EC 721 Advanced Digital Communications
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Modulation with memory

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Continuous Phase FSK (CPFSK)

To represent CPFSK, let's use PAM signal for each k-bits block

$$d(t) = \sum_n I_n g(t - nT)$$

is called delta function

Where I_n represent the amplitude values ±1, ±3,…, ±(M-1) and each of them maps k-bit blocks of information sequence

g(t) is rectangular pulse, amplitude 1/2T and duration T second.

d(t) signal is used to frequency modulate the carrier and the carrier -modulated signal is

$$s(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[2\pi f_c t + 4\pi T f_d \int_{-\infty}^{t} d(\tau)\, d\tau\right]$$

f_d: peak frequency deviation

The function $\int_{-\infty}^{t} d(\tau)\, d\tau$ Not consists of jump, this makes the result phase shift with memory

Continuous Phase FSK (CPFSK)

For simplicity

Let g(t) be a rectangular pulse of amplitude 1/2T at [0.T]

$$s(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[2\pi f_c t + \phi(t;\mathbf{I})\right]$$

where

$$\begin{aligned}
\phi(t;\mathbf{I}) &= 4\pi T f_d \int_{-\infty}^{t} d(\tau)\, d\tau \\
&= 4\pi T f_d \int_{-\infty}^{t} \left[\sum_n I_n g(\tau - nT) \right] d\tau \\
&= 4\pi T f_d \left(\sum_{k=-\infty}^{n-1} I_k \frac{T}{2T} + I_n \frac{(t-nT)}{2T} \right), \quad t \in [nT,(n+1)T] \\
&= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t-nT) I_n
\end{aligned}$$

Continuous Phase FSK (CPFSK)

The phase of carrier in *nT ≤ t ≤ (n+1)T is*

$$\phi(t;\mathbf{I}) = 2\pi T f_d \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n$$
$$= \theta_n + 2\pi h I_n q(t - nT)$$

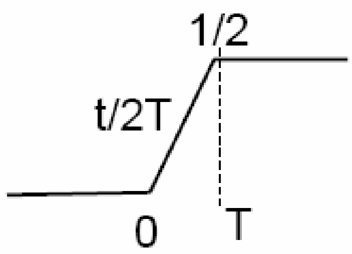
Special case of CPM explained in next slide

where

$h = 2Tf_d$ Modulation index

$\theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k$ Represents the accumulator (memory) up to time $(n-1)T$

$$q(t) = \begin{cases} 0 & (t < 0) \\ t/2T & (0 \le t \le T) \\ 1/2 & (t > 0) \end{cases}$$



Continues-phase Modulation (CPM)

The carrier phase of continuous-phase modulated signal is

$$\phi(t;\mathbf{I}) = 2\pi \sum_{k=-\infty}^{n} I_k h_k q(t-kT) \qquad nT \le t \le (n+1)T$$

where

$\{I_k\}$ $\pm 1, \pm 3, \ldots, \pm(M-1)$ sequence of M-ary information symbols
$\{h_k\}$ is a sequence of modulation index
q(t) is normalized waveform shape.

When h_k is not fixed, the CPM signal is called multi-h

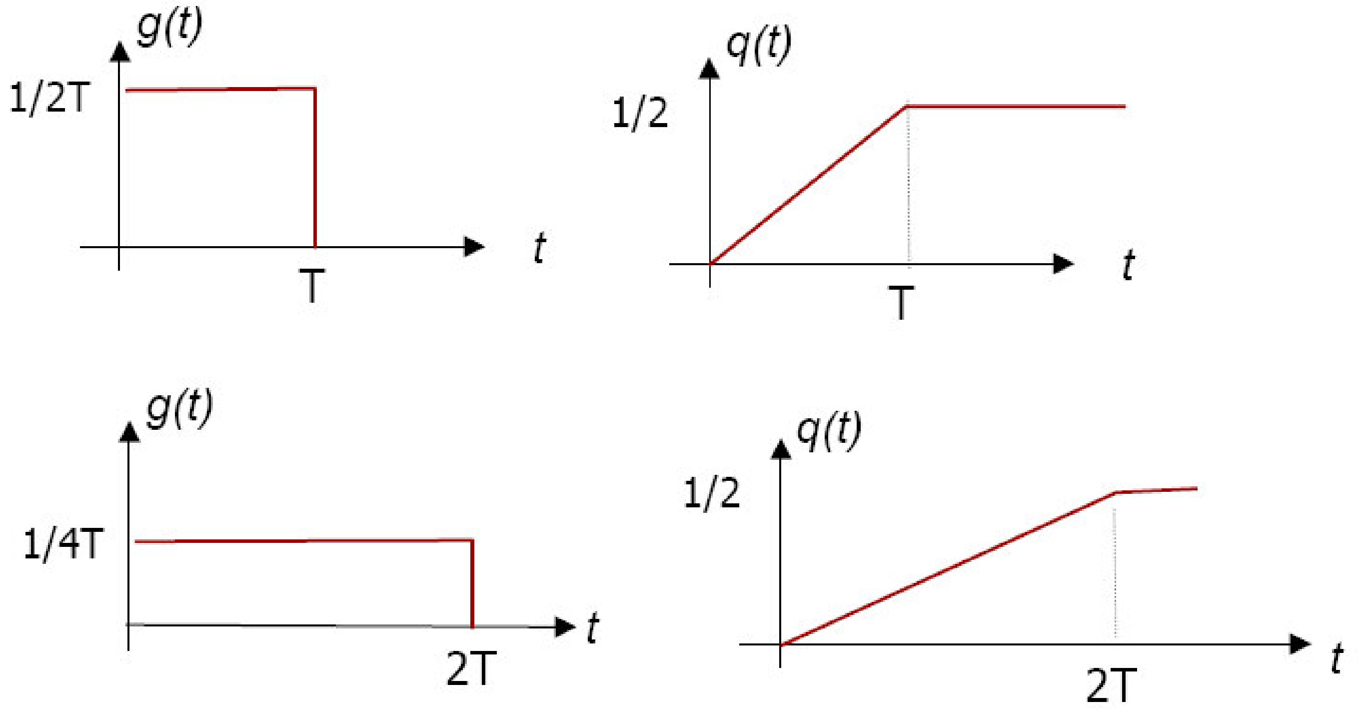
The normalized waveform q(t) can be represented as

$$q(t) = \int_0^t g(\tau)\, d\tau$$

- CPM signal is called full response CPM, if g(t)=0 for t>T
- CPM signal is called partial response CPM, if g(t)≠0 for some t>T

Continues-phase Modulation (CPM)

Example of full response CPM; Some shapes of g(t) and q(t)

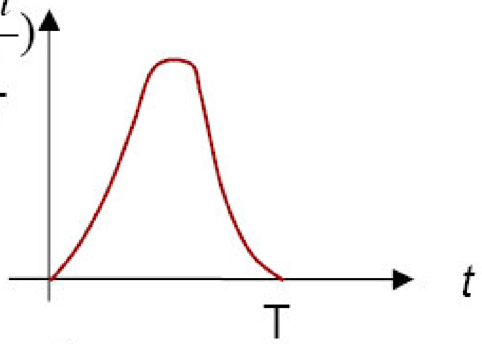


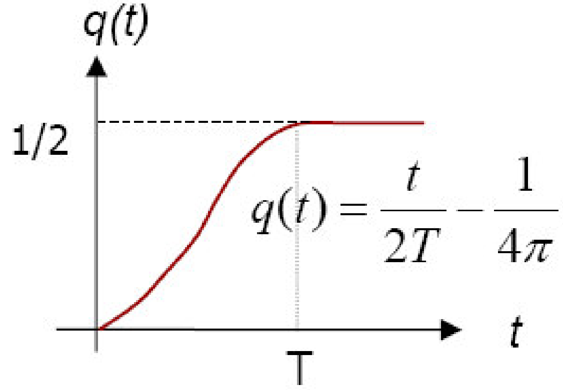
$$g(t) = \begin{cases} \dfrac{1}{2LT} & 0 \leq 1 \leq LT \\ 0 & (o.w) \end{cases}$$

Continues-phase Modulation (CPM)

Example of partial response CPM;Some shapes of g(t) and q(t)

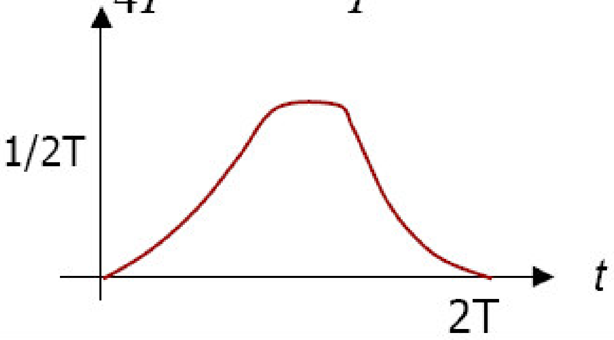
$$g(t)=\frac{1}{2T}\left(1-\cos\frac{2\pi t}{T}\right)$$

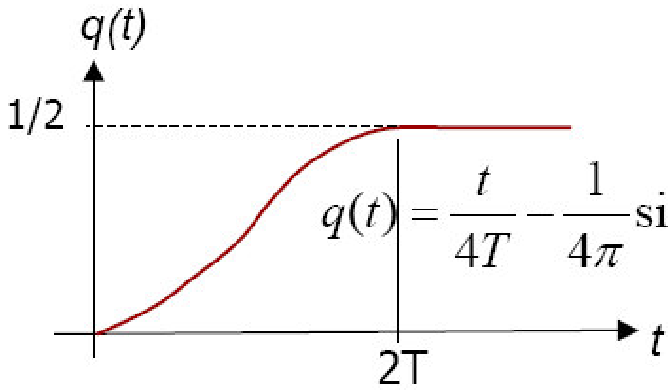




$$q(t)=\frac{t}{2T}-\frac{1}{4\pi}\sin\frac{2\pi t}{T}$$

$$g(t)=\frac{1}{4T}\left(1-\cos\frac{\pi t}{T}\right)$$





$$q(t)=\frac{t}{4T}-\frac{1}{4\pi}\sin\frac{\pi t}{T}$$

$$g(t)=\begin{cases}\frac{1}{2LT}\left(1-\cos\frac{2\pi t}{LT}\right) & 0\le 1\le LT\\ 0 & (o.w)\end{cases}$$

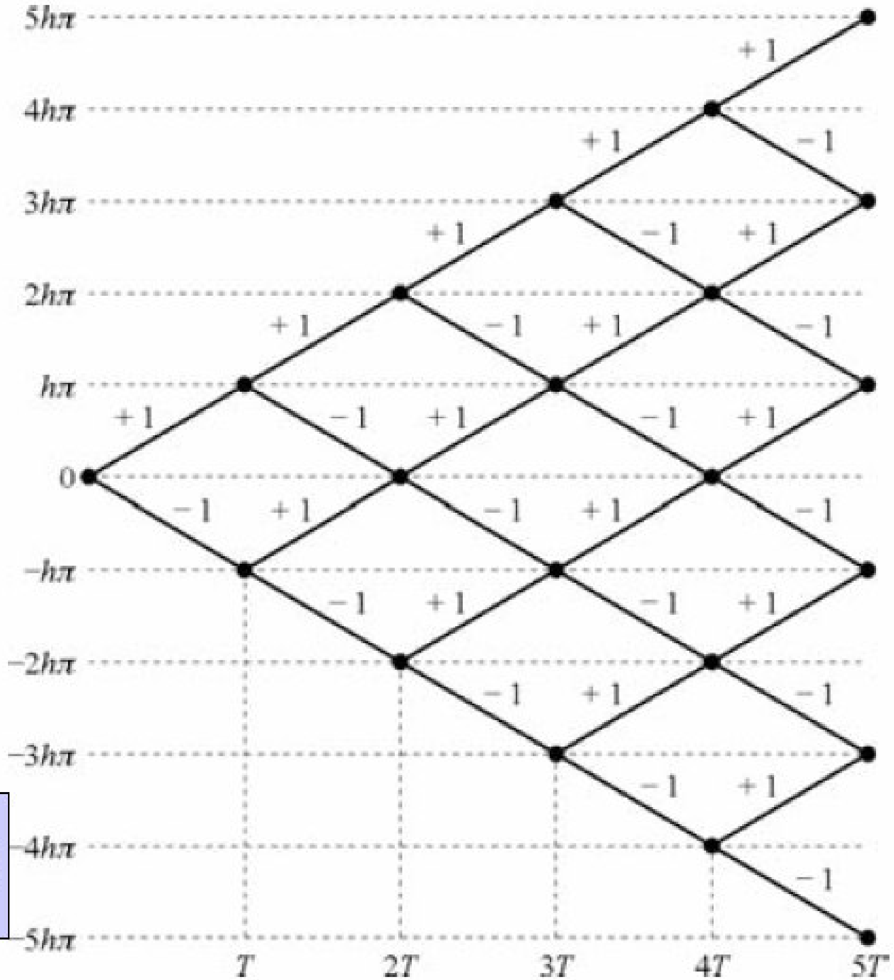
Continues-phase Modulation (CPM)

Example 1.

The case of Binary CPFSK with $I_n = \pm 1$ and g(t) is a full response rectangular function.

The set of phase trajectories starting t=0

$$\phi(t;\mathbf{I}) = \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n q(t - nT)$$

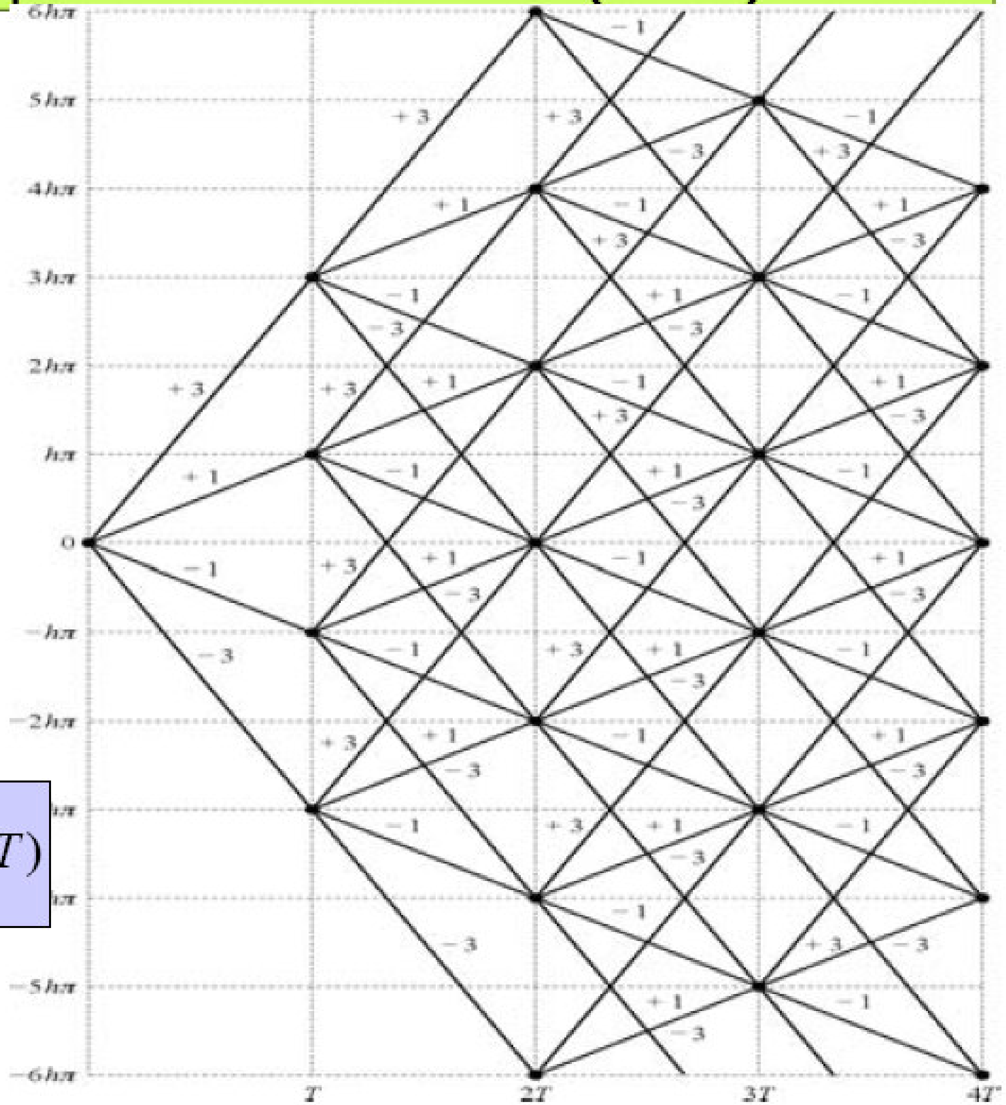


Continues-phase Modulation (CPM)

Example 2:

The case of Quaternary CPFSK with $I_n = \pm 1$ ± 3 and full response rectangular function, the set of phase trajectories starting t=0

$$\phi(t;\mathbf{I}) = \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n q(t - nT)$$



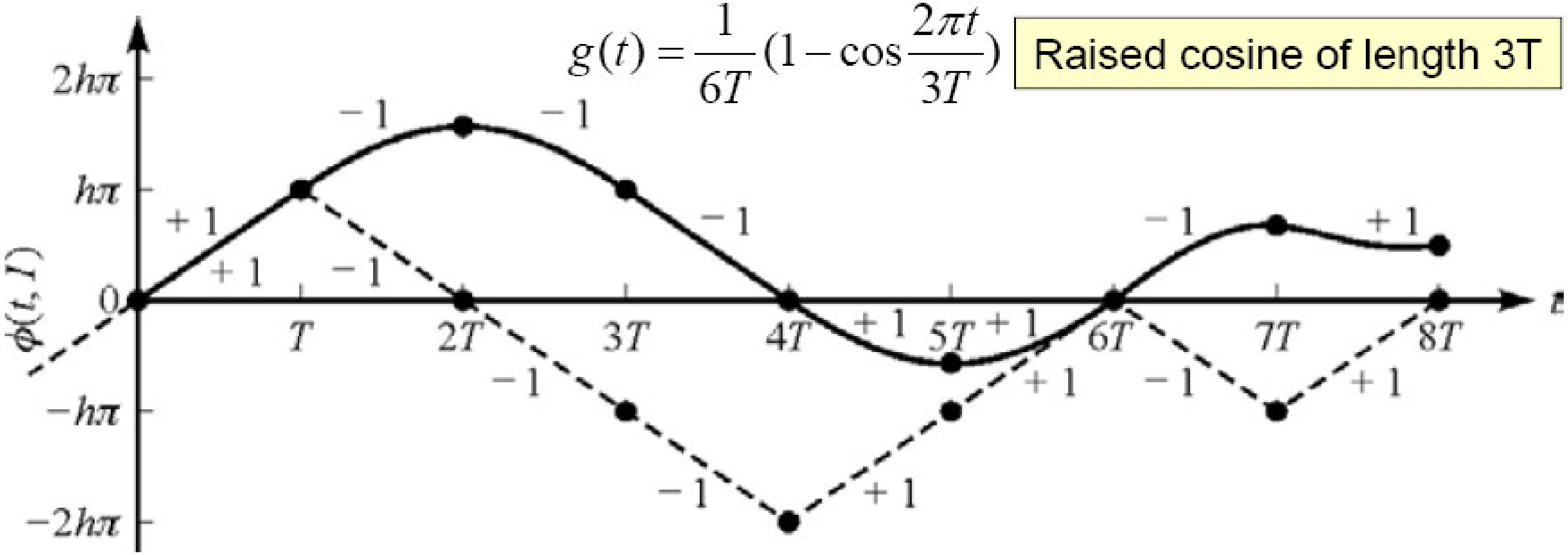
Continues-phase Modulation (CPM)

Example:

A phase trajectories generated by the sequence I_n=(1,-1,-1,-1,1,1,01,1) for the partial response

$$g(t) = \frac{1}{6T}(1 - \cos\frac{2\pi t}{3T})$$

Raised cosine of length 3T



Phase trajectories for binary CPFSK (dashed) and binary, partial response CPM based on raised cosine pulse of length 3*T* (solid). [*From Sundberg (1986), © 1986 IEEE.*]

Continues-Phase Modulation (CPM)

Phase State trellis

Simple way to represent the phase trajectories is concern only those phase values at $t=nT$. Range from $\phi=0$ to $\phi=\pi$.

$$\phi(nT,\mathbf{I}) \in \Theta_s = \{0, h\pi, 2h\pi, 3h\pi, \ldots\}$$

Example: For a full response CPM and h=m/p

$$\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \ldots, \frac{(p-1)\pi m}{p}\right\}$$

For m is even

There are p terminal phase state

$$\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \ldots, \frac{(2p-1)\pi m}{p}\right\}$$

For m is odd

There are $2p$ terminal phase state

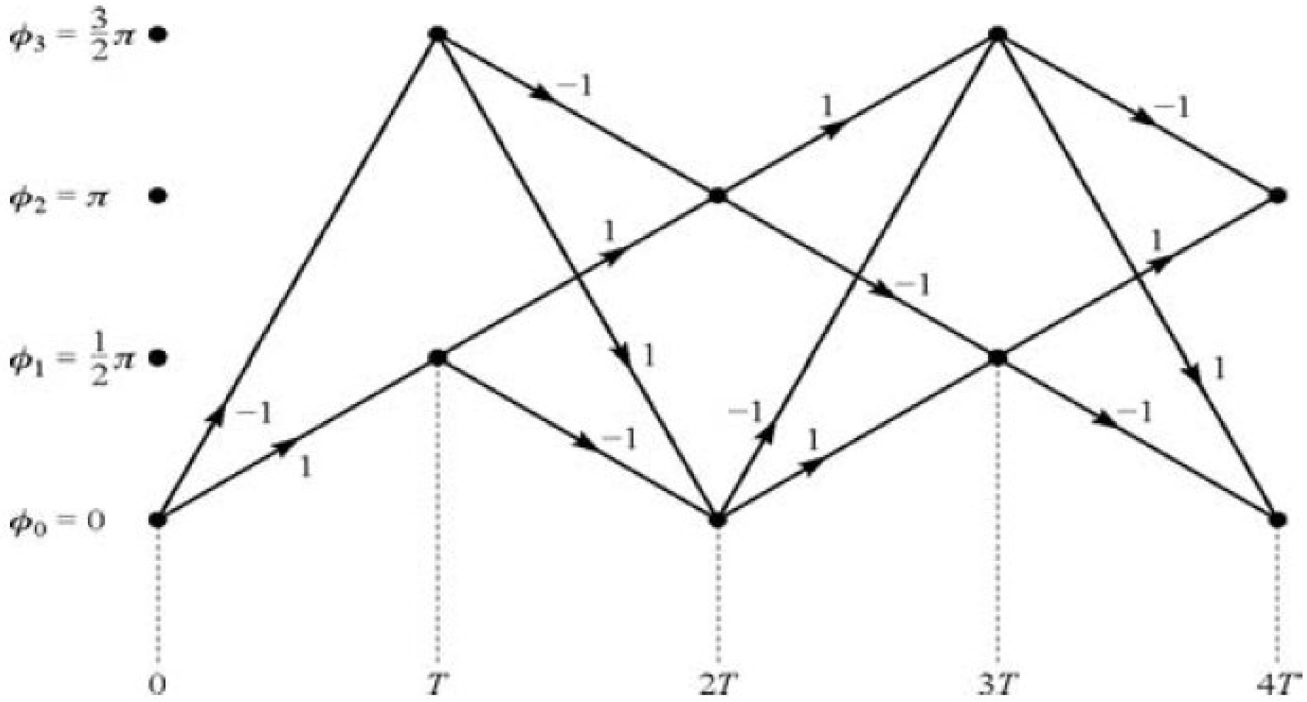
The maximum number of phase state is

$$S_t = \begin{cases} pM^{L-1} & m \text{ even} \\ 2pM^{L-1} & m \text{ odd} \end{cases}$$

M is alphabet size
L is an integer # extends the pulse shape

Continues-phase Modulation (CPM)

- Example: The phase state of the binary CPFSK (full response) with h=1/2 and S_t=4



Difference between phase state trellis and phase trellis is:

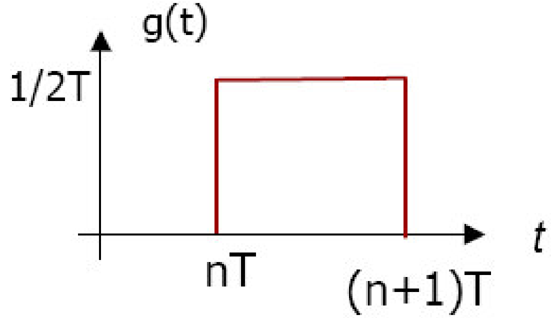
- The connection between state are made by drawing straight lines for phase state trellis. This is not true for phase trajectories from one state to another

Minimum-shift Keying (MSK)

MSK is special case of CPFSK and modulation index h=1/2.

The phase of the carrier in the interval $nT \le t \le (n+1)T$ is

$$\phi(t;\mathrm{I}) = \frac{1}{2}\pi \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t-nT)$$
$$= \theta_n + \frac{1}{2}\pi I_n \left(\frac{t-nT}{T}\right), \quad nT \le t \le (n+1)T$$



The modulated carrier signal is

$$s(t) = A\cos\left[2\pi f_c t + \theta_n + \frac{1}{2}\pi I_n\left(\frac{t-nT}{T}\right)\right]$$
$$= A\cos\left[2\pi\left(f_c + \frac{1}{4T}I_n\right)t - \frac{1}{2}n\pi I_n + \theta_n\right]$$

Minimum-shift Keying (MSK)

- The binary CPFSK signal will have two frequencies in the interval $nT \leq t \leq (n+1)T$

$$f_1 = f_c - \frac{1}{4T} \qquad f_2 = f_c + \frac{1}{4T}$$

- The binary CPFSK signal can be also written as

$$s_i(t) = A\cos\left[2\pi f_i t + \theta_n + \frac{1}{2}n\pi(-1)^{i-1}\right], \quad i = 1, 2$$

The frequency separation $\Delta f = f_2 - f_1 = 1/2T$

This the minimum frequency separation that is necessary to ensure the orthogonality of the signal s1(t) and s2(t) over a signaling interval of the length T. That is why binary CPFSK with h=1/2 is MSK

Minimum-shift Keying (MSK)

Also, MSK map be represents as a form of four-phase PSK.

The equivalent low-pass digitally modulated signal is

$$v(t) = A \sum_{n=-\infty}^{\infty} \left[I_{2n} g(t - 2nT) + j I_{2n+1} g(t - (2n+1)T) \right]$$

where

$$g(t) = \begin{cases} \sin \dfrac{\pi t}{2T}, & t \in [0, 2T) \\ 0, & o.w. \end{cases}$$

Viewed as a four-phase PSK signal with pulse shape is one-half cycle of sinusoidal. The even numbered binary valued symbols I_{2n} of the information sequence are transmitted via cos of the carrier. The odd-numbered symbols $\{I_{2n+1}\}$ are transmitted via the sin carrier

Minimum-shift Keying (MSK)

Transmission rate for each is 1/2T bits/ss, combine transmission rate will be 1/T bit/s.

MSK

$$s(t) = A\sum_{n=-\infty}^{\infty}\left[I_{2n}g(t-2nT)\cos 2\pi f_c t + I_{2n+1}g(t-(2n+1)T)\sin 2\pi f_c t\right]$$

s(t), a constant amplitude and frequency modulated signal

Minimum-shift Keying (MSK)

$$A\sum_{n=-\infty}^{\infty}[I_{2n}g(t-2nT)\cos 2\pi f_c t]$$

In phase signal component

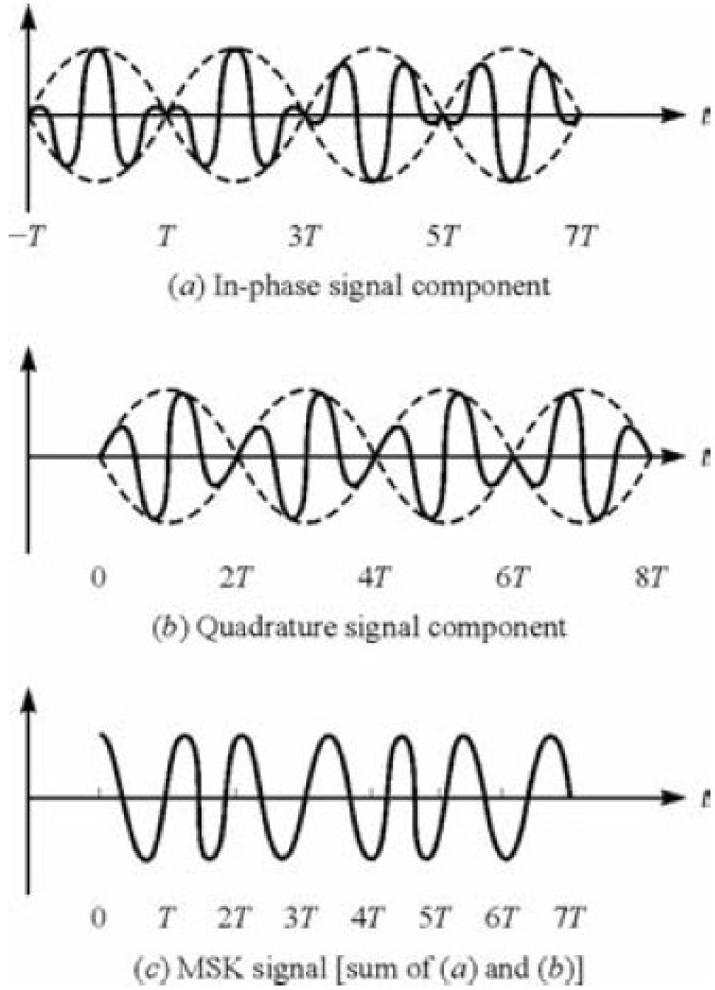
$$A\sum_{n=-\infty}^{\infty}[I_{2n+1}g(t-(2n+1)T)\sin 2\pi f_c t]$$

Quadrature signal component

s(t), a constant amplitude and frequency modulated signal

Frequency at $[nT,(n+1)T)$

$$f_c+\frac{1}{4T}I_n$$



Comparison of MSK and QPSK

MSK

$$s(t) = A\sum_{n=-\infty}^{\infty}\left[I_{2n}g(t-2nT)\cos 2\pi f_c t + I_{2n+1}g(t-(2n+1)T)\sin 2\pi f_c t\right]$$

where

$$g(t) = \begin{cases} \sin\frac{\pi t}{2T}, & t\in[0,2T) \\ 0, & o.w. \end{cases}$$

Continuous phase

Offset QPSK

$$s(t) = A\sum_{n=-\infty}^{\infty}\left[I_{2n}g(t-2nT)\cos 2\pi f_c t + I_{2n+1}g(t-(2n+1)T)\sin 2\pi f_c t\right]$$

where

$$g(t) = \begin{cases} 1, & t\in[9,2T) \\ 0, & o.w. \end{cases}$$

Possibly ±90 degree phase jump at each T

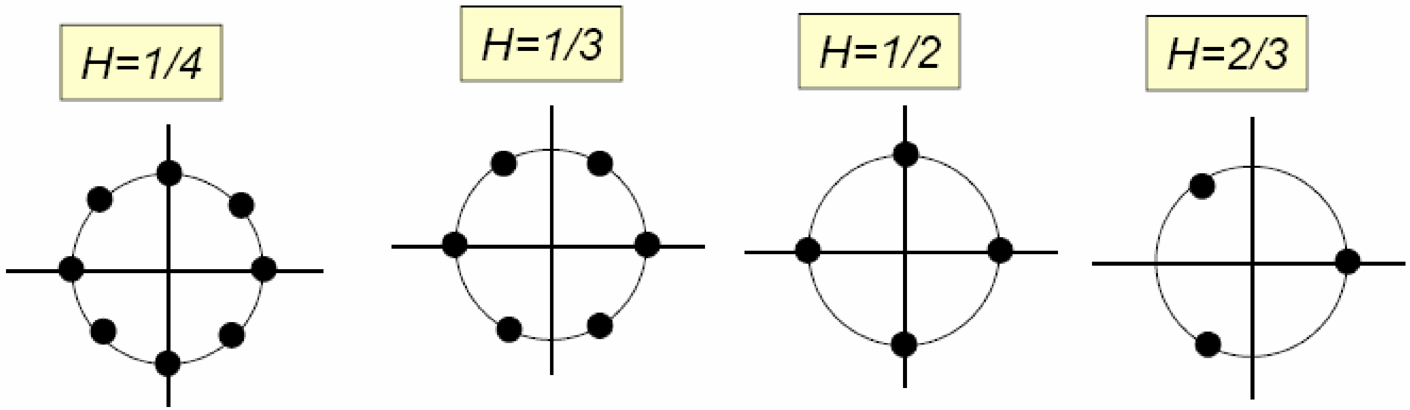
QPSK

$$s(t) = \sum_{n=-\infty}^{\infty}\left[-\frac{I_{2n}+I_{2n+1}}{2}g(t-2nT)\cos 2\pi f_c t + -\frac{I_{2n}-I_{2n+1}}{2}g(t-2nT)\sin 2\pi f_c t\right]$$

Signal space diagram of CPM

- Continues phase signal can not be represented by discrete points in signal space, like PAM, PSK
- It can be described by the trajectories from one phase state to another

- Here is signal phase trajectories diagram for CPFSK signal for h=1/4,h=1/3,h=1/2, and h=2/3



□ *Multi-amplitude CPM*

- *CPM (as a constant-amplitude signal) carries information in terms of its* **"frequency/continuous-phase" change**.
- *As instructed by QAM to PM, can we put information on the "amplitude" of CPM, such as a two-amplitude CPFSK?*

$$s(t) = 2A\cos\left[2\pi f_c t + \phi_2(t;\mathbf{I})\right] + A\cos\left[2\pi f_c t + \phi_1(t;\mathbf{J})\right],$$

$$\text{where } \begin{cases} \phi_2(t;\mathbf{I}) = \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n[(t-nT)/(2T)] \\ \phi_1(t;\mathbf{J}) = \pi h \sum_{k=-\infty}^{n-1} J_k + 2\pi h J_n[(t-nT)/(2T)] \end{cases}.$$

- *Different from QAM, which basically places information on two orthogonal quadrature components, the two amplitude-modulated components are not statistically independent.*

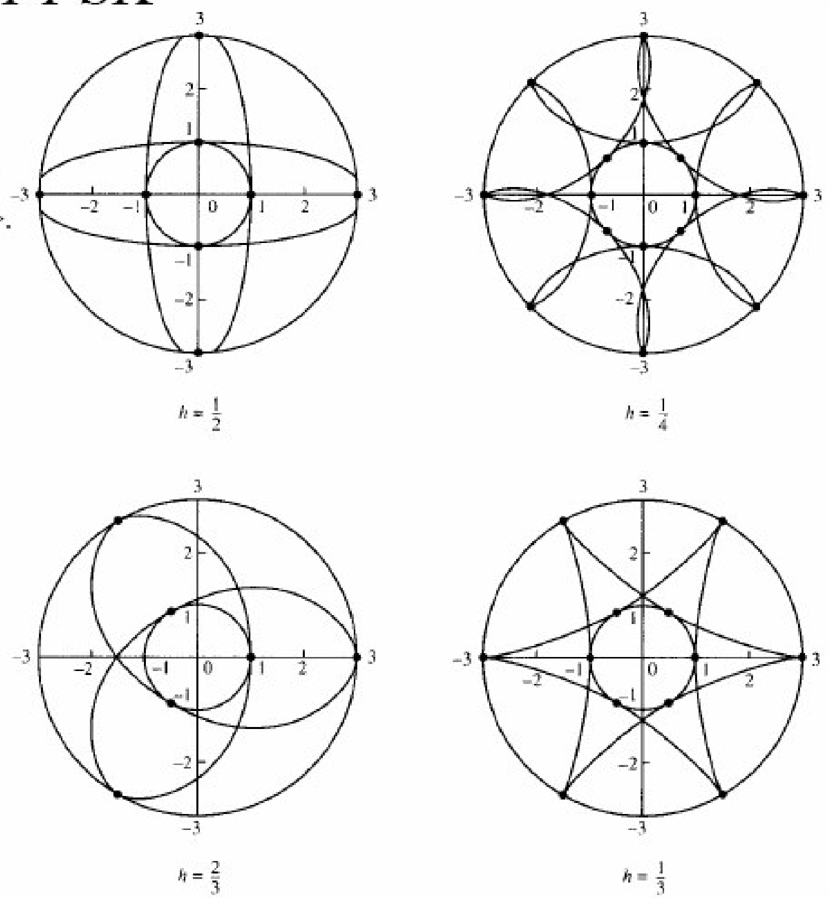
  For example: 2-dimensional MSK

$$s(t) = 2A\begin{cases} \pm\cos(2\pi f_+ t) \\ \pm\cos(2\pi f_- t) \\ \pm\sin(2\pi f_+ t) \\ \pm\sin(2\pi f_- t) \end{cases} + A\begin{cases} \pm\cos(2\pi f_+ t) \\ \pm\cos(2\pi f_- t) \\ \pm\sin(2\pi f_+ t) \\ \pm\sin(2\pi f_- t) \end{cases}$$

- *Signal space (phase trajectory) diagram for 2-component (2-amplitude) CPFSK*

$$\begin{cases} S_{Amplitude} = \{1,3\}. \\ S_{Phase} = \left\{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\right\}. \end{cases}$$

$$\begin{cases} S_{Amplitude} = \{1,3\}. \\ S_{Phase} = \left\{0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots, \frac{7}{4}\pi\right\}. \end{cases}$$



$$\begin{cases} S_{Amplitude} = \{1,3\}. \\ S_{Phase} = \left\{0, \frac{2}{3}\pi, \frac{4}{3}\pi\right\}. \end{cases}$$

$$\begin{cases} S_{Amplitude} = \{1,3\}. \\ S_{Phase} = \left\{0, \frac{1}{3}\pi, \frac{2}{3}\pi, \ldots, \frac{5}{3}\pi\right\}. \end{cases}$$

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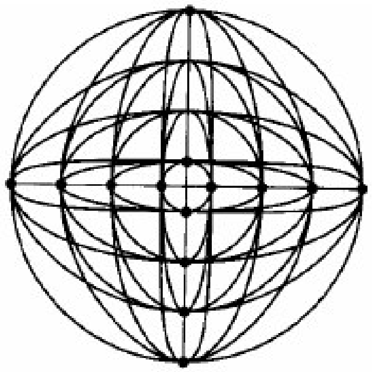
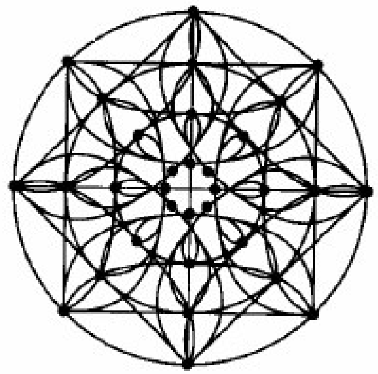
⅄ *Signal space diagram for 3-component CPFSK*

$$\begin{cases} S_{Amplitude} = \{1,3,5,7\}. \\ S_{Phase} = \left\{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\right\}. \end{cases}$$



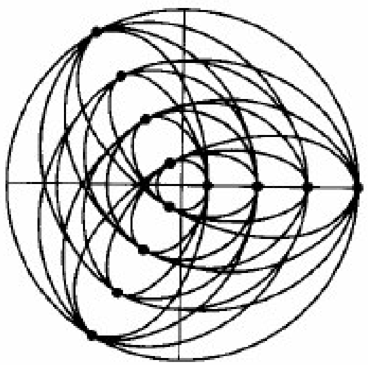
$h = \frac{1}{2}$



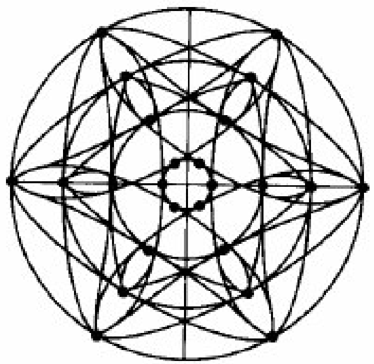
$h = \frac{1}{4}$

$$\begin{cases} S_{Amplitude} = \{1,3,5,7\}. \\ S_{Phase} = \left\{0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots, \frac{7}{4}\pi\right\}. \end{cases}$$

$$\begin{cases} S_{Amplitude} = \{1,3,5,7\}. \\ S_{Phase} = \left\{0, \frac{2}{3}\pi, \frac{4}{3}\pi\right\}. \end{cases}$$



$h = \frac{2}{3}$



$h = \frac{1}{3}$

$$\begin{cases} S_{Amplitude} = \{1,3,5,7\}. \\ S_{Phase} = \left\{0, \frac{1}{3}\pi, \frac{2}{3}\pi, \ldots, \frac{5}{3}\pi\right\}. \end{cases}$$

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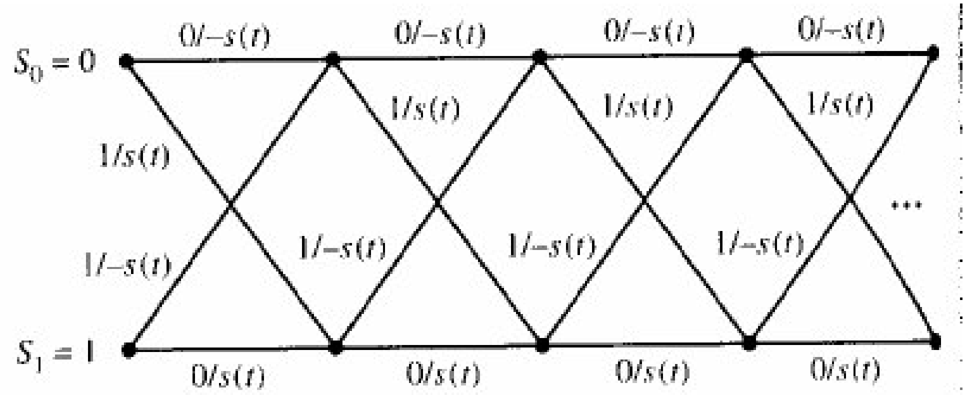
□ *Maximum-likelihood sequence detector*

- *Example study : NRZI*



$$\begin{cases} s(t) = A \\ -s(t) = -A \end{cases}$$

From the previous discussion,

$$r_k = \pm A + n_k,$$

where n_k is zero - mean Guassian distributed with variance $N_0/2$, and k is the index for time.

- *PDF of a sequence of demodulation outputs*

$$P(r_1,\dots,r_K \mid s_1,\dots,s_K) = \frac{1}{(\pi N_0)^{K/2}} \exp\left[-\sum_{k=1}^{K} \frac{(r_k - s_k)^2}{N_0} \right]$$

- $s_1, \dots, s_K$ *have memory, so it is advantageous to detect the original signals based on a sequence of outputs.*
- *If ML rule is employed, the resultant detector is called* the maximum-likelihood sequence detector.

□ *ML sequence detector for NRZI signals*

$$\begin{aligned} d_{ML}(r_1,\dots,r_K) &= \arg\max_{(s_1,\dots,s_K)\in\{-A,A\}^K} P(r_1,\dots,r_K \mid s_1,\dots,s_K) \\ &= \arg\max_{(s_1,\dots,s_K)\in\{-A,A\}^K} \frac{1}{(\pi N_0)^{K/2}} \exp\left[-\sum_{k=1}^{K} \frac{(r_k - s_k)^2}{N_0} \right] \\ &= \arg\min_{(s_1,\dots,s_K)\in\{-A,A\}^K} \sum_{k=1}^{K} (r_k - s_k)^2, \text{ Euclidean distance} \end{aligned}$$

We therefore needs to search for all possible combinations of $(s_1,\dots,s_K)$, which consist of 2^K possibilities.

- ***ML sequence detector for multi-dimensional signals with memory***

$$d_{ML}(\vec{r}_1,\ldots,\vec{r}_K) = \arg\max_{(\vec{s}_1,\ldots,\vec{s}_K)\in S^K} P(\vec{r}_1,\ldots,\vec{r}_K \mid \vec{s}_1,\ldots,\vec{s}_K)$$

$$= \arg\max_{(\vec{s}_1,\ldots,\vec{s}_K)\in S^K} \frac{1}{(\pi N_0)^{KN/2}} \exp\left[-\sum_{k=1}^{K}\sum_{j=1}^{N}\frac{(r_{kj}-s_{kj})^2}{N_0}\right]$$

$$= \arg\min_{(\vec{s}_1,\ldots,\vec{s}_K)\in S^K} \sum_{k=1}^{K}\sum_{j=1}^{N}(r_{kj}-s_{kj})^2, \text{ Euclidean distance,}$$

where $|S| = N$. We therefore needs to search for all possible combinations of $(\vec{s}_1,\ldots,\vec{s}_K)$, which consist of 2^{KN} possibilities.

  - *The complexity of "searching" the optimal solution becomes a burden.*

- ***Viterbi (demodulation) Algorithm***
  - ***A sequential trellis search*** *algorithm that performs ML sequence detection*
    - *Transforming a* ***search*** *over 2K vector points into a* ***sequential search*** *over a (vector) trellis*
    - *"sequential" = break the vectors into components and perform the search based on each component (in sequence) of the vectors*

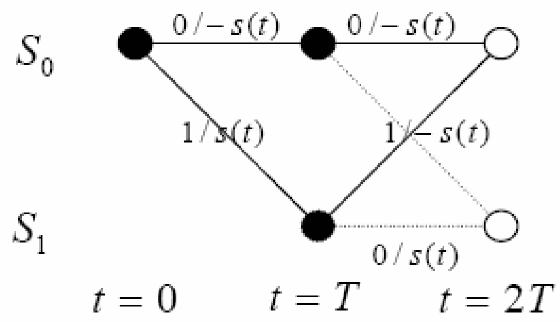
Notations : $s_k = \begin{cases} s_{k-1}, & \text{if } I_k = 0 \\ -s_{k-1}, & \text{if } I_k = 1 \end{cases} = (1-2I_k)s_{k-1} \in \{-A, A\}$ is the channel symbol, which has memory.

$I_k \in \{0,1\}$ is the digital input, which does not have memory. 5-

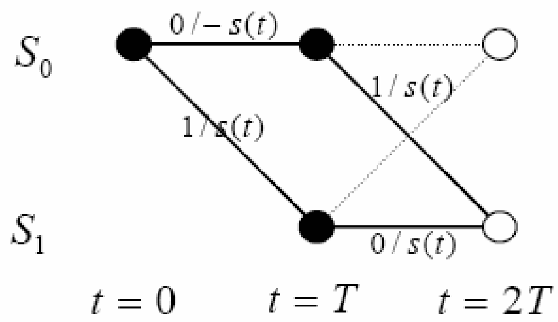
- *The signal memory order of NRZI signals is 1 (L=1).*
  - *The current channel symbol only depends on the previous channel symbol.*
  - *Assume the initial state is S_0. Then the trellis will reach its regular form after the reception of the first two signals.*

- *Explaining the Viterbi Algorithm (from S_0 at t = 0).*
  - *There are two paths entering each node at t = 2T.*

path $(I_1, I_2) = (0,0)$ or $(1,1)$
$\rightarrow$ node S_0 at $t = 2T$,
denoted by $S_0(2T)$.



path $(I_1, I_2) = (0,1)$ or $(1,0)$
$\rightarrow$ node S_1 at $t = 2T$,
denoted by $S_1(2T)$.



– *Euclidean distance for each path*

Euclidean distance for path (0,0) entering node $S_0(2T) = D_0(0,0) = (r_1 - (-A))^2 + (r_2 - (-A))^2$

Euclidean distance for path (1,1) entering node $S_0(2T) = D_0(1,1) = (r_1 - A)^2 + (r_2 - (-A))^2$

– *Viterbi algorithm.*

- *Discard, among the above two paths, the one with larger Euclidean distance.*
- *The remaining path is called* survivor *at $t = 2T$.*

Veterbi Algorithm (Example)