EC 421 Statistical Communication Theory

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Lecture # 5

Physical interpretation of process parameters

FOR AN ERGODIC PROCESS $X(t)$

- $\overline{X(t)} = \langle x(t) \rangle$ proportional to DC component
- $\overline{X(t)}^2 = \langle x(t) \rangle^2$ proportional to power in DC component
- $\overline{R_{xx}}(0) = \langle X^2(t) \rangle$ proportional to total average power
- $\sigma^2 = \overline{X^2(t)} - \overline{X(t)}^2$ total average AC power
- σ root-mean-square (RMS) value of AC power

$X(t)$ represents a voltage, a current or any other physical phenomenon.

Wide-Sense Stationary (WSS) Processes

- A random process X(t) is **WSS** if:

$$(i)\ \mu_X(t) = E[X(t)] = \text{constant}$$
$$(ii)\ R_X(t, t-\tau) = E[X(t)X(t-\tau)] = R_X(\tau)$$

- In other words, a random process *X(t) is WSS if its two* statistics, its **mean** and **autocorrelation**, do not vary with a shift in the time origin.

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Properties of Autocorrelation Function of a WSS Process *X(τ)*

$$1.\ R_X(\tau) = R_X(-\tau)$$

- Symmetric in τ about zero

$$2.\ |R_X(\tau)| \le R_X(0) \quad \text{for all } \tau,$$

- Maximum value occurred at the origin

$$3.\ R_X(\tau) \leftrightarrow S_X(f)$$

- Autocorrelation and PSD form a pair of the Fourier transform

$$4.\ R_X(0) = E[X(t)^2]$$

- The value at origin is equal to the average power of the signal

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Cross-Correlation Function of a WSS Processes $X(\tau)$ and $Y(\tau)$

1. $R_{XY}(-\tau) = R_{YX}(\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{xx}(0)R_{YY}(0)}$
3. $|R_{XY}(\tau)| \le \frac{1}{2}[R_{xx}(0) + R_{YY}(0)]$

- Symmetric in τ about zero
- Maximum value occurred at the origin

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Auto-Covariance and Cross-Covariance of a WSS Processes $X(\tau)$ and $Y(\tau)$

- Auto-Covariance of $X(t)$:

$$C_{XX}(\tau) = E\left[\left\{X(t) - E[x(t)]\right\}\left\{X(t+\tau) - E[X(t+\tau)]\right\}\right]$$
$$= R_{XX}(\tau) - \mu_X^2$$

- Cross-Covariance of $X(t)$ and $Y(t)$:

$$C_{XY}(\tau) = E\left[\left\{X(t) - E[x(t)]\right\}\left\{Y(t+\tau) - E[Y(t+\tau)]\right\}\right]$$
$$= R_{XY}(\tau) - \mu_X \mu_Y$$

- If $X(t)$ and $Y(t)$:
  - Orthogonal, $R_{XY}(\tau) = 0$
  - Uncorrelated, $C_{XY}(\tau) = 0$

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POWER SPECTRAL DENSITY (PSD) OF A WSS RANDOM PROCESS

- For a given WSS process *X(t), the PSD of X(t) is the* Fourier transform of its autocorrelation, i.e.,

$$S_X(f) = \mathcal{F}(R_X(\tau)) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \mathcal{F}^{-1}(S_X(f)) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} df$$

- The above equations are known as: *Wiener-Khinchin relations.*

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PROPERTIES OF PSD

1. $S_X(f) \geq 0$ — Always real valued
2. $S_X(f) = S_X(-f)$ — for *X(t) real-valued*
3. $S_X(f) \leftrightarrow R_X(\tau)$ — A pair of Fourier transform
4. $P = R_X(0) = \int_{-\infty}^{+\infty} S_X(f) df$ — Relationship between average power and psd

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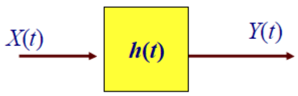
Transmission over Linear Time Invariant (LTI) Systems

- Response of LTI system to a random input *X(t)*:



$$Y(t) = X(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

- Properties of the output:
  1) Mean $\mu_Y(t) = h(t) * \mu_X(t)$

If *X(t) is* WSS*, so does Y(t),* *mean*:

$$\mu_Y = \mu_X H(0)$$

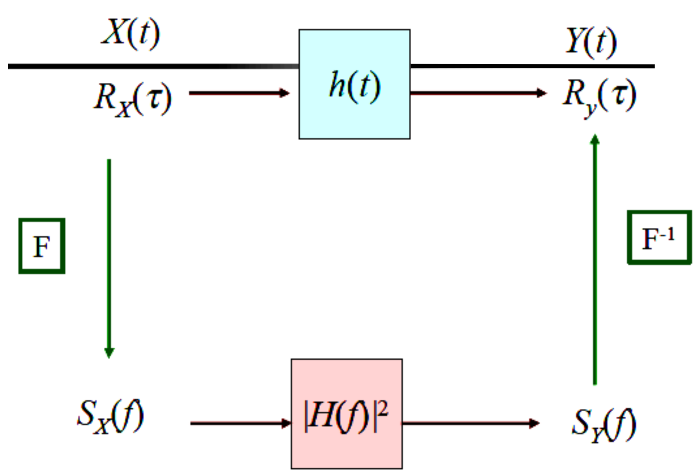
  3) Autocorrelation:

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

  4) PSD:

$$S_Y(f) = S_X(f) \,|\, H(f) \,|^2$$

Transmission over LTI Systems



Description in frequency domain

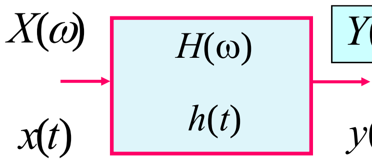
A <u>deterministic</u> signal $x(t)$ can be <u>fully</u> described in the frequency domain by the Fourier Transform (FT):

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-\mathrm{j}\,\omega t)\,\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(\mathrm{j}\,\omega t)\,\mathrm{d}\omega$$

is the <u>inverse Fourier Transform</u>.

Main purpose: linear time invariant (LTI) systems:



$$Y(\omega) = X(\omega)\,H(\omega)$$

$$y(t) = \int x(\tau)h(t-\tau)\,\mathrm{d}\tau$$