EC 421 Statistical Communication Theory

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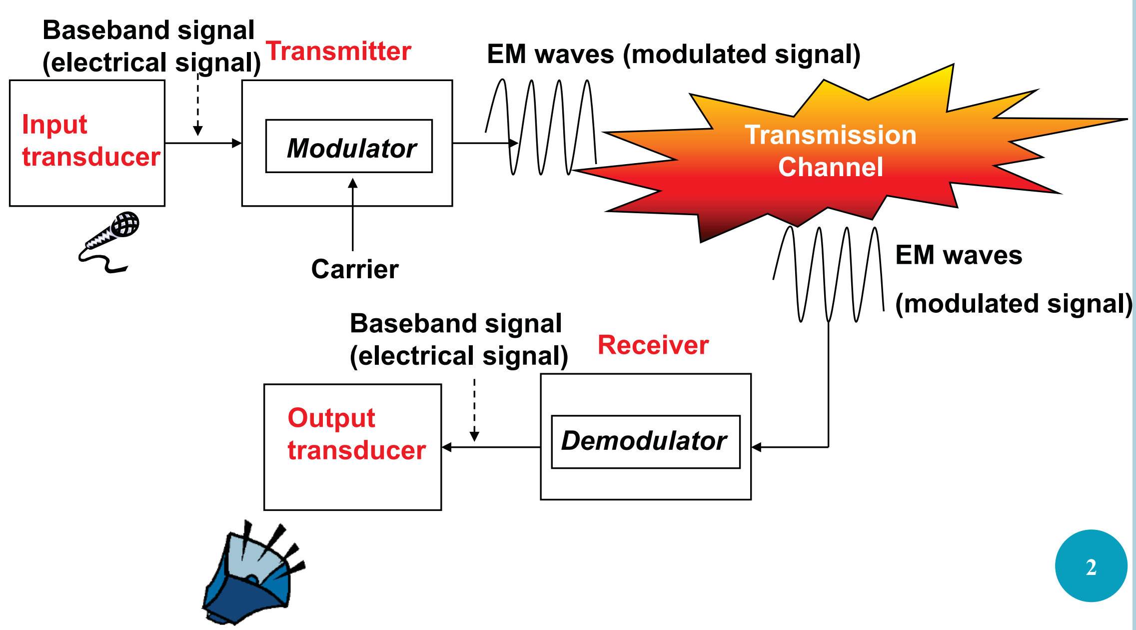
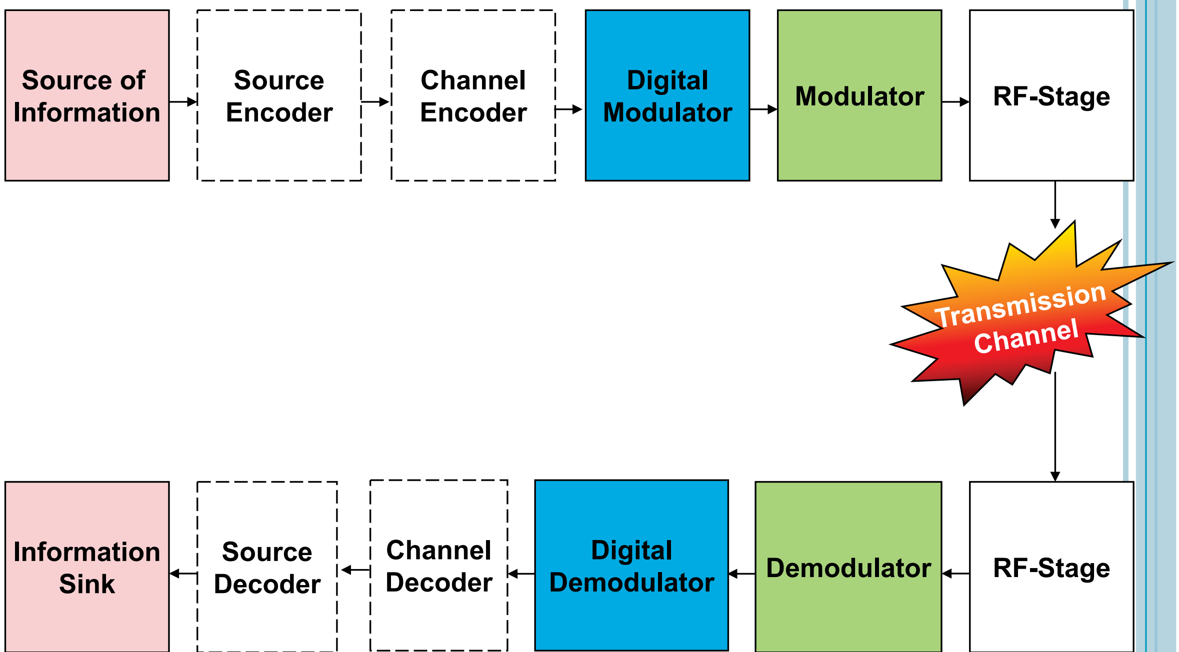
Lecture # 1

Basic analog communications system



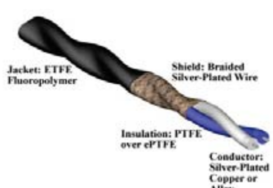
DIGITAL COMMUNICATION SYSTEM



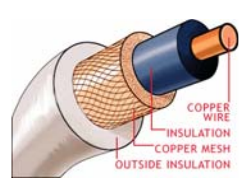
TRANSMISSION CHANNEL

- Wire-line Channel

Twisted pair



Coaxial cable

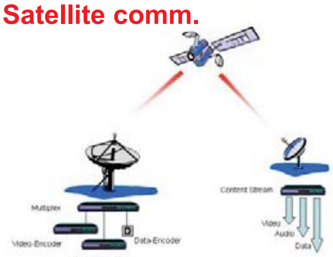


Fiber optics

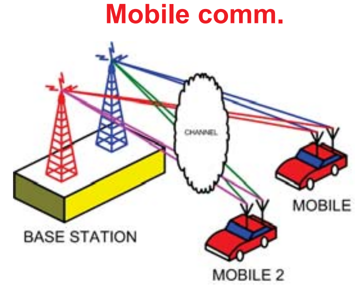
- Wireless Communication Channel

Wireless networks

Satellite comm.



Mobile comm.

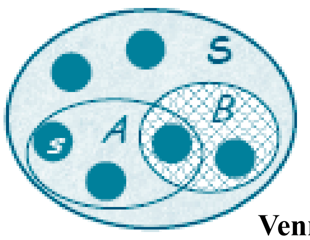


Outcomes, events and sample space

- **Random experiment:** repeating the experiment produces a different (*a priori* unknown) outcome each time.
- **An event *A*** is a set of outcomes:

  A = { *s* : such that *s* is an even number }
- **Sample space** is the set of all possible outcomes.

"outcome" ∈ Event ⊂ Sample Space



Venn diagram

Examples of random experiments

- **Roll a die once and record the result of the top-face:**
- ***S*** = { 1, 2, 3, 4, 5, 6}
- *A* = "the outcome is even" = {2, 4, 6}
- *B* = "the outcome is larger than 3" = {4, 5, 6}
- *C* = "the outcome is odd" = {1, 3, 5}
- **Roll a die once and see if the top-face is even**
- ***S*** = { even, odd } = { *A*, *C* }

Axioms of Probability

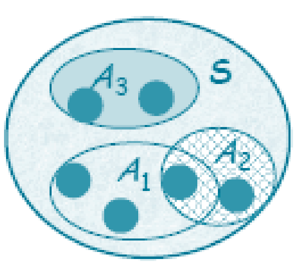
- Probability of any event *A* is non-negative: ***P*[*A*] ≥ 0**.
- The probability that "the outcome belongs to the sample space" is 1: ***P*[*S*] = 1**.
- The probability of "the union of mutually-exclusive events" is the sum of their probabilities:
  - If $A_1 \cap A_2 = \emptyset$, $\Rightarrow$ $\mathbf{P[A_1 \cup A_2] = P[A_1] + P[A_2]}$

Mutual Exclusivity

- **In general**:

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$$

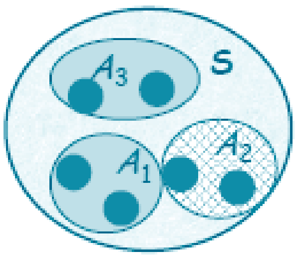
- This formula works for both mutually exclusive and non-mutually-exclusive events



Mutual Exclusivity

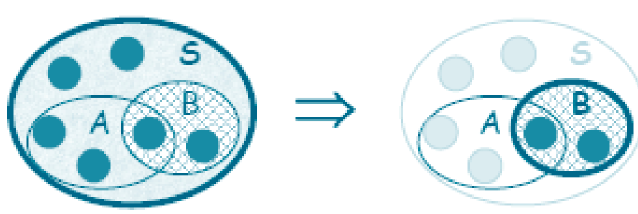
- The probability of "the union of mutually-exclusive events" is the sum of their probabilities:

$$\text{If } A_i \cap A_j = \varnothing,\ i \neq j \quad \Rightarrow \quad P\left[\bigcup_j A_j\right] = \sum_j P[A_j]$$



Conditional Probabilities

- When two events are related, in a sense that one tells us something about the other.
- **Given that an event *B* has occurred, what is the probability of *A*?**
  - Given that *B* has occurred, this reduces the sample space: $S \rightarrow B \subset S$

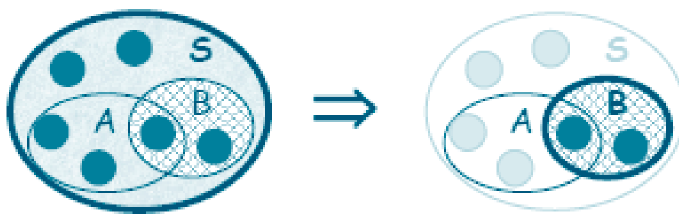


CONDITIONAL PROBABILITIES

- **We need to:**
  - Compute the intersection of *A* and *B*.
  - Normalize the probabilities by *P*[*B*].

$$P[A/B] = P[A \cap B] \;/\; P[B]$$

$$P[A/S] = P[A \cap S] \;/\; P[S]$$



EXAMPLE ON CONDITIONAL PROBABILITY

- Select a ball from an urn containing ***two black*** balls labeled 1 and 2, and ***two white*** balls labeled 3 and 4.
- Assuming equi-probable outcomes, find P[A/B].

$$S = \{(1,b),(2,b),(3,W),(4,W)\}$$

  - *A* = {(1,*b*),(2,*b*)}, "*black ball selected*".
  - *B* = {(2,*b*),(4,*W*)}, "*even-numbered ball selected*".

$$P[A/B] = \frac{P[A \cap B]}{P[B]} = \frac{0.25}{0.5} = 0.5$$

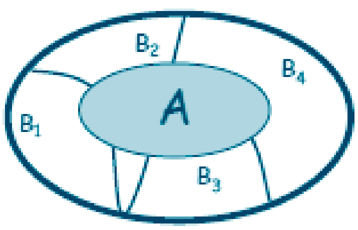
Total Probability

- ***The Law of Total Probability:***
- If B_1 , B_2 ,… B_n form a "partition" of ***S***, then for any event *A*:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots (A \cap B_n)$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots P[A \cap B_n]$$

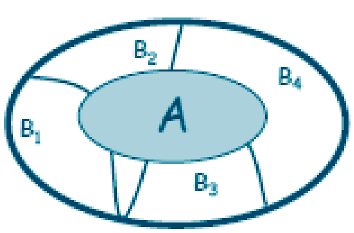
$$P[A] = P[A/B_1].P[B_1] + P[A/B_2].P[B_2] \ldots$$



Bay's Rule

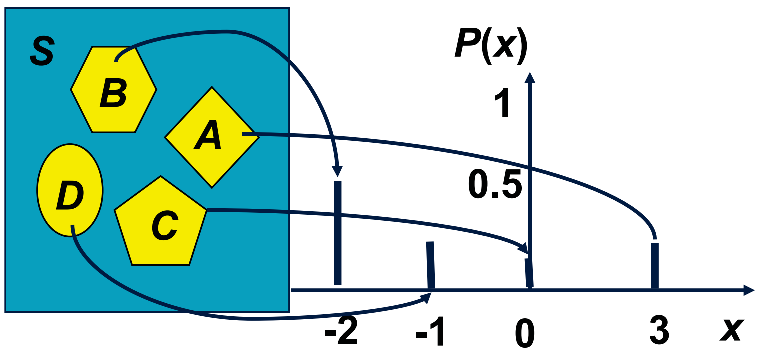
- If B_1 , B_2 ,…B_n form a "partition" of ***S***, then for any event *A*:

$$P[B_j/A] = \frac{P[A/B_j] \cdot P[B_j]}{\sum_{i=1}^{n} P[A/B_i] \cdot P[B_i]}$$



Random Variables

> ***Random Variable***: assigns a number to each outcome of a random experiment, or, equivalently, to each unit in a population.



Event	RV Value	P(x)
A	3	0.2
B	-2	0.5
C	0	0.1
D	-1	0.2

What is a Random Variable?

"Numerical outcome of a random circumstance"

Two different broad classes of random variables:

1. A **continuous random variable** can take any value in an interval or collection of intervals.
2. A **discrete random variable** can take one of a countable list of distinct values.

Discrete Random Variables

X the random variable.

k = a number the discrete r.v. could assume.

$P(X = k)$ is the probability that X equals k.

Discrete random variable: can only result in a countable set of possibilities – often a finite number of outcomes, but can be infinite.

Example: *It is Possible to Toss Forever*

Repeatedly tossing a fair coin, and define:
X = number of tosses until the first head occurs

Any number of flips is a possible outcome.

$P(X = k) = (1/2)^k$

Example: How Many Girls are Likely?

Family has 3 children. Probability of a girl is ?
What are the probabilities of having 0, 1, 2, or 3 girls?

Sample Space: For each birth, write either B or G. There are eight possible arrangements of B and G for three births. These are the *simple events.*

Sample Space and Probabilities: The eight simple events are equally likely.

Random Variable X: number of girls in three births. For each simple event, the value of X is the number of G's listed.

EXAMPLE: HOW MANY GIRLS? (CONT)

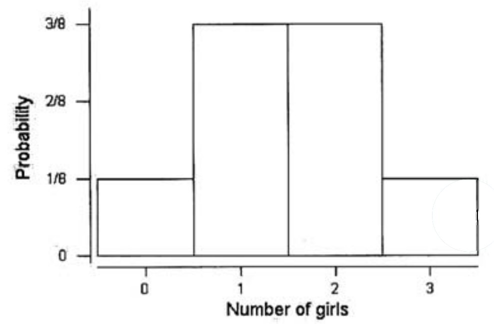
Value of *X* for each simple event:

Simple Event	BBB	BBG	BGB	GBB	BGG	GBG	GGB	GGG
Probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
X = # of Girls	0	1	1	1	2	2	2	3

Probability distribution function for Number of Girls *X*:

k	0	1	2	3
$P(X = k)$	1/8	3/8	3/8	1/8

Graph of the pdf of *X*:



CONDITIONS FOR PROBABILITIES FOR DISCRETE RANDOM VARIABLES

Condition 1

The sum of the probabilities over all possible values of a discrete random variable must equal 1.

Condition 2

The probability of any specific outcome for a discrete random variable must be between 0 and 1.

CUMULATIVE DISTRIBUTION FUNCTION (CDF) OF A DISCRETE RANDOM VARIABLE

Cumulative distribution function (cdf) for a random variable *X* is a rule or table that provides the probabilities $P(X \le k)$ for any real number *k*.

Cumulative probability = probability that *X* is less than or equal to a particular value.

Example: ***Cumulative Distribution Function for the Number of Girls (cont)***

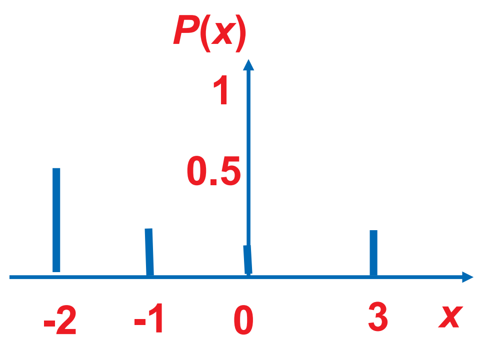
k	0	1	2	3
$P(X \le k)$	1/8	4/8	7/8	1

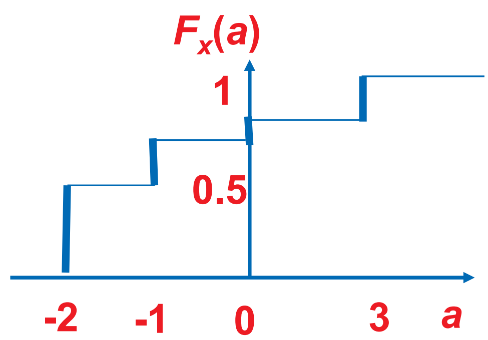
CDF PROPERTIES

- $F_x(a)$ is a non-decreasing function
- $0 \le F_x(a) \le 1$
- $F_x(-infinity) = 0$
- $F_x(infinity) = 1$
- $$F_x(a) = \lim_{\varepsilon \to 0} \int_{-\infty}^{a+\varepsilon} f_x(x)\,dx$$

Cumulative Distribution Function (CDF)

- The cumulative distribution function (CDF) of the RV, x, is given by $F_x(a)=P_x(x \leq a)$

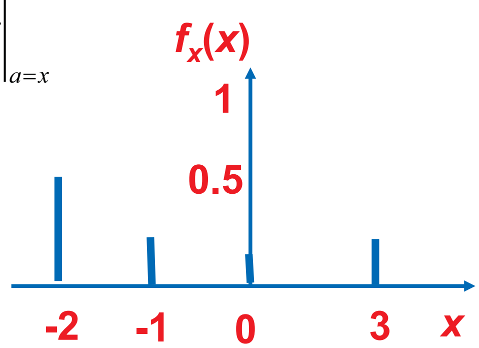




Probability Density Function (CDF)

- The probability density function(PDF) of the RV x is given by $f(x)$
- *Shows how probability is distributed across the axis*

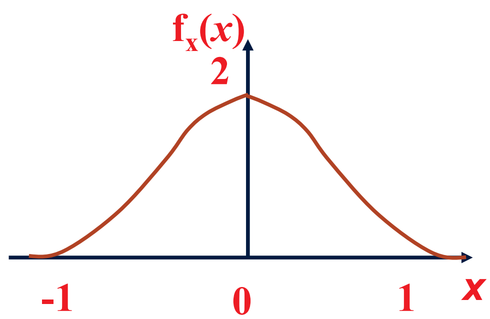
$$f_x(x) = \left.\frac{dF_x(a)}{da}\right|_{a=x} = \left.\frac{dP_x(x \leq a)}{da}\right|_{a=x}$$

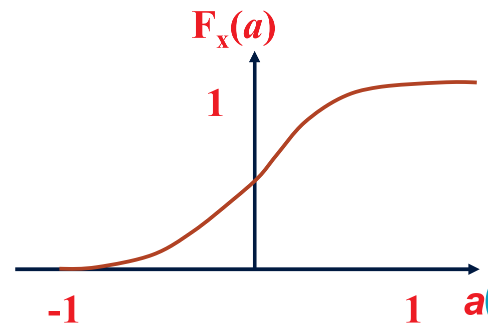


Probability Density Function (PDF) Properties

- $f_x(x)$ is non-negative, $f_x(x) \geq 0$
- The total probability adds up to one

$$\int_{-\infty}^{\infty} f_x(x)\,dx = F_x(\infty) = 1$$



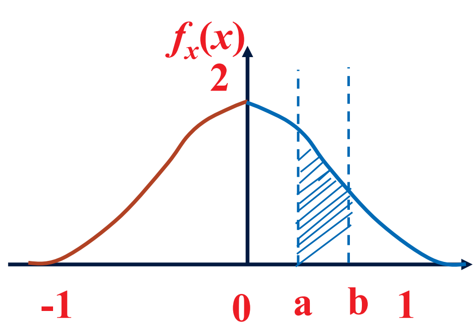


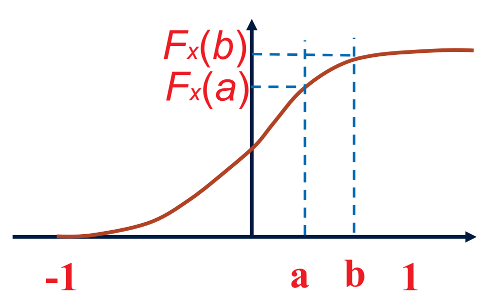
Calculating Probability

- To calculate the probability for a range of values

$$P_x(a < x \leq b) = P_x(x \leq b) - P_x(x \leq a)$$

$$= F_x(b) - F_x(a) = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b+\varepsilon} f_x(x)\,dx$$





Mean of a Random Variable

- **Discrete Random Variable**

$$E(X) = \sum_{i=1}^{n} x_i \, P(X = x_i)$$

$E(.)$ is called expectation of $(.)$

- **Continuous Random Variable**

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$f_X(x)$ is probability density function of x

Moments of a Random Variable

- **Discrete Random Variable**

$$E(X^k) = \sum_{i=1}^{n} x_i^k \, P(X = x_i)$$

- **Continuous Random Variable**

$$E(X^k) = \int_{-\infty}^{\infty} x^k \, f_X(x) \, dx$$

Central Moments

- **Discrete Random Variable**

$$\mathrm{E}((X-E(X))^k) = \sum_{i=1}^{n}(x_i - E(X))^k P(X = x_i)$$

- **Continuous Random Variable**

$$\mathrm{E}(X - E(x))^k) = \int_{-\infty}^{\infty}(x - E(x))^k f_X(x)\,dx$$

Variance of the random variable

$$\mathrm{Var(X)} = \sum_{i=1}^{n}(x_i - E(X))^2 P(X = x_i)$$

$$\mathrm{Std}(X)=\sqrt{\mathrm{Var}(X)}$$

- **Std : is the standard deviation.**
- **Variance is a measure of random variable's randomness**

Standard Deviation for a Discrete Random Variable

The **standard deviation** of a random variable is essentially the average distance the random variable falls from its mean over the long run.

If X is a random variable with possible values x_1, x_2, x_3, . . . , occurring with probabilities p_1, p_2, p_3, . . . , and **expected value** $E(X) = m$, then

$$\text{Variance of } X = V(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i$$

$$\text{Standard Deviation of } X = \sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

Continuous Random Variables

Continuous random variable: the outcome can be any value in an interval or collection of intervals.

Example of CRV = height, time, weight, and money.

Probability density function for a continuous random variable X is a curve such that the area under the curve over an interval equals the probability that X is in that interval.

$P(a \le X \le b)$ = area under density curve over the interval between the values a and b.

CONTINUOUS RANDOM VARIABLES

A density curve (or probability density function) is a graph of a continuous probability distribution. It must satisfy the following 2 properties.

- ✓ **Total area under the curve = 1.0**
- ✓ **Every point on the curve must have a vertical height that is 0 or greater (the curve cannot fall below the x-axis)**