EC 421 Statistical Communication Theory

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Lecture # 2

Binomial Random Variables

Class of *discrete* random variables = **Binomial --** results from a binomial experiment.

A **binomial random variable** is defined as X=number of successes in the n trials of a binomial experiment.

Conditions for a binomial experiment:

1. There are ***n* "trials"** where n is determined in advance and is not a **RANDOM** value.
2. ***Two possible outcomes*** on each trial, called "success" and "failure" and denoted S and F.
3. ***Outcomes are independent*** from one trial to the next.
4. ***Probability of a "success"***, denoted by p, remains **same** from one trial to the next. Probability of "failure" is $1 - p$.

FINDING BINOMIAL PROBABILITIES

$$P(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

Example*: *Probability of Two Wins in Three Plays

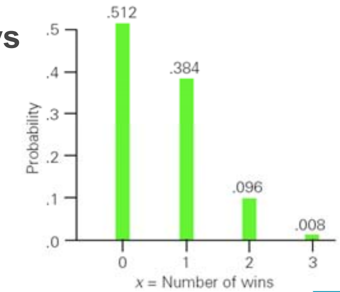
p = probability win = 0.2; plays of game are *independent*.

***X* = number of wins in three plays**

What is ***P*(*X* = 2)**?

$$P(X = 2) = \frac{3!}{2!(3-2)!} .2^2 (1-.2)^{3-2}$$

$$= 3(.2)^2 (.8)^1 = 0.096$$



EXPECTED VALUE AND STANDARD DEVIATION FOR A BINOMIAL RANDOM VARIABLE

For a *binomial random variable X* based on *n* trials and success probability *p*,

Mean $\mu = E(X) = np$

Standard deviation $\sigma = \sqrt{np(1-p)}$

EXAMPLE: CRV - TIME SPENT WAITING FOR BUS

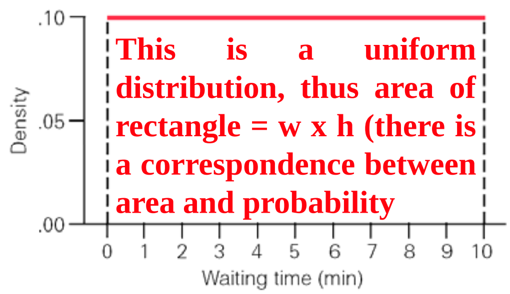
Bus arrives at stop every 10 minutes. Person arrives at stop at a random time, how long will s/he have to wait?

X = waiting time until next bus arrives.

X is a continuous random variable over 0 to 10 minutes.

Note: Height is 0.10 so total area under the curve is (0.10)(10) = 1

This is an example of a ***Uniform random variable density curve***

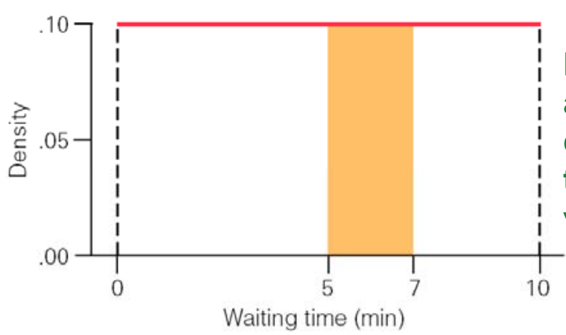


EXAMPLE: WAITING FOR BUS (CONT)

What is the probability of waiting time *X* was in the interval from 5 to 7 minutes?

Probability = area under curve between 5 and 7

= (base)(height) = (2)(.1) = .2



$P(a \leq X \leq b)$ is the area under the density curve over the interval between values *a* and *b*

Normal Random Variables

If a population of **measurements follows a normal curve**, and if *X* is the measurement for a randomly selected individual from that population, then

X is said to be a ***normal random variable***

X is also said to have a ***normal distribution***

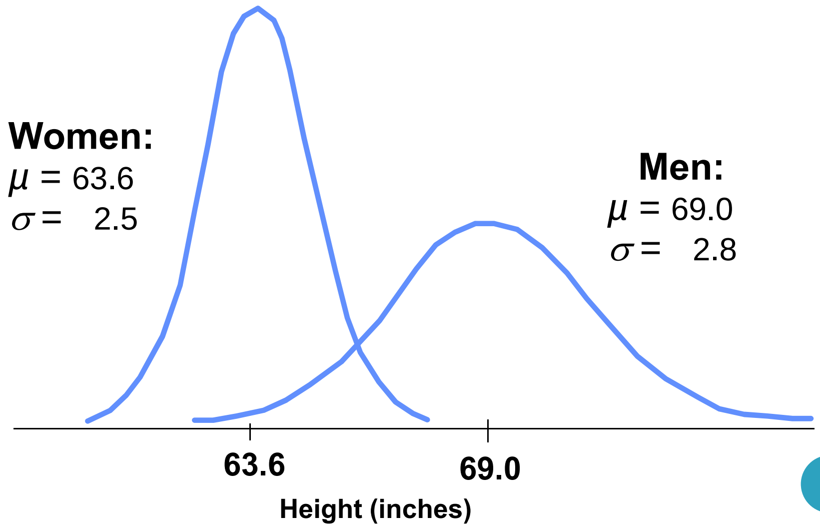
Any normal random variable can be completely characterized by its mean, *m*, and standard deviation, *s*.

Normal Distributions (of normal random variables)

Just as there are many different uniform distributions (with different ranges of values), there are also many different normal distributions, with each one depending on 2 parameters: the population mean and population SD.

The standard normal distribution is a normal probability distribution that has a mean = 0 and a SD = 1.0, and the total area under its density curve = 1.0

Example Normal Curves (Heights; males; females)



Standard Scores

The formula for converting any value *x* to a *z*-score is

$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}$$

A *z*-score measures the number of standard deviations that a value falls from the mean.

EXAMPLE STANDARD SCORES FOR HEIGHT

For a population of college women, the *z*-score corresponding to a height of 62 inches is

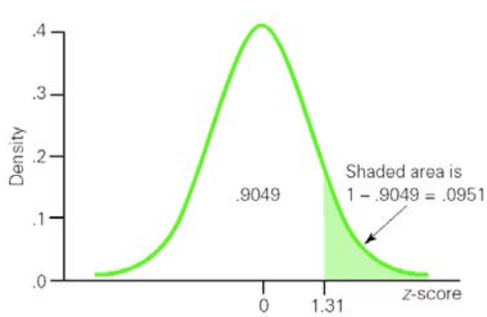
$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard deviation}} = \frac{62 - 65}{2.7} = -1.11$$

This *z*-score tells us that 62 inches is 1.11 standard deviations below the mean height for this population.

EXAMPLE: PROBABILITY Z > 1.31

$P(Z > 1.31) = 1 - P(Z \leq 1.31)$

$= 1 - .9049 = .0951$

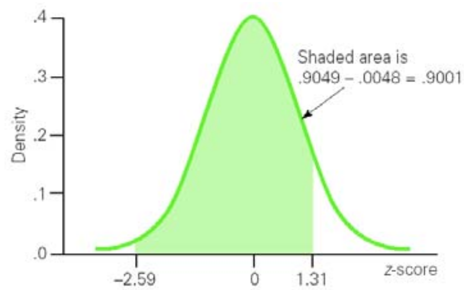


EXAMPLE: PROBABILITY Z IS BETWEEN –2.59 AND 1.31

$P(-2.59 \le Z \le 1.31)$

$= P(Z \le 1.31) – P(Z \le -2.59)$

$= .9049 – .0048 = .9001$

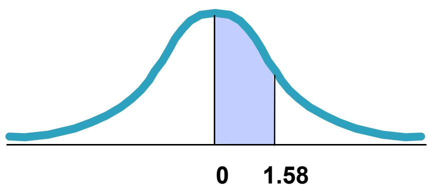


STANDARD NORMAL (Z) DISTRIBUTION

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	* .4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	* .4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Example:

- If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water and if one thermometer is randomly selected, find the probability that it reads freezing water between 0 degrees and 1.58 degrees.



P (0 < x < 1.58) =

Standard Normal (z) Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
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0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
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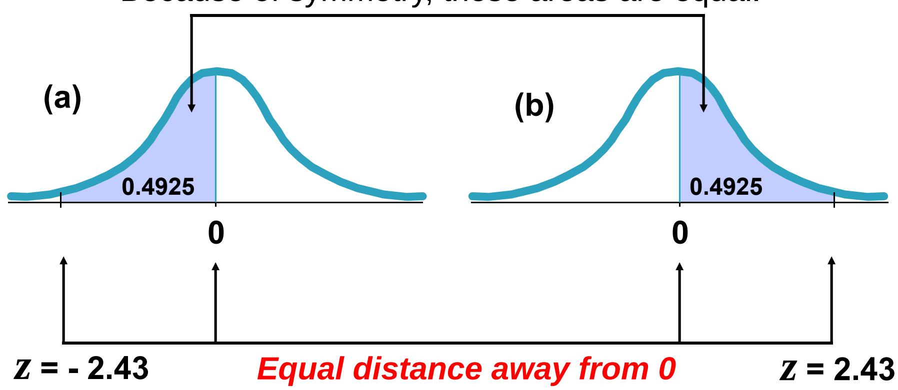
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	* .4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

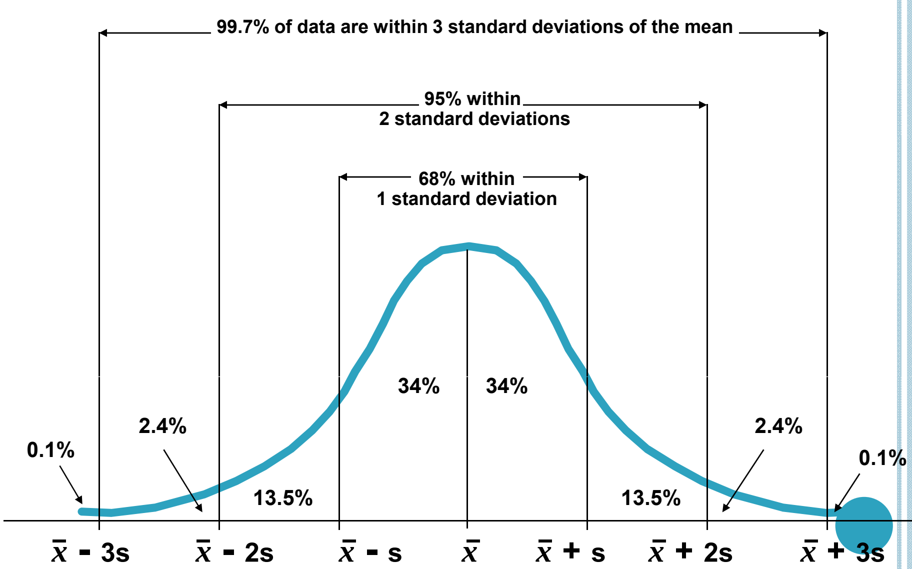
Using Symmetry to Find the Area to the Left of the Mean

Because of symmetry, these areas are equal.

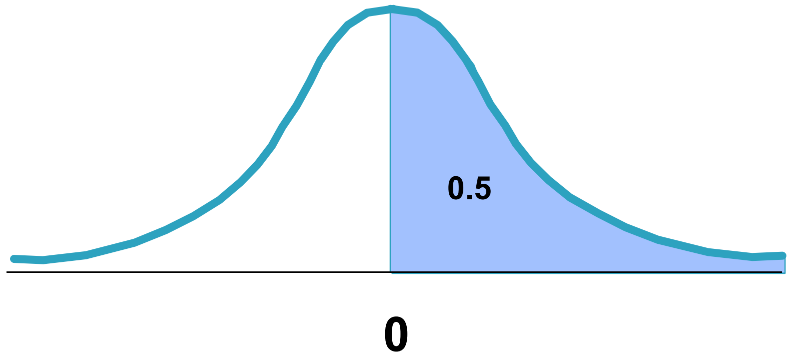


NOTE: Although a *z*-score can be negative, the area under the curve (or the corresponding probability) can never be negative.

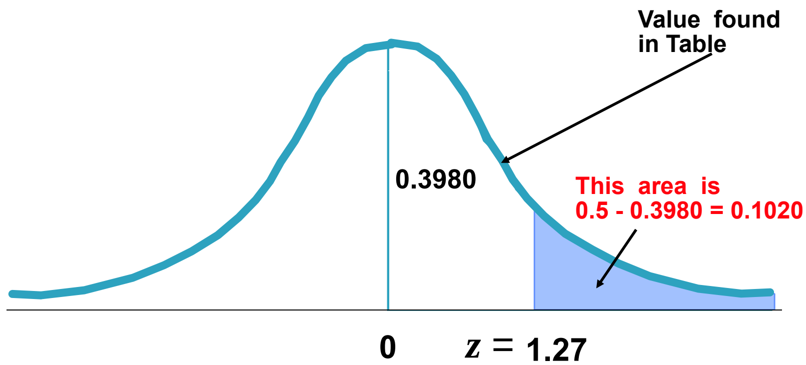
The Empirical Rule
Standard Normal Distribution: $\mu = 0$ and $s = 1$



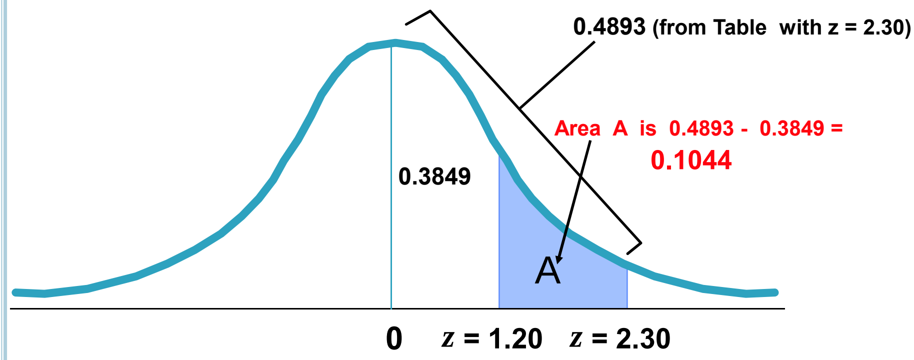
PROBABILITY OF HALF OF A DISTRIBUTION



FINDING THE AREA TO THE RIGHT OF z = 1.27



FINDING THE AREA BETWEEN $z = 1.20$ AND $z = 2.30$



CUMULATIVE DISTRIBUTION FUNCTION

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

$$F_X(x) = 1 - \frac{1}{2}\mathrm{erfc}\left\{\frac{(x - m_x)}{\sqrt{2\sigma_x^2}}\right\}$$

- The area under the pdf curve is calculated using Q(.) function:

$$Q(x) = \frac{1}{2}\mathrm{erfc}\left\{\frac{x}{\sqrt{2}}\right\}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\exp(-z^2/2)\,dz;\ x > 0$$

PROPERTIES OF *Q*(.) FUNCTION

$$P(X>a)=\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_a^{\infty}\exp\left[-\frac{(x-\propto_x)^2}{2\sigma_x^2}\right]dx$$

Let $z=(x-\propto_x)/\sigma_x$

then

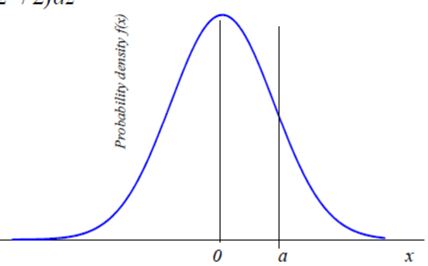
$$P(X>a)=\int_{\frac{(a-\propto_x)}{\sigma_x}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-z^2/2)\,dz$$

$$=Q\left(\frac{a-\propto_x}{\sigma_x}\right)$$

$$P(X<a)=1-Q\left(\frac{a-\propto_x}{\sigma_x}\right)$$

$$\int_a^b f_X(x)=P(a<x<b)$$

$$\int_{-\infty}^{\infty} f_X(x)=1$$



Q-Function Table

z	$Q(z)$	z	$Q(z)$
0.0	0.50000	2.0	0.02275
0.1	0.46017	2.1	0.01786
0.2	0.42074	2.2	0.01390
0.3	0.38209	2.3	0.01072
0.4	0.34458	2.4	0.00820
0.5	0.30854	2.5	0.00621
0.6	0.27425	2.6	0.00466
0.7	0.24196	2.7	0.00347
0.8	0.21186	2.8	0.00256
0.9	0.18406	2.9	0.00187
1.0	0.15866	3.0	0.00135
1.1	0.13567	3.1	0.00097
1.2	0.11507	3.2	0.00069
1.3	0.09680	3.3	0.00048
1.4	0.08076	3.4	0.00034
1.5	0.06681	3.5	0.00023
1.6	0.05480	3.6	0.00016
1.7	0.04457	3.7	0.00011
1.8	0.03593	3.8	0.00007
1.9	0.02872	3.9	0.00005