EC 421 Statistical Communication Theory

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Lecture # 6

Gaussian process

1. **Gaussian pdf** (probability density function)

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{(x-\overline{X})^2}{2\sigma_X^2}\right)$$

2. **Bi-variate** pdf (jointly Gaussian)

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]\right\}.$$

Correlation coefficient $\rho \triangleq \frac{E[(X-\overline{X})(Y-\overline{Y})]}{\sigma_X\sigma_Y}$

3. ***N*-variate pdf** (jointly Gaussian) $-1 < \rho < 1$

Properties of Gaussian processes

1. Gaussian processes are completely specified by their mean $E[X(t)]$ and autocorrelation function $R_{XX}(t_i, t_j)$.
2. A wide-sense stationary Gaussian process is also strict-sense stationary.
3. If the jointly Gaussian processes $X(t)$ and $Y(t)$ are uncorrelated, then they are independent.
4. If the Gaussian process $X(t)$ is passed through a linear time-invariant system, then the corresponding output process $Y(t)$ is also a Gaussian process. ←

Linear weighted sum of Gaussian RVs → Another Gaussian RV

White Noise

- **White** → occupies all frequencies → *PSD is independent on the operating frequency.*
- Dimensions of N_o is watt per Hertz, $N_o = KT$ (*K: Boltman const., T: equivalent noise temperature of the receiver*)

$$S_N(f) = \frac{N_o}{2}$$

$$R_N(\tau) = \frac{N_o}{2}\delta(\tau)$$

- Any two different samples of white noise no matter how close they are will be **uncorrelated**.
- If white noise is **Gaussian** then they will also be **independent**

White Gaussian Noise Processes

- **Definition. A random process $N(t)$ is a white Gaussian noise if:**
- *$N(t)$ is a WSS process satisfying:*

1. $\mu_N = E[N(t)] = 0$ (zero mean)
2. For any time instants $t_1 < t_2 < \ldots < t_k$, $N(t_1)$, $N(t_2)$, …, $N(t_k)$ are **independent** Gaussian random variables.

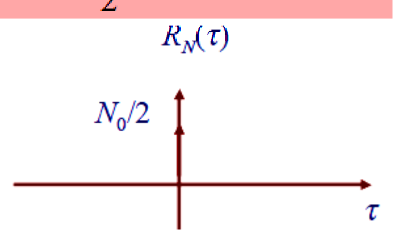
The characteristics of *N(τ)*

(1) $N(t)$ is a **Gaussian random variable** for any time instance t.

(2) **Zero mean**: $\mu_N = 0$

(3) The **autocorrelation function**:

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau), \quad -\infty < \tau < \infty$$

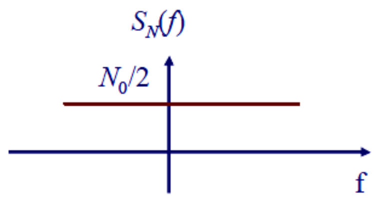


THE CHARACTERISTICS OF *N(τ)*

(4) The **power spectral density** is flat:

$$S_N(f) = \frac{N_0}{2}, \quad -\infty < f < \infty$$

- where N_0 is the power per unit bandwidth of *N(t).*



(5) The **average power** is infinity:

$$P_N = E[N^2(t)] = \infty$$

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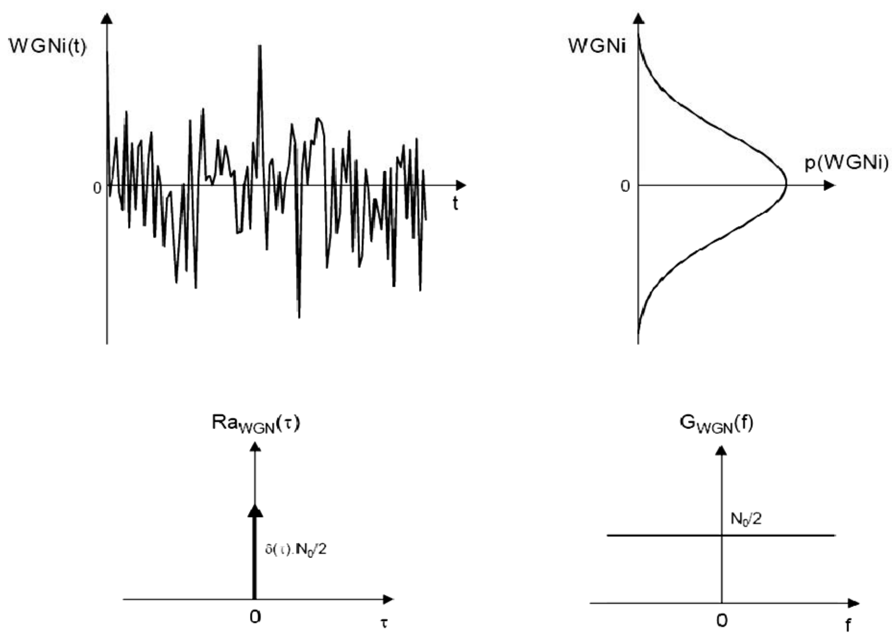
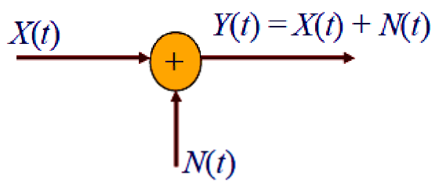
WHITE GAUSSIAN NOISE (WGN)



EXAMPLE

- Sum with a White Gaussian Noise. Suppose that *N*(*t*) *is a white Gaussian noise with power spectrum* $N_0/2$. We also assume that *X(t) and N(t) are joint WSS and independent. Find* the mean and the power spectral density of *Y(t).*



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SOLUTION

- **Mean:** $\mu_Y(t) = E[X(t) + N(t)]$

$$= E[X(t)] + E[N(t)]$$

$$= \mu_X + 0 = \mu_X$$

- **Autocorrelation:**

$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[(X(t) + N(t))(X(t+\tau) + N(t+\tau))]$$

$$= R_X(t,\tau) + R_{XN}(t,\tau) + R_{NX}(t,\tau) + R_N(t,\tau)$$

$$= R_X(\tau) + R_N(\tau) = R_Y(\tau)$$

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Solution

- Thus, $Y(t)$ is WSS. The PSD of $Y(t)$:

$$S_Y(f) = F(R_Y(\tau)) = F(R_X(\tau) + R_N(\tau))$$

$$= S_X(f) + S_N(f) = S_X(f) + \frac{N_0}{2}$$

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Ideal low pass filtered (ILPF) white noise

- A white Gaussian noise with zero mean and variance $N_0/2$ is applied to an ideal low pass filter of bandwidth B and amplitude response of one.
- PSD of output $Y(t)$ is: $S_Y(f) = \begin{cases} \frac{N_o}{2} & -B \le f \le B \\ 0 & otherwise \end{cases}$
- The autocorrelation is: $R_Y(\tau) = N_0 B \sin c(2B\tau)$
- Autocorrelation **maximum** at τ equal **zero**, it is equal N_0B, and passes through zero at $\tau = n/2B$ for n= integer values and variance N_0B.
- If noise is sampled at rate 2B then they are uncorrelated and being Gaussian then statistically independent

RC LOW PASS FILTERED WHITE NOISE

- The *H*(*f*) of RC filter is $H(f) = \frac{1}{1 + j2\pi fRC}$
- The PSD of the o/p is $S_Y(f) = \frac{N_0 / 2}{1 + (2\pi fRC)^2}$
- The autocorrelation of the output is $R_Y(\tau) = \frac{N_o}{4RC} \exp(-\frac{|\tau|}{RC})$
- If noise is samples at rate 0.217/*RC* then they are **uncorrelated** and being **Gaussian** then **statistically independent**

EXAMPLE SINE WAVE PLUS WHITE NOISE

$$X(t) = A\cos(2\pi f_c t + \theta) + N(t)$$

θ is uniformly distributed and *N*(*t*) is WGN.

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau)$$

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{N_0}{2}$$

NOISE EQUIVALENT BANDWIDTH

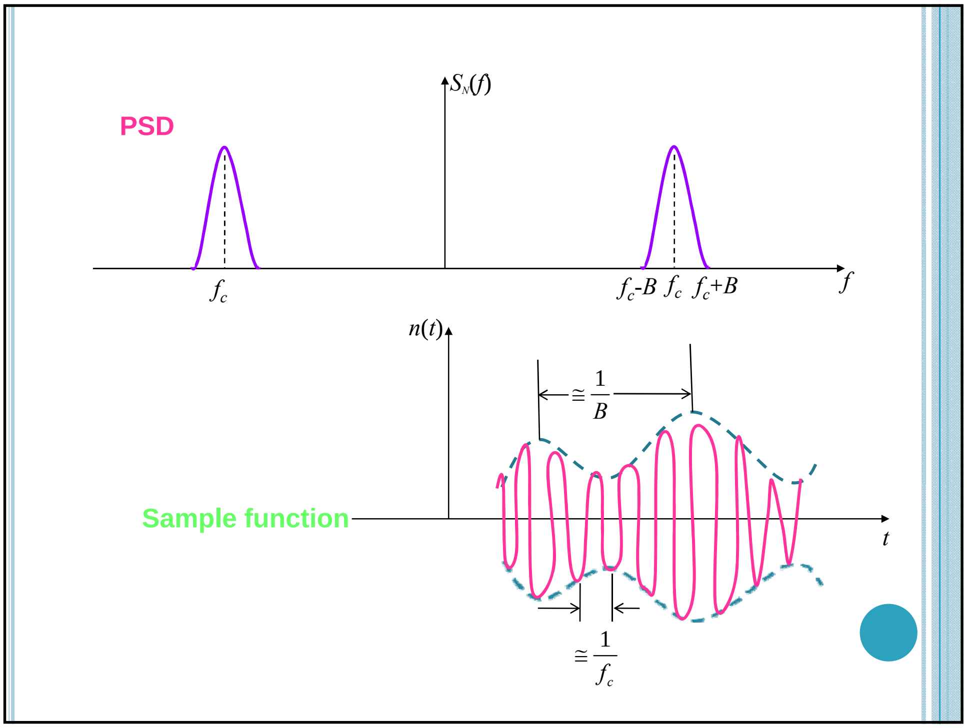
- Output average power of **I-LPF**→ $N_o B$
- Output average power of **RC-LPF**→ $N_o/4RC$
- Output average power of **any filter** → $N_o B H^2(0)$
- Equivalent Bandwidth =

$$B = \frac{\int_0^{\infty} |H(f)|^2 df}{H^2(0)}$$

NARROWBAND NOISE (NBN)

- **Filters** at the receiver have **enough bandwidth** to pass the **desired signal** but not too big to pass **excess noise**.
- **Narrowband (NB)**→ center frequency is much bigger that the bandwidth.
- **Noise** at the output of such filters are called **narrowband noise (NBN)**.
- **NBN** has spectral concentrated about some mid-band frequency $\pm f_c$
- The **sample function** of such NBN $n(t)$ **appears as** a **sine wave of frequency** f_c which modulates slowly in amplitude and phase.

$$S_N(f) = \frac{N_0}{2}|H(f)|^2$$



$S_N(f)$
PSD
f_c
f_c-B f_c f_c+B
f
$n(t)$
$\cong \frac{1}{B}$
Sample function
t
$\cong \frac{1}{f_c}$