EC 421 Statistical Communication Theory

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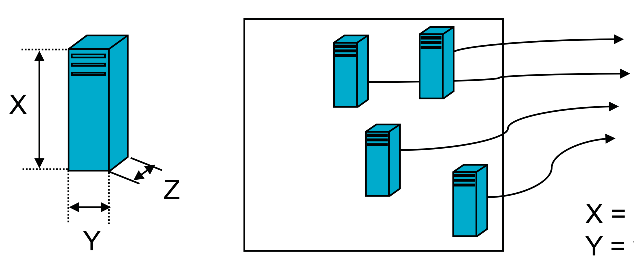
Lecture # 3

Jointly Distributed Variables

- Many problems in statistics and probability involve more than a single random variable.
- Therefore, sometimes it is necessary to study several random variables simultaneously.

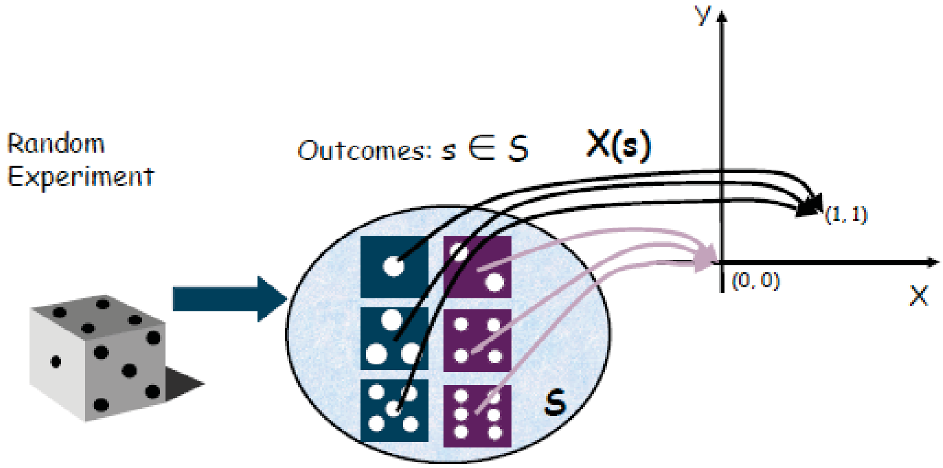


X	Y	Z
1	0	0
0	1	1
0	1	0
1	0	1

X = height (1 in spec, 0 out of spec)
Y = width (1 in spec, 0 out of spec)
Z = depth (1 in spec, 0 out of spec)

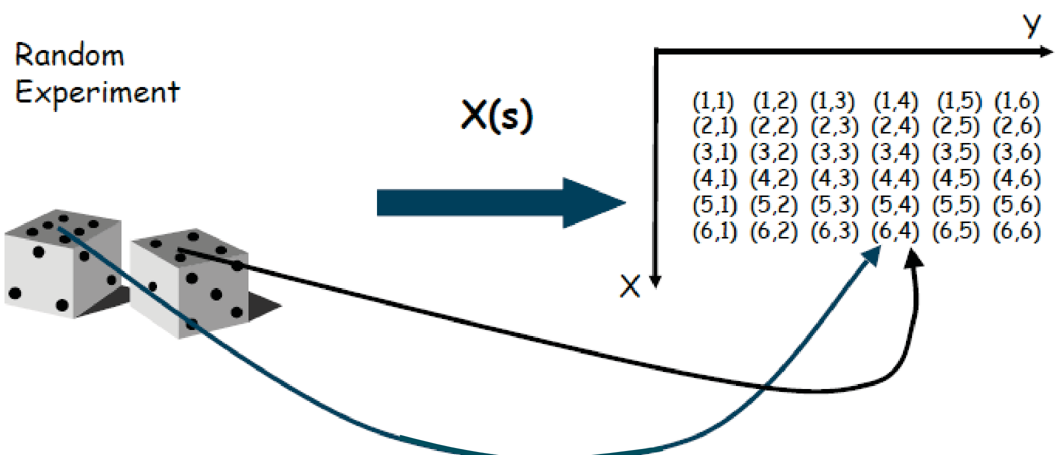
Multiple Random Variables

- Each outcome is mapped into a single vector.
- However, more than one outcome can be mapped into the same vector.



Multiple Random Variables

- Performing direct mapping.



Cumulative Distribution Function (CDF)

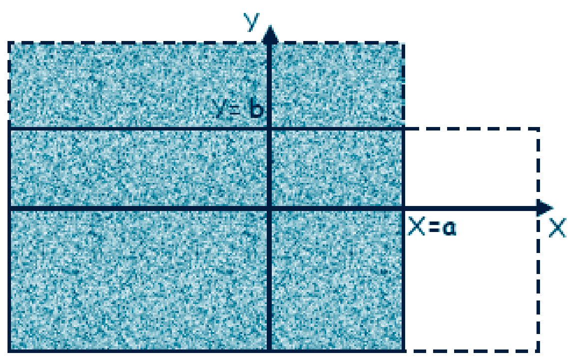
- CDF $F_{XY}(a, b)$ of the joint random variables X and Y is defined as the probability of the event,

$\{\{X \leq a\} \cap \{Y \leq b\}\}$:

$$F_{XY}(a, b) = P[\{X \leq a\} \cap \{Y \leq b\}]$$
$$\forall\ -\infty \leq a \leq +\infty \text{ and } -\infty \leq b \leq +\infty$$

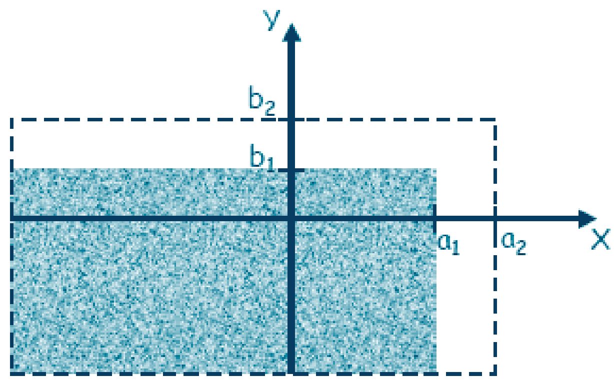
Joint CDF F_{XY}

- $F_{XY}(a, b) = P[\{X \leq a\} \cap \{Y \leq b\}]$.



PROPERTIES OF JOINT CDF F_{XY}

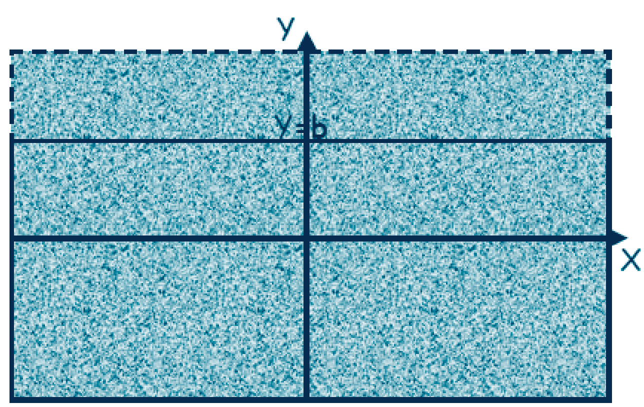
- If $a_1 \le a_2$ and $b_1 \le b_2 \Rightarrow F_{XY}(a_1, b_1) \le F_{XY}(a_2, b_2)$, i.e. F_{XY} is a non-decreasing function.



MARGINAL CDF

$$\lim_{a \to \infty} F_{xy}(a,b) = P[\{y \le b\}]$$

$$\lim_{a \to \infty} F_{xy}(a,b) = F_y(b)$$



Marginal CDF

- $F_X(a)$ and $F_Y(b)$ are the marginal CDF functions of the joint random variables X and Y.

$$\lim_{a\to\infty} F_{xy}(a,b) = F_y(b)$$

$$\lim_{b\to\infty} F_{xy}(a,b) = F_x(a)$$

Joint Dist. – Two Continuous RV's

Joint PDF

Let X and Y represent two continuous rv's

$f_{XY}(x,y) >= 0$ for all x,y

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$P[(X,Y)\in A] = \iint_A f(x,y)\,dx\,dy$$

Marginal PDF

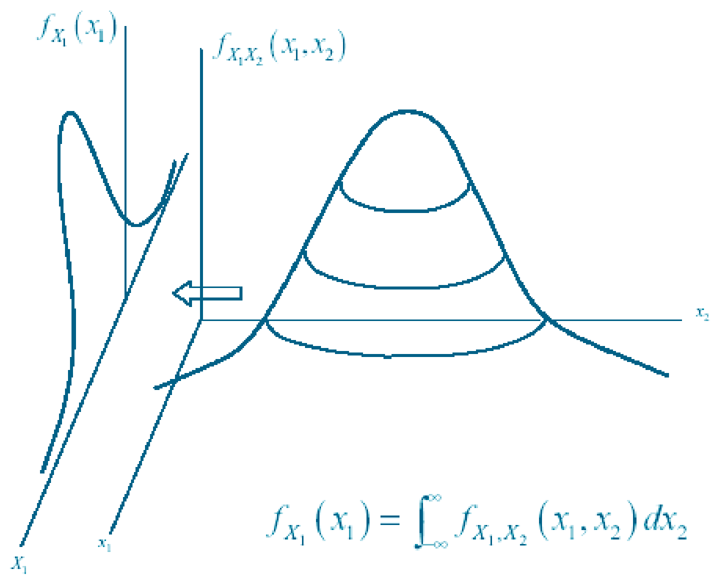
To obtain a marginal pdf for say $X=x1$,

P(x1,y) - you compute prob. for all possible y values.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Interpretation of Marginal PDF as a Projection



$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_2$$

Jointly Distributed Random Variables

- Joint Cumulative Distribution Function
  - **Discrete**

$$F(x,y) = P(X \le x_i, Y \le y_j)$$

$$F(x,y) = \sum_{i:x_i \le x} \sum_{j:y_j \le y} p_{ij}$$

  - **Continuous**

$$F(x,y) = \int_{w=-\infty}^{x} \int_{z=-\infty}^{y} f(w,z)\,dz\,dw$$

Jointly Distributed Random Variables

- Joint Probability Distributions Satisfy:
  - **Discrete**

$$P(X = x_i, Y = y_j) = p_{ij} \geq 0$$

$$\text{satisfying} \quad \sum_i \sum_j p_{ij} = 1$$

  - **Continuous**

$$f(x, y) \geq 0 \quad \text{satisfying} \quad \iint_{\text{state space}} f(x, y)\,dxdx = 1$$

Independence and Covariance

- Two random variables *X* and *Y* are said to be *independent* if:
  - **Discrete**

$$p_{ij} = p_{i+}p_{+j} \quad \text{for all values } i \text{ of } X \text{ and } j \text{ of } Y$$

  - **Continuous**

$$f(x, y) = f_X(x) f_Y(y) \quad \textit{for all } x \textit{ and } y$$

Independence and Covariance

- **Covariance**

$$\mathrm{Cov}(X,Y) = E((X-E(X))(Y-E(Y)))$$
$$= E(XY) - E(X)E(Y)$$

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E((X-E(X))(Y-E(Y))) \\ &= E(XY - XE(Y) - E(X)Y + E(X)E(Y)) \\ &= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y)\end{aligned}$$

  - Can take any **positive** **OR** **negative** numbers.
  - **Independent** random variables have a covariance of **zero**.

Independence and Covariance

- **Correlation:**

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

  - Values between -1 and 1, and **independent** random variables have a correlation of **zero**.

Joint Distributions of Independent Random Variables

- Independent events: $P(A\ B) = P(A) \cdot P(B)$
- If two variables are (fully, or statistically) independent, then
  - **Discrete**: $p(x,y) = p_X(x) \cdot p_Y(y)$, for ALL possible (x,y) pairs.
  - **Continuous**: $f(x,y) = f_X(x) \cdot f_Y(y)$, for ALL possible (x,y) pairs also.
- If two variables do not satisfy the above for all (x,y) then they are said to be dependent. Therefore, if you can find even **ONE** pair not satisfying the above, you just proved dependence.

Expected Values using Joint Distributions

- Let X and Y be jointly distributed rv's
- If two variables are:
  - ***X*, *Y* Discrete**:

$$\mathrm{E}[h(x,y)] = \sum_x \sum_y h(x,y) \bullet p(x,y)$$

  - ***X*, *Y* Continuous**:

$$\mathrm{E}[h(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \bullet f(x,y)\,dx\,dy$$

Joint Moments

- **Moments of a random variables**:

$$E[x^n y^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^n y^n f_{X,Y}(x,y)dxdy$$

$$E[X] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{X,Y}(x,y)dxdy = \int_{-\infty}^{+\infty} x\left[\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\right]dydx = \int_{-\infty}^{+\infty} x f_X(x)dx$$

$$E[Y] = \int_{-\infty}^{+\infty} y f_y(y)dy$$

$$E[XY] = E[X].E[Y]$$ *If X and Y are independent and in this case*

$$f_{X,Y}(x,y) = f_X(x).f_Y(y)$$

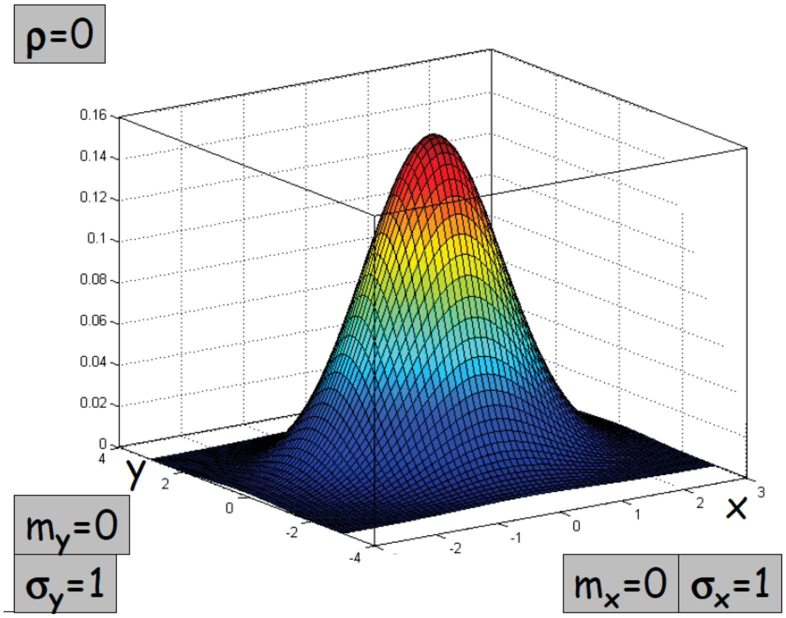
Jointly Normal (Gaussian) RVs

- The joint pdf of "jointly normal" random variables *X* and *Y* with means m_x and m_y and standard deviations σ_x & σ_y:

$$f_{XY}(x,y) = C\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_x)^2}{\sigma_x^2} - 2\rho\frac{(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\}$$

$$C = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \qquad |\rho| < 1$$

Jointly Normal (Gaussian) RVs



Functions of Random Variables

R.V.	Function of R.V.
X = test score	$Y = 10X - 3$ (bonus, $)
X = # cars produced	$Y = 4X$ (# tires)

$f(x)$ → $h(y)$

$Y = g(X)$

Transformation of Variables

Theorem

X = discrete R.V. with probability distribution $f(x)$.

$Y = u(X)$: one-to-one transformation.

i.e., $y = g(x) \longrightarrow \quad x = g^{-1}(y) = h(y)$

Then,

$$F_Y(y) = P(Y \le y)$$
$$= P(X \le h(y))$$

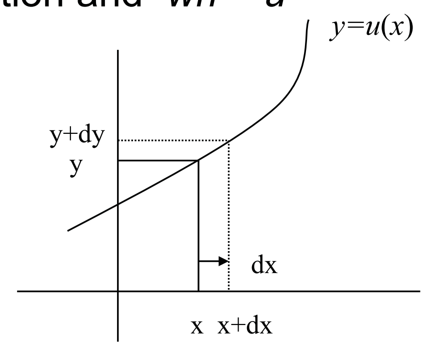
$\Rightarrow \quad h(y) = g^{-1}(y)$

Transformation of Variables

X = continuous R.V. with pdf $f(x)$.

$Y = g(X)$: one-to-one transformation and $wh = u^{-1}$

$y=g(x) \longrightarrow \quad x = h(y)$



If differentiable mono-tone increasing

$$F_Y(y) \int_{-\infty}^{h(y)} f_X(x)dx$$

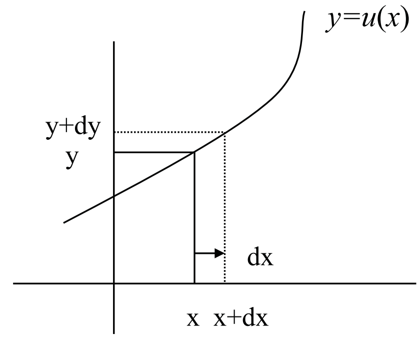
$$f_Y(y) = f_X(h(y))\frac{dh}{dy}$$

Transformation of Variables

If differentiable mono-tone decreasing

$$F_Y(y) \int_{h(y)}^{\infty} f_X(x)dx$$

$$f_Y(y) = -f_X(h(y))\frac{dh}{dy}$$



We can express both results:

$$f_{Y(y)} = f_X(h(y))\left|\frac{dh}{dy}\right|$$

Transformation of Variables

Example: (Uniform and Exponential Random Variables)

Suppose X follows Uniform (0, 1).

Let Y= -βln(1-X). Find the distribution of Y.

$$\begin{aligned} F_Y(y) &= P\{Y < y\} \\ &= P\{-\beta \ln(1-X) < y\} \\ &= P\{1-X > e^{-y/\beta}\} \\ &= P\{X < 1-e^{-y/\beta}\} \\ &= 1-e^{-y/\beta}, y > 0 \end{aligned}$$

$$g(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\beta}e^{-y/\beta}, y > 0$$

$$x = 1-e^{-y/\beta} = h(y)$$

$$J = \frac{dx}{dy} = \frac{1}{\beta}e^{-y/\beta}$$

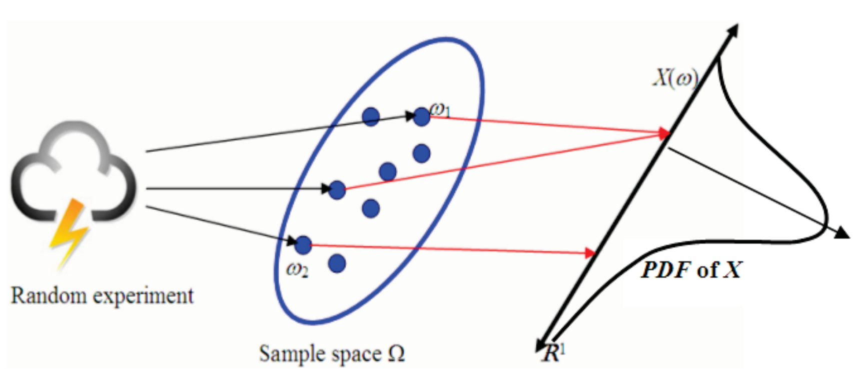
$$\begin{aligned} g(y) &= f(h(y))\,|J| \\ &= \frac{1}{\beta}e^{-y/\beta} \end{aligned}$$

Random Signals and Noise

- **Random Signal:** *Information signal*
  - (voice, audio, picture, video, text file, data, presence of a radar target, etc.).
- **Noise:** *Unwanted disturbance of information signal*,
  - interference or cross-talk (disturbance by other information signal).
- Common tools to describe both random signal and noise: **Stochastic process.**

Random Signals and Noise

- A random process is a process (i.e., variation in time or one dimensional space) whose behavior is not completely predictable and can be characterized by statistical laws.



Examples of an ensemble of sample functions