidth compatible BCH codes

Coding gain in dB with MPSK

n	k	t	$P_B = 10^{-5}$	$P_B = 10^{-9}$
31	26	1	1.8	2.0
63	57	1	1.8	2.2
63	51	2	2.6	3.2
127	120	1	1.7	2.2
127	113	2	2.6	3.4
127	106	3	3.1	4.0

Design example of coded systems …

- Examine that combination of 8-PSK and (63,51) BCH codes meets the requirements:

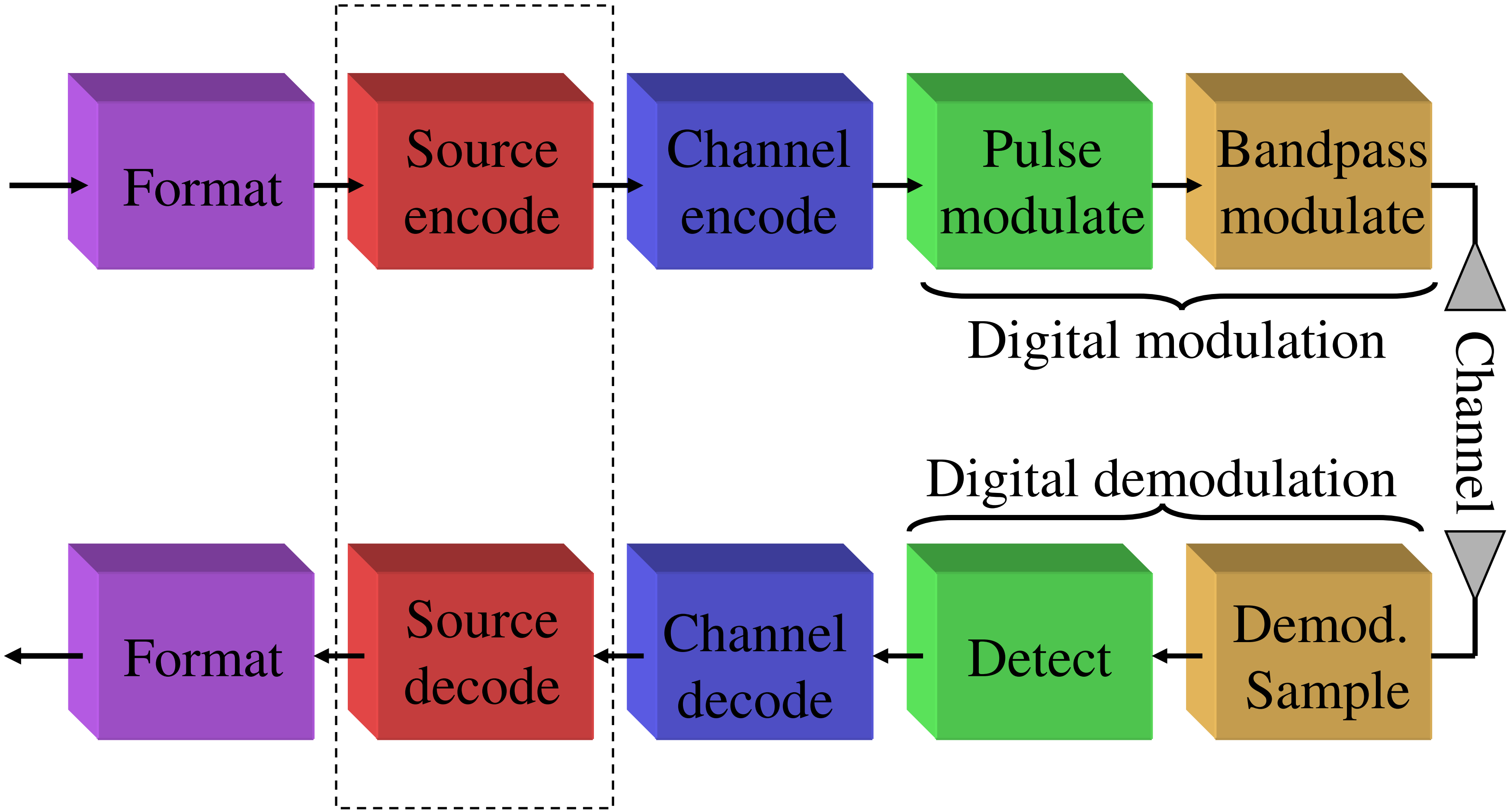
$$R_s = \left(\frac{n}{k}\right)\frac{R_b}{\log_2 M} = \left(\frac{63}{51}\right)\frac{9600}{3} = 3953 \text{ [sym/s]} < W_C = 4000 \text{ [Hz]}$$

$$\frac{E_s}{N_0} = \frac{P_r}{N_0}\frac{1}{R_s} = 50.47 \quad \Rightarrow \quad P_E(M) \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right] = 1.2\times 10^{-4}$$

$$p_c \approx \frac{P_E(M)}{\log_2 M} = \frac{1.2\times 10^{-4}}{3} = 4\times 10^{-5}$$

$$P_B \approx \frac{1}{n}\sum_{j=t+1}^{n} j\binom{n}{j} p_c^j (1-p_c)^{n-j} \approx 1.2\times 10^{-10} < 10^{-9}$$

Block diagram of a DCS

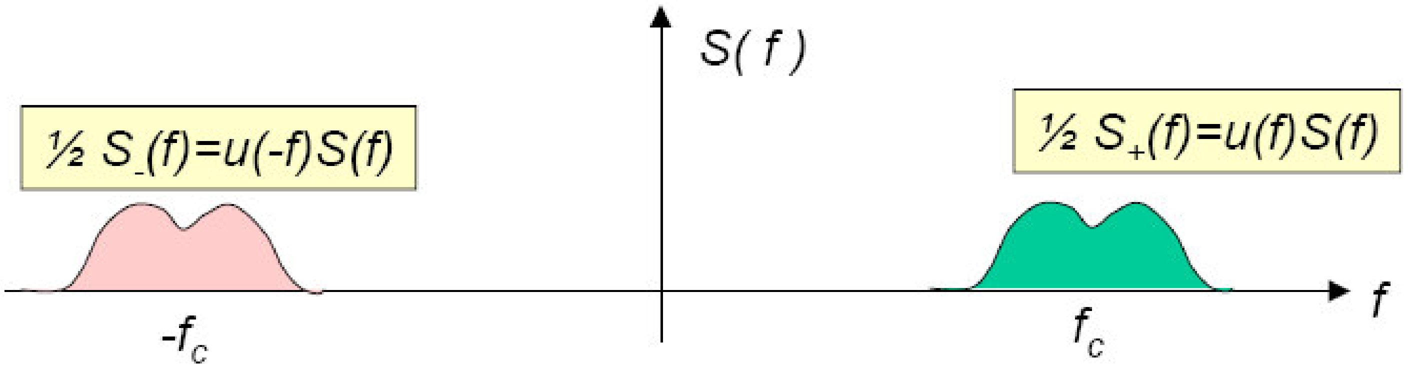


Representation of Band-Pass Signal

- The transmitted signal is usually a real valued band-pass signal and let's call *s(t)*

- Mathematical model of a real-valued narrowband band-pass signal is

$$S(f) \neq 0 \quad \text{for} \quad f_c - f_B \leq |f| \leq f_c + f_B \quad \text{and} \quad f_c \gg f_B$$



S(f) is Fourier transform of *s(t)*. *u(f)* is the unit step function in frequency domain

Representation of Band-Pass Signal

Goal is to develop a mathematical representation in time domain of $S_+(f)$ and $S_-(f)$

The time domain representation of $S_+(f)$ is $s_+(t)$, which is called pre-envelope of s(t)

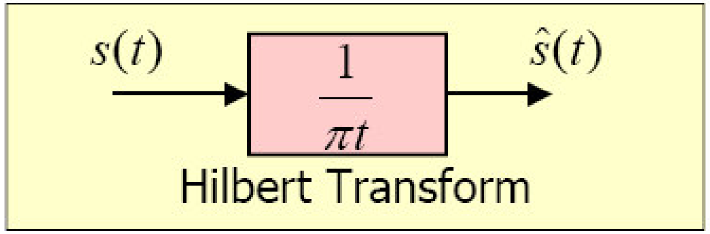
$$\begin{aligned}
s_+(t) &= \int_{-\infty}^{\infty} S_+(f) e^{j2\pi ft} df \\
&= \int_{-\infty}^{\infty} [2u(f)S(f)] e^{j2\pi ft} df \\
&= F^{-1}[2u(f)] * F^{-1}[S(f)] \\
&= \left[\delta(t) + \frac{j}{\pi t} \right] * s(t) \\
&= s(t) + \frac{j}{\pi t} * s(t) \\
&= s(t) + j\hat{s}(t)
\end{aligned}$$

where

$$\begin{aligned}
\hat{s}(t) &= \frac{1}{\pi t} * s(t) \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} d\tau
\end{aligned}$$

Representation of Band-Pass Signal

$\hat{s}(t)$ may be considered the output of the filter such as



- The frequency response of the filter is

$$H(f)=\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t}e^{-j2\pi ft}dt$$

$$H(f)=-j\,\text{sgn}(f)=\begin{cases}-j & f>0\\ 0 & f=0\\ j & f<0\end{cases}$$

$$|H(f)|=1 \text{ for } f\neq 0$$

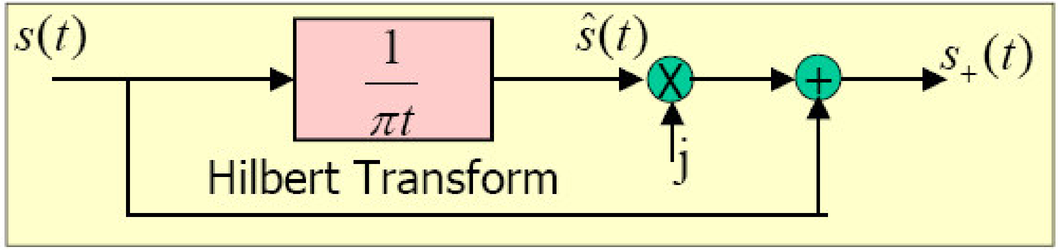
$$\Theta(f)=\begin{cases}-\frac{\pi}{2} & \text{for } f>0\\ \frac{\pi}{2} & \text{for } f<0\end{cases}$$

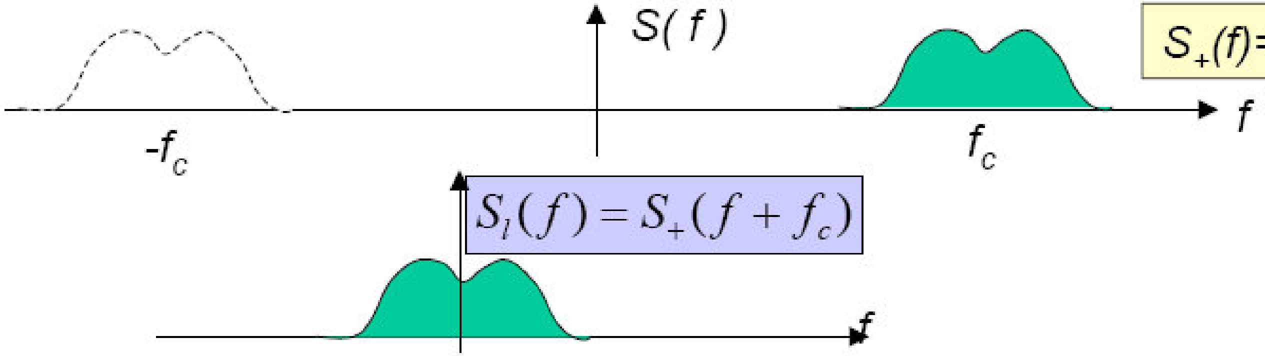
Fourier Transform of Hilbert Transform

$$F\left[\frac{j}{\pi t}\right]=\text{sgn}(f)$$

$$F\left[\frac{1}{\pi t}\right]=-j\,\text{sgn}(f)$$

Representation of Band-Pass Signal





$S_+(f) = 2u(f)S(f)$

$S_l(f) = S_+(f + f_c)$

The low-pass representation of $S_+(f)$ is $S_l(f) = S_+(f + f_c)$ in time domain

$s_l(t) = s_+(t)e^{-j2\pi f_c t} = \left[s(t) + j\hat{s}(t)\right]e^{-j2\pi f_c t}$

$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$

$\hat{s}(t) = x(t)\sin(2\pi f_c t) + y(t)\cos(2\pi f_c t)$

In complex form

$s_l(t) = x(t) + jy(t)$

$x(t)$ and $y(t)$:quadrature components of $s_l(t)$

Representation of Band-Pass Signal

- Another representation of the signal

$$s(t) = \mathrm{Re}\left\{\left[x(t) + jy(t)\right]e^{j2\pi f_c t}\right\}$$
$$= \mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right]$$

Or we can represent

$$s_l(t) = a(t)e^{j\theta(t)}$$

$$a(t) = \sqrt{x^2(t) + y^2(t)}$$

$$\theta(t) = \tan^{-1}\frac{y(t)}{x(t)}$$

$$s(t) = \mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right]$$
$$= \mathrm{Re}\left[a(t)e^{j\theta(t)}e^{j2\pi f_c t}\right]$$
$$= a(t)\cos\left[2\pi f_c t + \theta(t)\right]$$

a(t) is called envelope of *s(t)* and *θ(t)* is called the phase of *s(t)*

Representation of Band-Pass Signal

- The energy in the signal *s(t)* is defined as

$$\varepsilon = \int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} \left\{\mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right]\right\}^2 dt$$

- Using representation of *s(t)* in cosine form

$$\varepsilon = \int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} \left\{a(t)\cos\left[2\pi f_c t + \theta(t)\right]\right\}^2 dt$$

Then

$$\varepsilon = \frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty} \left\{a^2(t)\cos\left[4\pi f_c t + 2\theta(t)\right]\right\} dt$$

a(t) is the envelope and varies slowly relative to cosine function

$$\varepsilon = \frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \left|s_l^2(t)\right| dt$$

Representation of Linear Band-Pass signal

- A linear filter or system can be represented either *h(t)* or of *H(f).*
- Since *h(t)* is real

$$H(f) = H^*(-f)$$

Let's define $H_l(f - f_c)$

And $H_l^*(-f - f_c)$

$$H_l(f - f_c) = \begin{cases} H(f) & f > 0 \\ 0 & f < 0 \end{cases}$$

$$H_l^*(-f - f_c) = \begin{cases} 0 & f > 0 \\ H^*(-f) & f < 0 \end{cases}$$

Then, we have

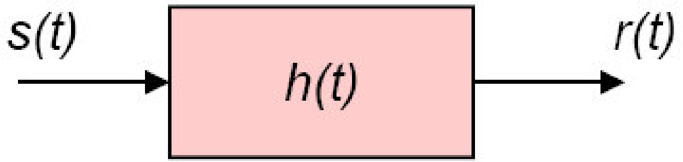
$$H(f) = H_l(f - f_c) + H_l^*(-f - f_c)$$

The Inverse transform of *H(f)*

$$\begin{aligned} h(t) &= h_l(t)e^{j2\pi f_c t} + h_l^*(t)e^{-j2\pi f_c t} \\ &= 2\,\mathrm{Re}\left[h_l(t)e^{j2\pi f_c t}\right] \end{aligned}$$

$h_l(t)$, inverse transform of $H_l(f)$, is impulse response of low-pass system and is comlex

Response of Band-pass System to a Band-pass signal



Let's have

- *s(t)* narrowband band-pass signal and is the equivalent low-pass signal $s_l(t)$.
- Band-pass filter (system) the impulse response *h(t)* and its equivalent low-pass impulse response $h_l(t)$

The output of the band-pass filter is *r(t)* also band-pass signal

$$r(t) = \mathrm{Re}\left[r_l(t) e^{j2\pi f_c t} \right]$$

Response of Band-pass System to a Band-pass signal

r(t) can be given as

$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

In frequency domain $R(f) = S(f)H(f)$

$$R(f) = \frac{1}{2}\left[S_l(f-f_c) + S_l^*(-f-f_c)\right]\left[H_l(f-f_c) + H_l^*(-f-f_c)\right]$$

For narrow band signal *s(t)* and narrow band system *h(t)*

$$S_l(f-f_c)H_l^*(-f-f_c) = 0 \qquad S_l^*(-f-f_c)H_l(f-f_c) = 0$$

$$R(f) = \frac{1}{2}\left[S_l(f-f_c)H_l(f-f_c) + S_l^*(-f-f_c)H_l^*(-f-f_c)\right]$$

$$R(f) = \frac{1}{2}\left[R_l(f-f_c) + R_l^*(-f-f_c)\right] \qquad R_l(f) = S_l(f)H_l(f)$$

$$r_l(t) = \int_{-\infty}^{\infty} s_l(\tau)h_l(t-\tau)d\tau$$

Representation Digitally modulated signal

- Modulator maps the digital information into analog waveform that match the characteristic of the channel
- It takes blocks of $k=log_2M$ binary digits at a time from the information sequence $\{a_n\}$ and represents one of the deterministic value $M=2^k$.
- The modulated waveform is $\{s_m(t),\ m=1,2,\ldots,M\}$ for transmission over the channel

Memoryless Modulation: The mapping from sequence $\{a_n\}$ to the waveforms $\{s_m(t)\}$ is performed without any constraint on previously transmitted waveform.

Memory Modulation: The mapping from sequence $\{a_n\}$ to the waveforms $\{s_m(t)\}$ is performed depend on the one or more previously transmitted waveform.

Memoryless Modulation Medhods

Pulse-Amplitude Modulation (PAM) signal

PAM is also called Amplitude-shift Keying ASK)

PAM signal waveform representation

$$
\begin{aligned}
s_m(t) &= \mathrm{Re}\left\{A_m g(t) e^{j2\pi f_c t}\right\} \\
&= A_m g(t)\cos 2\pi f_c t, \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T
\end{aligned}
$$

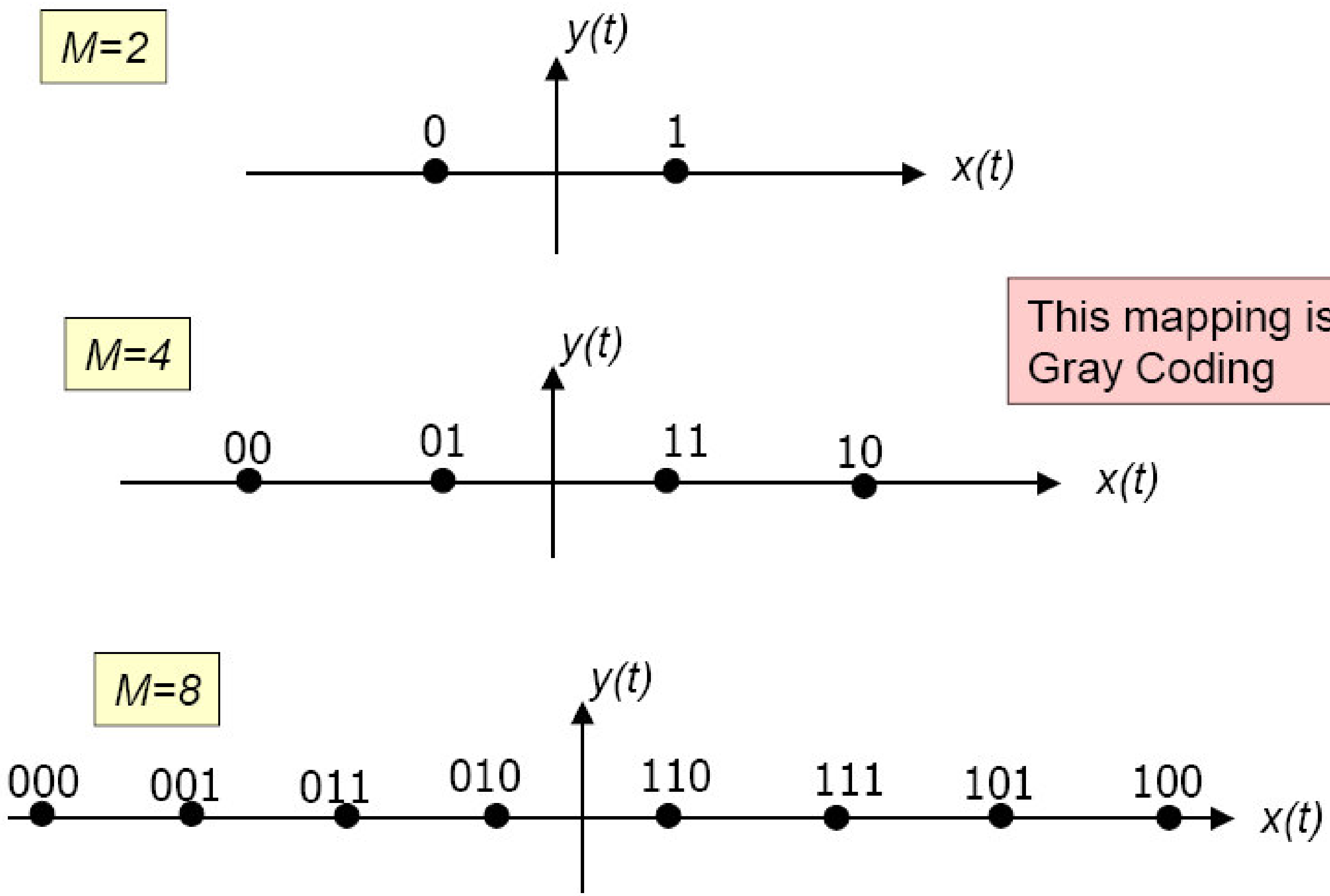
where $\{A_m,\ m = 1, 2, \ldots, M\}$ denotes the set of M possible amplitude and g(t) is signal pulse shape

A_m takes the discrete values

$$A_m = (2m - 1 - M)d, \quad m = 1, 2, \ldots, M$$

where $2d$ is the distance between adjacent signal amplitudes

Pulse-Amplitude Modulation(PAM) signal



This mapping is called Gray Coding

Pulse-Amplitude Modulation(PAM) signal

If R show the # bit per second {R [bit/s]}. The time interval will be

$$T_b = \frac{1}{R}$$

is called bit interval

The symbol rate is R/k , then the symbol interval will be

$$T = \frac{k}{R} = kT_b$$

The M PAM signal energies

$$\begin{aligned}\varepsilon_m &= \int_0^T s_m^2 dt \\ &= \frac{1}{2} A_m^2 \int_0^T g^2(t) dt \\ &= \frac{1}{2} A_m^2 \varepsilon_g\end{aligned}$$

Where ε_g denotes the energy of pulse g(t)

Pulse-Amplitude Modulation(PAM) signal

- Let's define $s_m(t)$ with unit-energy signal waveform

$$s(t) = s_m f(t)$$

where

$$f(t) = \sqrt{\frac{2}{\varepsilon_g}} g(t) \cos 2\pi f_c t$$

Unit-energy waveform

$$s_m = A_m \sqrt{\frac{1}{2}\varepsilon_g}$$

$$m = 1, 2, \ldots, M$$

- Euclidean distance between any pair of signal points is

$$d_{mn}^{(e)} = \sqrt{(s_m - s_n)^2}$$

$$= \sqrt{\frac{1}{2}\varepsilon_g} \, |A_m - A_n| \quad = d\sqrt{2\varepsilon_g} \, |m - n|$$

The minimum distance

$$d_{\min}^{(e)} = d\sqrt{2\varepsilon_g}$$

Digital Phase-Modulated signals

- Digital PM is also called Phase-shift keying (PSK)

- The M signal waveforms can be represented in PM

$$s_m(t) = \mathrm{Re}\left\{ g(t) e^{j2\pi f_c t} e^{j\frac{2\pi(m-1)}{M}} \right\}, \quad m = 1, 2, \ldots, M$$

Or

$$\begin{aligned} s_m(t) &= g(t)\cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right], \quad m = 1, 2, \ldots, M \\ &= g(t)\cos\left[\frac{2\pi(m-1)}{M}\right]\cos(2\pi f_c t) - g(t)\sin\left[\frac{2\pi(m-1)}{M}\right]\sin(2\pi f_c t) \end{aligned}$$

Where g(t) is the signal pulse shape and $\theta_m = 2\pi\,(m-1)/M$, m=1,2,..,M are the M possible phases of the carrier.

Digital Phase-Modulated signals

Digital PM signal has same energy and

$$\varepsilon = \int_0^T s_m^2 dt$$
$$= \frac{1}{2}\int_0^T g^2(t)dt \quad = \frac{1}{2}\varepsilon_g$$

$s_m(t)$ can be expressed as a linear combination of two orthogonal signal

$$s(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

where

$$\mathrm{f}(t) = [f_1(t) \quad f_2(t)]^T$$

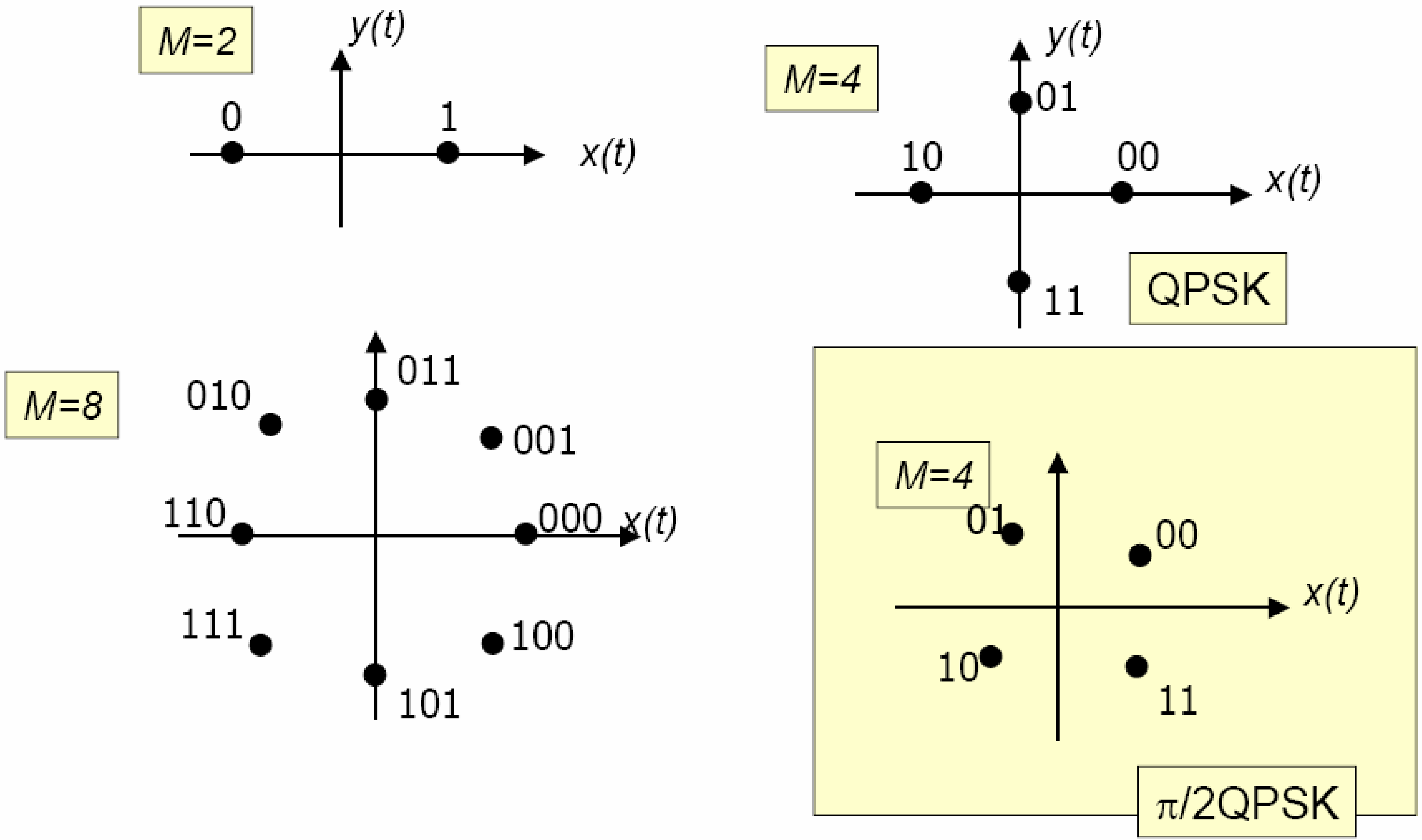
$$\mathrm{f}(t) = \left[\sqrt{\frac{2}{\varepsilon_g}} g(t)\cos 2\pi f_c t \quad -\sqrt{\frac{2}{\varepsilon_g}} g(t)\sin 2\pi f_c t\right]^T$$

$$\mathrm{s}_m = [s_{m1} \quad s_{m2}]$$

$$\mathrm{s}_m = \left[\sqrt{\frac{\varepsilon_g}{2}}\cos\left[\frac{2\pi(m-1)}{M}\right] \quad \sqrt{\frac{\varepsilon_g}{2}}\sin\left[\frac{2\pi(m-1)}{M}\right]\right]$$

Digital Phase-Modulated signals

Signal Space Diagram of PSK



Digital Phase-Modulated signals

The Euclidean distance between two signal points are

$$d_{mn}^{(e)} = \| s_m - s_n \|$$
$$= \left\{ \varepsilon_g \left[1 - \cos \frac{2\pi(m-1)}{M} \right] \right\}^{1/2}$$

The minimum distance

$$d_{min}^{(e)} = \left\{ \varepsilon_g \left[1 - \cos \frac{2\pi}{M} \right] \right\}^{1/2}$$

Quadrature Amplitude Modulation

- The signal waveform is

$$
\begin{aligned}
s_m(t) &= \mathrm{Re}\left\{[A_{mc} + jA_{ms}]g(t)e^{j2\pi f_c t}\right\} \\
&= A_{mc}g(t)\cos 2\pi f_c t - A_{ms}g(t)\sin 2\pi f_c t, \quad m = 1,2,\ldots,M, \quad 0 \le t \le T
\end{aligned}
$$

- Or

$$
\begin{aligned}
s_m(t) &= \mathrm{Re}\left\{[V_m e^{j\theta_m}]g(t)e^{j2\pi f_c t}\right\} \\
&= V_m g(t)\cos(2\pi f_c t + \theta_m), \quad m = 1,2,\ldots,M, \quad 0 \le t \le T
\end{aligned}
$$

- where

$$V_m = \sqrt{A_{mc}^2 + A_{ms}^2}$$

$$\theta_m = \tan^{-1}\frac{A_{ms}}{A_{mc}}$$

Quadrature Amplitude Modulation

$s_m(t)$ can be expressed as a linear combination of two orthogonal signal

$$s(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

where $\mathrm{f}(t) = [f_1(t) \quad f_2(t)]^T$

$$\mathrm{f}(t) = \left[\sqrt{\frac{2}{\varepsilon_g}} g(t) \cos 2\pi f_c t \quad -\sqrt{\frac{2}{\varepsilon_g}} g(t) \sin 2\pi f_c t \right]^T$$

$$\mathrm{s}_m = [s_{m1} \quad s_{m2}]$$

$$\mathrm{s}_m = \left[A_{mc}\sqrt{\frac{\varepsilon_g}{2}} \quad A_{ms}\sqrt{\frac{\varepsilon_g}{2}} \right]$$

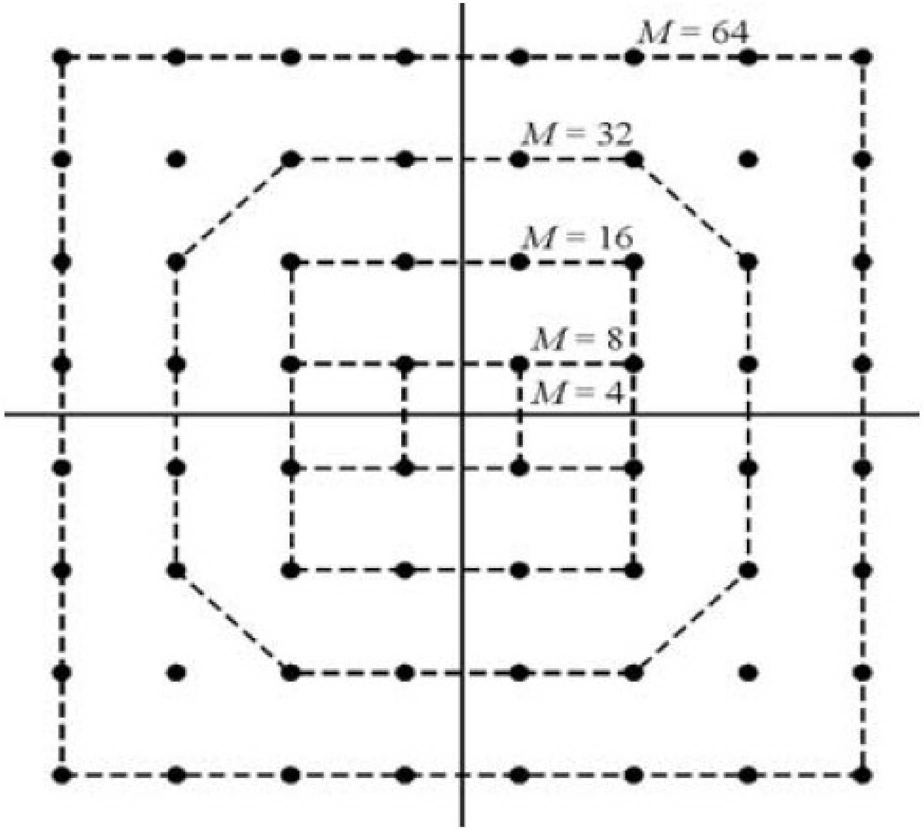
The Euclidean distance between two signal points are

$$d_{mn}^{(e)} = \| s_m - s_n \| = \left\{ \frac{1}{2} \varepsilon_g [(A_{mc} + A_{nc})^2 + (A_{ms} + A_{ns})^2] \right\}^{1/2}$$

The minimum distance

$$d_{min}^{(e)} = d\sqrt{2\varepsilon_g}$$

Quadrature Amplitude Modulation



Signal space diagram of QAM