EC 421 Statistical Communication Theory

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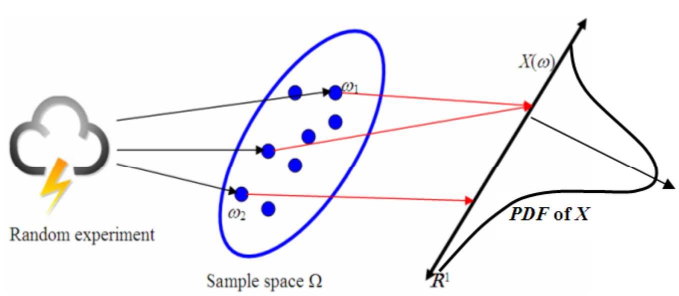
Lecture # 4

Random Signals and Noise

- A random process is a process (i.e., variation in time or one dimensional space) whose behavior is not completely predictable and can be characterized by statistical laws.



Random Processes

- Random signals cannot be explicitly described prior to their occurrence.
- Noises cannot be described by deterministic functions of time.
- However, when observed over a long period, a random signal or noise exhibit certain regularities that can be described in terms of probabilistic description of a collection of functions of times, which is called a random process.

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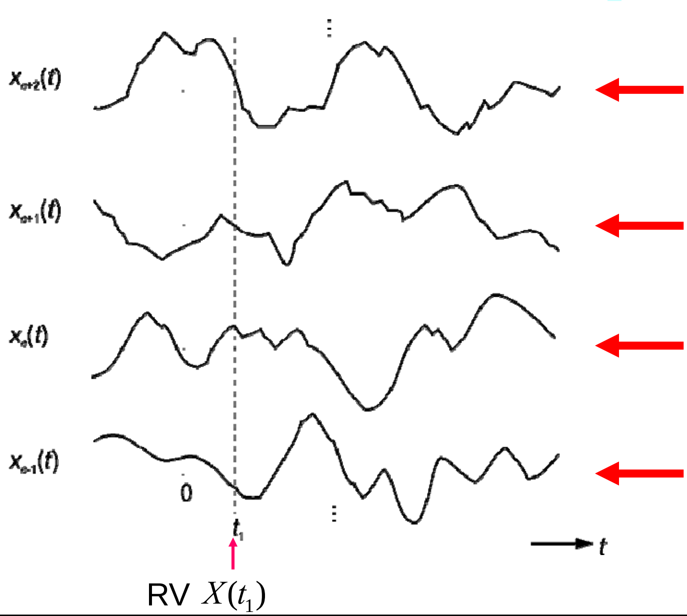
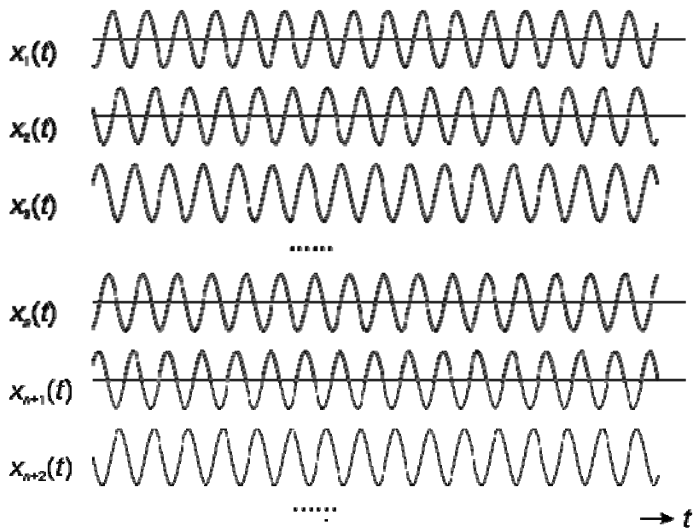
Examples of an ensemble of sample functions



Both x and t continuous

Select a realization by means of random experiment

RANDOM PHASED SINE WAVE



$$X(t) = \cos(\omega_0 t - \Theta) \qquad \Theta \text{ uniform } (0, 2\pi]$$

DISTRIBUTION AND DENSITY FUNCTIONS

Distribution and probability density function of RV X :

$$F_X(x) \hat{=} \mathrm{P}\{X \leq x\} \longrightarrow f_X(x) \hat{=} \frac{\mathrm{d}\, F_X(x)}{\mathrm{d}x}$$

Distribution and probability density function of Stochastic Process $X(t)$ **:**

$$F_X(x_1;t_1) \hat{=} \mathrm{P}\{X(t_1) \leq x_1\} \longrightarrow f_X(x_1;t_1) \hat{=} \frac{\partial F_X(x_1;t_1)}{\partial x_1}$$

Time parameter is involved, i.e.
$F_X(x_1;t_1)$ and $f_X(x_1;t_1)$ will, in general, depend on time

Second order joint distribution and probability density functions

Joint distribution and probability density function of two RVs X and Y:

$$F_{X,Y}(x,y) \hat{=} \mathrm{P}\{X \le x, Y \le y\} \longrightarrow f_{X,Y}(x,y) \hat{=} \frac{\mathrm{d}\,F_{X,Y}(x,y)}{\mathrm{d}x\,\mathrm{d}y}$$

Second order distribution and probability density function of $X(t)$:

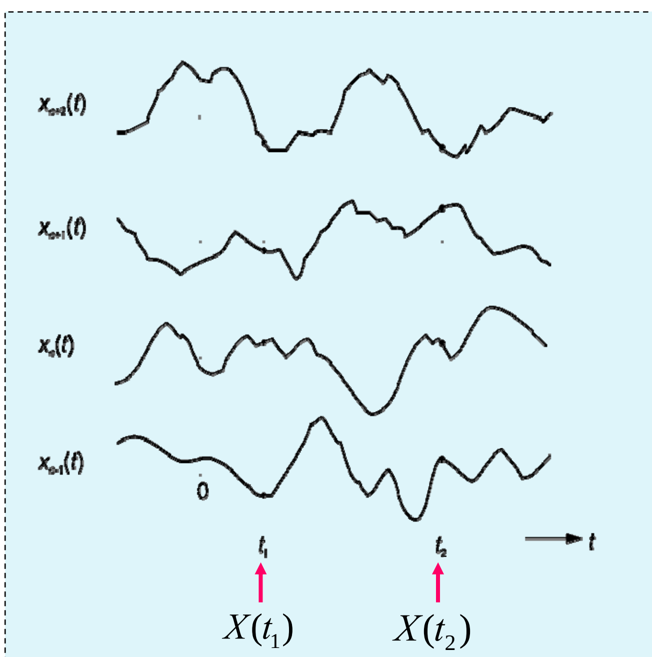
$$F_X(x_1,x_2;t_1,t_2) \hat{=} \mathrm{P}\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

$$\longrightarrow f_X(x_1,x_2;t_1,t_2) \hat{=} \frac{\partial^2 F_X(x_1,x_2;t_1,t_2)}{\partial x_1 \partial x_2}$$

Joint distribution and probability density function of two Stochastic Processes $X(t)$ and $Y(t)$:

$$F_{X,Y}(x_1,y_1;t_1,t_1') \hat{=} \mathrm{P}\{X(t_1) \le x_1, Y(t_1') \le y_1\}$$

$$\longrightarrow f_X(x_1,y_1;t_1,t_1') \hat{=} \frac{\partial^2 F_X(x_1,y_1;t_1,t_1')}{\partial x_1 \partial y_1}$$



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Higher order joint distribution and probability density functions

$$F_X(x_1, ..., x_N; t_1, ..., t_N) \triangleq \mathrm{P}\{X(t_1) \le x_1, ..., X(t_N) \le x_N\}$$

$$f_X(x_1, ..., x_N; t_1, ..., t_N) \triangleq \frac{\partial^N F_X(x_1, ..., x_N; t_1, ..., t_N)}{\partial x_1 ... \partial x_N}$$

Two processes $X(t)$ and $Y(t)$ are statistically independent iff

$$f_{X,Y}(x_1, ..., x_N; y_1, ..., y_M; t_1, ..., t_N; t'_1, ..., t'_M)$$

For all arbitrary choices of the time parameters t_1, ..., t_N, t_1', ..., t_N'

Stationarity

First order stationarity: $f_X(x_1; t_1) = f_X(x_1; t_1 + \tau), \qquad \forall \tau$

Consequence:

$$\overline{X(t)} \equiv \mathrm{E}[X(t)] \triangleq \int x \cdot f_X(x; t)\, \mathrm{d}x = \text{constant}$$

Statistical average (ensemble average) *i.e. independent of time*

N-th order stationarity:

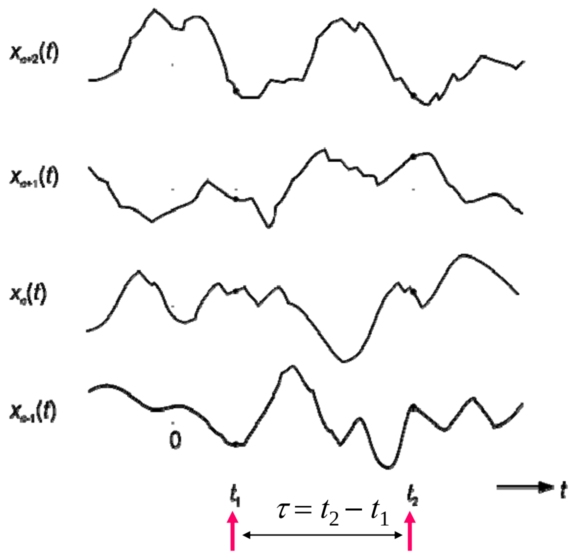
$$f_X(x_1, ..., x_N; t_1, ..., t_N) = f_X(x_1, ..., x_N; t_1 + \tau, ..., t_N + \tau)$$

$\forall \tau$

Stationarity of certain order N → Stationarity of all orders $k \le N$

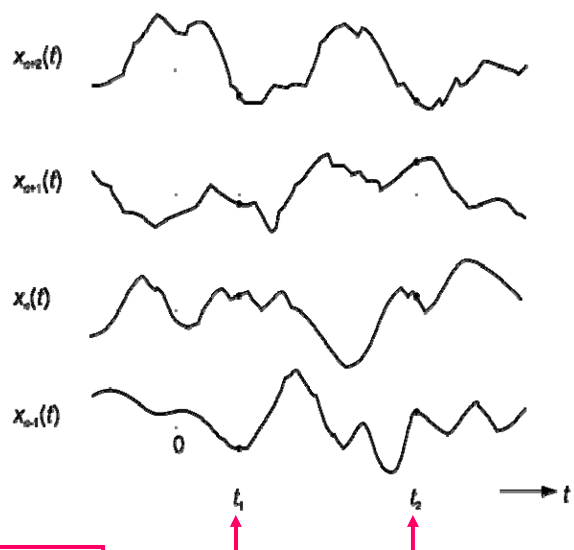
Strict-sense stationary: → Stationary of any arbitrary order

STATIONARITY



Second order stationarity $X(t_1)$ $X(t_2)$

AUTOCORRELATION FUNCTION



$\mathrm{E}[X(t_1) \cdot X(t_2)]$ $X(t_1)$ $X(t_2)$

STATIONARITY

- A random process $X(t)$ is called *strict-sense stationary (SSS)* if its statistics are invariant to a shift of origin.
- A random process $X(t)$ is called *wide-sense stationary (WSS)* if:
  - Its mean is constant.
  - Its autocorrelation depends only on the time-difference (τ), and consequently its auto-covariance also depends only on time difference (τ).

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AUTOCORRELATION FUNCTION

Correlation of RVs $X_1=X(t_1)$ and $X_2=X(t_2)$

$$R_{XX}(t_1,t_2) \triangleq \mathrm{E}[X(t_1)X(t_2)] = \iint x_1 x_2 f_X(x_1,x_2;t_1,t_2)\, dx_1\, dx_2$$

For second order stationary process:

$$R_{XX}(t,t+\tau) =$$
$$\mathrm{E}[X(t)] =$$

→ independent of t

Definition:

Wide-sense stationarity (WSS)

$$\mathrm{E}[X(t)] = \overline{X(t)} = \text{constant}$$
$$\mathrm{E}[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Second order stationarity:

wide-sense stationarity

Time statistics of a random process

1[st] order CDF: $F_X(x_1;t_1)$

1[st] order PDF: $f_X(x_1;t_1)$

mean: $\mathrm{E}[X(t_1)]$ = constant

1[st] order stationarity

2[nd] order PDF: $f_X(x_1,x_2;t_1,t_2)$

2[nd] order stationarity

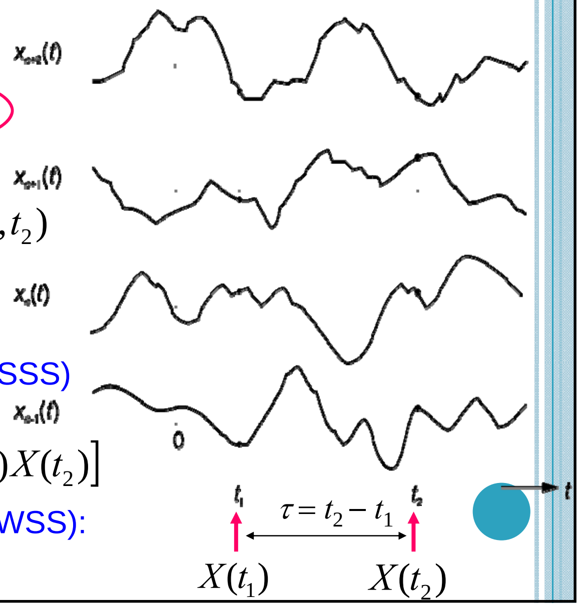
…

Strict-sense stationarity (SSS)

ACF: $R_{XX}(t_1,t_2)=\mathrm{E}[X(t_1)X(t_2)]$

Wide-sense stationarity (WSS):

$$R_{XX}(t,t+\tau)=R_{XX}(\tau)$$



Properties of $R_{XX}(\tau)$ of WSS processes 6

1) $|R_{XX}(\tau)| \le R_{XX}(0)$

   i.e. $|R_{XX}(\tau)|$ attains its maximum value for $\tau = 0$.

2) $R_{XX}(-\tau) = R_{XX}(\tau)$

   i.e. $R_{XX}(\tau)$ is an even function of τ.

3) $R_{XX}(0) = \mathrm{E}[X^2(t)]$

4) If $X(t)$ has no periodic component then $R_{XX}(\tau)$ comprises a constant term equal to $\overline{X(t)}^2$, i.e. $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X(t)}^2$.

5) If $X(t)$ has a periodic component then $R_{XX}(\tau)$ will comprise a periodic component as well and which has the same periodicity.

ERGODICITY

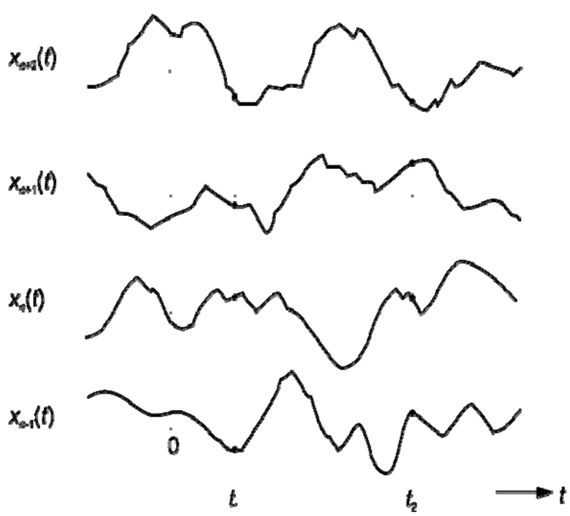
Time average: $\mathbf{A}[X(t)] \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt \longrightarrow$ RV

Ergodic process defined by

$$A[X(t)] = E[X(t)] = \overline{X(t)}$$
$$A[X(t)X(t+\tau)] = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Process has to be wide-sense stationary

ERGODICITY



Ergodic Process.

- A random process $X(t)$ is said to be *ergodic* if its **time averages** are the same for all functions and **equal to** the corresponding **ensemble averages**.

- Testing the ergodicity of a random process is usually very difficult.

- A reasonable assumption in the analysis of most communication signals is that a random waveform is ergodic in the mean and in the autocorrelation.

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Wide-sense stationarity and ergodicity

Wide-sense stationarity (WSS):

$$E[X(t)] = \overline{X(t)} = \text{constant}$$
$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Ergodicity:

$$A[X(t)] = E[X(t)] = \overline{X(t)}$$
$$A[X(t)X(t+\tau)] = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

CROSS-CORRELATION FUNCTION (CCF)

The cross-correlation function $R_{XY}(t, t + t)$ of two processes $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t, t+\tau) \triangleq \mathrm{E}[X(t) \cdot Y(t+\tau)]$$

$X(t)$ and $Y(t)$ are called jointly wide-sense stationary if both are individually WSS and $R_{XY}(t, t + \tau) = R_{XY}(\tau)$, i.e. independent of t

If $R_{XY}(t, t + \tau) = 0$ then $X(t)$ and $Y(t)$ are orthogonal

Processes $X(t)$ and $Y(t)$ are jointly ergodic if individually ergodic and

$$\mathrm{A}[X(t) \cdot Y(t+\tau)] = \mathrm{E}[X(t) \cdot Y(t+\tau)] = R_{XY}(\tau)$$

(Processes have to be jointly WSS)